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On the Complexity of Some Geometric Problems in Unbounded Dimension

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This paper examines the complexity of several geometric problems due to unbounded dimension. The problems considered are: (i) minimum cover of points by unit cubes, (ii) minimum cover of points by unit balls, and (iii) minimum number of lines to hit a set of balls. Each of these problems is proven not to have a polynomial approximation scheme unless P = NP. Specific lower bounds on the error ratios attainable in polynomial time are given, assuming P $\neq$ NP. In particular, it is shown that covering by two cubes is in P while covering by three cubes is NP-complete.

1. Introduction

Many results in computational geometry obtained in recent years are concerned with problems in the plane. In some cases generalizations to higher dimensions are known. In this note we are concerned with the dependence of the complexity of certain problems on the dimension of the space. Undoubtedly the most important result related to this question is that the linear programming problem is in the class P. Here, if the dimension of the space, d, is bounded (but the number of linear inequalities n is not), the problem can easily be solved in strongly polynomial time, that is, in $p(n)$ arithmetic operations where $p(n)$ is a polynomial. If d is unbounded then the problem can be solved in polynomial time in terms of the length of the binary representation of the input (Khachiyan, 1979) but is not known to have a strongly polynomial algorithm. The 1-center problem (that is, finding the smallest ball containing n given points) is closely related to linear programming. Here too the problem has a strongly polynomial algorithm when the dimension is

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bounded and a polynomial algorithm when the dimension is unbounded. These results hold even for the weighted version of the problem (Chandrasekaran, 1982) where one is asked to find a point which minimizes the largest weighted distance from any of n points given with positive weights.

There are at least two ways in which the complexity associated with unbounded dimension is manifested. First, there are problems whose restrictions to any bounded dimension are in the class P while the original problems are NP-complete. An example was recently found in the context of separation by hyperplanes. The following problem was shown in (Megiddo, 1988) to be NP-complete: given two disjoint sets of points in a Euclidean space, decide the existence of two hyperplanes that together separate the sets from each other. Note that this problem has a strongly polynomial algorithm if the dimension of the space is bounded. The second kind of complexity is in the context of approximation algorithms. It is shown in (Hochbaum and Maass, 1985) that there exist geometric optimization problems whose restrictions to any bounded dimension are NP-hard, yet these problems have polynomial approximation schemes (Garey and Johnson, 1979). Thus, for any fixed dimension and any $\epsilon > 0$ there exists a polynomial algorithm that provides an ϵ-optimal solution. The degree of the polynomial grows superpolynomially with the dimension. In this note we show that for some of these problems, if the dimension is unbounded then for certain positive values of ϵ, the existence of a polynomial algorithm for ϵ-optimal solutions implies P = NP.

The problems discussed in this note are as follows. In Section 2 we consider the problem of covering points by unit cubes whose edges are parallel to the axes. We show that it is NP-complete to recognize whether three cubes suffice, but it takes only polynomial time to recognize whether two cubes suffice. In Section 3 we consider the problem of covering points by unit balls. We show that it is NP-complete to recognize whether two balls suffice. It follows from the polynomiality of the linear programming problem that it takes polynomial time to recognize whether one ball suffices. In Section 4 we consider the problem of hitting balls by straight lines. We show that it is NP-complete to recognize whether one line suffices.

2. Covering Points by Cubes

In this section we consider the following problem:

Problem 2.1. Covering by cubes. Given a set $S = \{p_1, \cdots, p_m\}$ of points $p_i = (p_{i1}, \cdots, p_{id})^T \in R^d$ and an integer k, find k unit cubes $C_1, \cdots, C_k \subset R^d$ (whose edges are parallel to the coordinate axes) so that $S \subset \bigcup_j C_j$, or recognize that no such cubes exist.

We shall prove that Problem 2.1 with $k = 2$ is in P. The proof will follow from the following proposition:

Proposition 2.2. *A set of points $S = \{p_1, \cdots, p_m\}$ can be packed in a unit cube (with edges parallel to the coordinate axes) if and only if every pair of points $p_i, p_j \in S$ can be packed in such a cube.*

Proof: The proof follows from the fact that S can be packed as required if and only if for every j $(j = 1, \cdots, d)$ $\max_i p_{ij} - \min_i p_{ij} \leq 1$.

In view of Proposition 2.2 it is natural to consider a graph $G = G(S)$, which we call the *cube covering graph*, whose vertices correspond to the points $p_1, \cdots, p_m$ where p_i and p_j $(p_i \neq p_j)$ are joined with an edge if and only if $\|p_i - p_j\|_\infty \leq 1$. Thus, two such points are joined with an edge if and only if they can be packed in a unit cube as above. Proposition 2.2 then says that S can be packed in a cube if and only if the graph $G(S)$ is complete. In the problem of partitioning a graph by cliques (Garey and Johnson, 1979) one is asked to find k subsets of vertices such that each subset induces a complete subgraph and such that each vertex is in exactly one of the subsets. We thus have the following characterization:

Corollary 2.3. *The set S can be covered by k unit cubes as required if and only if the graph $G(S)$ can be partitioned by k cliques.*

The problem of partitioning a graph by k cliques is in P for $k = 2$ since in this case the problem is equivalent to recognizing whether the graph is the complement of a bipartite graph. In that case the bipartite structure of the graph reveals a partition of the set S into two sets which can each be packed in a unit cube with edges parallel to the axes. Conversely, a partition by cliques induces a bipartite structure on the complement graph. We thus have

Corollary 2.4. *There is a polynomial-time algorithm for Problem 2.1 with $k = 2$.*

The problem of partitioning by cliques is NP-complete for $k \geq 3$. However, although Corollary 2.3 characterizes the existence of k covering cubes in terms of partitioning by cliques, it still does not imply that covering by k cubes is NP-complete, unless one shows that every graph can be interpreted in the context of the problem of covering by cubes. We consider the following restricted version:

Problem 2.5. Covering by three cubes. Given a set $S = \{p_1, \cdots, p_m\}$ of points $p_i = (p_{i1}, \cdots, p_{id})^T \in R^d$ where $p_{ij} \in \{0, 1, 2\}$, find three unit cubes $C_1, C_2, C_3 \subset R^d$ (whose edges are parallel to the coordinate axes) so that $S \subset C_1 \cup C_2 \cup C_3$, or recognize that no such cubes exist.

Proposition 2.6. *In unbounded dimension, covering by three cubes is NP-complete.*

Proof: The proof is by reduction from the 3-colorability problem (Garey and Johnson, 1979): Given a graph $G = (V, E)$, does there exist a function $f : V \to \{1, 2, 3\}$ such that $f(u) \neq f(v)$ whenever $(u, v) \in E$? We show that any graph with n vertices can be interpreted as the complement of the cube covering graph of n points in R^n. Let G be any graph with n vertices denoted $V_1, \cdots, V_n$. For each i define $p_i = (p_{i1}, \cdots, p_{in})^T$ as follows. First, $p_{ii} = 2$. Second, for $j \neq i$, if V_i and V_j are adjacent in G then let $p_{ij} = 0$. Otherwise, $p_{ij} = 1$. It is easy to check that V_i and V_j $(i \neq j)$ are adjacent if and only if $\|p_i - p_j\|_\infty > 1$. Thus, p_i and p_j can be covered by one cube if and only if V_i and V_j can be colored with the same color.

Remark 2.7. It follows from Proposition 2.6 that for the problem of finding the minimum number of unit cubes (with edges parallel to the axes) required to cover a set of n given points, there is no polynomial-time approximation algorithm with error ratio less than $33\frac{1}{3}\%$, unless P = NP. In fact, an error ratio less than 100% cannot be achieved in polynomial time (unless P = NP) since the same holds for the graph coloring problem (see Theorem 6.11 in (Garey and Johnson, 1979)) and the reduction preserves the error ratio. The problem of finding the minimum number of covering unit cubes is NP-hard even in the plane (Fowler et al, 1981, Megiddo and Supowit, 1984). However, there exist polynomial-time approximation schemes for this problem in any fixed dimension (Hochbaum and Maass, 1985).

With regard to the problem of covering n points with a single cube whose edges are not necessarily parallel to the axes, we conjecture it is NP-complete in unbounded dimension.

3. Covering by Two Balls

In this section we consider the following problem:

Problem 3.1. Covering by two balls. Given a set $S = \{p_1, \cdots, p_m\}$ of points $p_i = (p_{i1}, \cdots, p_{id})^T \in R^d$, find two unit balls $B_1, B_2 \subset R^d$ so that $S \subset B_1 \cup B_2$, or recognize that no such balls exist.

Proposition 3.2. *Covering by two balls is NP-complete.*

Proof: The proof is by reduction is from 3-SAT. The reduction is based on the following observation. Consider the set $U = \{\pm e^i : i = 1, \cdots, d\} \subset R^d$ of $2d$ unit vectors (where e^i has 1 in the i-th position). It is easy to check that the smallest ball that contains all the vectors $e^1, \cdots, e^d$ (we use the words 'points' and 'vectors' interchangeably) is centered at the point $(1/d)\sum_i e^i$ and has radius $r_d = \sqrt{1 - (1/d)}$. Thus, with two balls of radius r_d

one can cover all the points in U. We wish to view r_d as having length 1. Thus all the distances and vectors in the following argument will have to be scaled by r_d^{-1} in order for the proof to hold for covering by unit balls. This presents no problem. There are precisely 2^{d-1} different ways to cover these points with two balls of radius r_d. Since e^i and $-e^i$ cannot belong to the same ball of radius r_d, one must split the set U into two sets U_1, U_2, each consisting of d mutually orthogonal vectors. Note that any subset of U consisting of d mutually orthogonal vectors can be covered by a ball of radius r_d. We use this fact to represent boolean variables as follows.

Given a set of clauses E_j $(j = 1, \cdots, m)$ with literals taken from the set $\{u_1, \bar{u}_1, \cdots, u_n, \bar{u}_n\}$, let $d = n+1$. We assume without loss of generality that each E_j consists of three distinct variables. A variable u_i $(i = 1, \cdots, n)$ is represented by two unit vectors: e^i and $-e^i$. The role of the vectors e^{n+1} and $-e^{n+1}$ will be clarified later. Let $S \subset U$ be any set of d mutually orthogonal vectors and let $\bar{S}$ denote its complement in U. Thus $e^i \in S$ if and only if $-e^i \notin S$. We say that u_i is "true" relative to S if $e^i \in S$. Otherwise, we say that u_i is "false" relative to S.

Our next step is to represent the clauses E_j $(j = 1, \cdots, m)$ of the satisfiability problem. A clause $E_j = x_j \vee y_j \vee z_j$ is represented by a point p^j which is defined as follows. In the definition we use a positive number α. The value of α has to be fixed so that

$$12\alpha^2 - 4d^{-1}\alpha + d^{-1} < r_d^2 < 12\alpha^2 + d^{-1} .$$

We first set $p^j_{n+1} = 3\alpha$. For $i \leq n$, if the variable u_i does not occur in E_j then we set $p^j_i = 0$. If the literal u_i occurs in E_j, we set $p^j_i = \alpha$, and if $\bar{u}_i$ occurs in E_j, we set $p^j_i = -\alpha$. Denote $P = \{p^j : j = 1, \cdots, m\}$. We claim that $E_1 \wedge \cdots \wedge E_m$ has a satisfying assignment if and only if the set $P \cup U$ can be covered by two balls of radius r_d.

(i) Suppose there is a satisfying assignment. We define a set $S \subset U$ as follows. For $i = 1, \cdots, n$, if u_i is true, we include e^i in S, and if u_i is false, we include $-e^i$ in S. Also, the vector e^{n+1} is included in S. Let q^S denote the center of gravity of S. Thus, $q^S_j = \pm 1/(n+1)$ $(j = 1, \cdots, n+1)$. As noted above, the set U is contained in the union of two balls of radius r_d centered at q^S and $-q^S$. We claim that the set P is contained in the first ball. Consider any point of the form p^j. Let ν denote the number of literals of E_j which are true in the assignment under consideration $(1 \leq \nu \leq 3)$. It is easy to check that

$$\begin{aligned}
\|p^j - q^S\|^2 =& \nu(\alpha - d^{-1})^2 + (3 - \nu)(\alpha + d^{-1})^2 + (n - 3)d^{-2} + (3\alpha - d^{-1})^2 \\
=& 12\alpha^2 - 4\nu d^{-1}\alpha + d^{-1} \\
\leq& 12\alpha^2 - 4d^{-1}\alpha + d^{-1} < r_d^2 .
\end{aligned}$$

(ii) Suppose there are two balls B_1, B_2 of radius r_d which cover the set $U \cup P$. As argued above, since the set U is covered by these two balls, they must be centered at points of the form q^S and $q^{\bar{S}} = -q^S$, where S is a set of d mutually orthogonal vectors from U and q^S is its center of gravity. Without loss of generality assume $e^{n+1} \in S$ and B_1 is centered at q^S. We now set u_i to be true if u_i is true relative to S (false if u_i is false relative to S). We claim that this is a satisfying assignment. Let us first prove that for every j ($j = 1, \cdots, m$) $p^j \notin B_2$. Consider any p^j and let τ ($0 \le \tau \le 3$) denote the number of literals of E_j which are true relative to S. It is easy to see that

$$\begin{aligned}\|p^j - q^{\bar{S}}\|^2 &= \tau(\alpha + d^{-1})^2 + (3-\tau)(\alpha - d^{-1})^2 + (n-3)d^{-2} + (3\alpha + d^{-1})^2 \\ &= 12\alpha^2 + 4\tau d^{-1}\alpha + d^{-1} \\ &\ge 12\alpha^2 + d^{-1} > r_d^2 \,.\end{aligned}$$

It therefore follows that $p^j \in B_1$. This means that

$$\begin{aligned}\|p^j - q^S\|^2 &= \tau(\alpha - d^{-1})^2 + (3-\tau)(\alpha + d^{-1})^2 + (n-3)d^{-2} + (3\alpha - d^{-1})^2 \\ &= 12\alpha^2 - 4\tau d^{-1}\alpha + d^{-1} < r_d^2 < 12\alpha^2 + d^{-1}\end{aligned}$$

from which it follows that $\tau \ge 1$.

Remark 3.3. It follows from Proposition 3.2 that for the problem of finding the minimum number of balls required to cover a set of n given points, there is no polynomial-time approximation algorithm with error ratio less than 50%, unless P = NP. As in Remark 2.7, the exact problem is NP-hard even in the plane (Fowler et al, 1981, Megiddo and Supowit, 1984) and there exist polynomial-time approximation schemes for this problem in any fixed dimension (Hochbaum and Maass, 1985).

4. Hitting Balls in One Shot

In this section we present another geometric problem where unbounded dimension implies NP-completeness.

Problem 4.1. One-shot. Given n balls in R^d, recognize whether there exists a straight line in R^d which intersects all of them.

It is interesting to note that the problem of recognizing whether there exists a point in the intersection of n given balls is in P since it can be solved as a linear programming problem. However, replacing the point by a line makes the problem harder:

Proposition 4.2. *One-shot is NP-complete.*

Proof: The proof is by reduction is from 3-satisfiability and is similar to the proof of Proposition 3.2. Consider the set $U = \{\pm e^i : i = 1, \cdots, d\} \subset R^d$ as in the proof of Proposition 3.2 and also denote $r_d = \sqrt{1-(1/d)}$. Let B_i^+ and B_i^- denote balls of radius r_d centered at e^i and $-e^i$, respectively. Note that the intersection of the balls $B_1^+, \cdots, B_d^+$ contains exactly one point, namely, the point $d^{-1}\sum_i e^i$. Moreover, for every vector of signs $\epsilon = (\epsilon_1, \cdots, \epsilon_d)$ $\epsilon_i \in \{+,-\}$ $(i = 1, \cdots, d)$, the intersection of the balls $B_1^{\epsilon_1}, \cdots, B_d^{\epsilon_d}$ contains only the point $q(\epsilon) \equiv d^{-1}\sum_i \epsilon_i e^i$. Also, $B_i^+ \cap B_i^- = \emptyset$ $(i = 1, \cdots, d)$. It follows that there are precisely 2^{d-1} straight lines $\ell(\epsilon) \equiv \{tq(\epsilon) : t \in R\}$ that hit all the balls $B_i^{\pm}$. Note that $\ell(\epsilon) = \ell(-\epsilon)$. We use this fact to represent boolean variables as follows. Given a set of clauses E_j $(j = 1, \cdots, m)$ with literals taken from the set $\{u_1, \bar{u}_1, \cdots, u_n, \bar{u}_n\}$, let $d = n + 1$. We assume without loss of generality that each E_j consists of three distinct variables. A variable u_i $(i = 1, \cdots, n)$ is represented by two balls: B_i^+ and B_i^-. Given a vector ϵ as above, we say that u_i is "true" relative to ϵ if $\epsilon_i = +$. Otherwise, we say that u_i is "false" relative to ϵ. A clause $E_j = x_j \vee y_j \vee z_j$ is represented by a ball B_j^* of radius r_d centered at a point p^j which is defined as in the proof of Proposition 3.2, but here the constant α is chosen so that

$$(12 - 4d^{-1})\alpha^2 < r_d^2 < 12\alpha^2 \ .$$

We claim that $E_1 \wedge \cdots \wedge E_m$ has a satisfying assignment if and only if there exists a single line which hits all the balls $B_1^{\pm}, \cdots, B_d^{\pm}, B_1^*, \cdots, B_m^*$.

(i) Suppose there is a satisfying assignment. For $i = 1, \cdots, n$, let $\epsilon_i = +$ if u_i is true, and let $\epsilon_i = -$ if u_i is false. Also let $\epsilon_{n+1} = +$. Consider the line $\ell(\epsilon)$. As noted above, the lines $\ell(\epsilon)$ hits all the balls $B_i^{\pm}$ $(i = 1, \cdots, m)$. Let $f(t)$ be the square of the distance from p^j to $t \cdot q(\epsilon)$. The squared distance between p^j and $\ell(\epsilon)$ is the minimum of the function

$$f(t) = \tau(\alpha - t)^2 + (3 - \tau)(\alpha + t)^2 + (n - 3)t^2 + (3\alpha - t)^2$$

where τ is the number of true literals in E_j. The minimum is at $t = 2\tau\alpha/(n+1)$ and its value is $(12 - 4\tau^2 d^{-1})\alpha^2$. Since for every j $(j = 1, \cdots, m)$, $\tau \geq 1$, it follows that this value is less than r_d^2 and hence the line $\ell(\epsilon)$ intersects the ball B_j^* $(j = 1, \cdots, m)$.

(ii) Suppose there exists a single line ℓ that hits all the balls. It follows from the preceding discussion that ℓ must equal one of the lines $\ell(\epsilon)$. Since $\ell(\epsilon) = \ell(-\epsilon)$, we may assume without loss of generality that $\epsilon_{n+1} = +$. Set u_i to be true if $\epsilon_i = +$ and false otherwise. We claim that this is a satisfying assignment. Consider any clause E_j. Since the line $\ell(\epsilon)$ intersects the ball B_j^* the minimum of the function $f(t)$ defined above must be less than or equal to r_d^2. In other words, $(12 - 4\tau^2 d^{-1})\alpha^2 \leq r_d^2$. By the choice of α this can be true only if $\tau \geq 1$ which means that E_j is true.

Remark 4.3. Proposition 4.2 has a consequence with regard to the optimization problem of minimizing the number of shots required to hit n given balls. Approximation algorithms with finite performance ratios are presented in (Hassin and Megiddo, 1990) for cases of this problem in bounded dimension. In view of Proposition 4.2, there is no polynomial-time approximation algorithm for this problem with guaranteed error of less than 100% unless P = NP.

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