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NOTE

ON FINDING A MINIMUM DOMINATING SET IN A TOURNAMENT

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Abstract. The problem of finding a minimum dominating set in a tournament can be solved in $n^{O(\log n)}$ time. It is shown that if this problem has a polynomial-time algorithm, then for every constant C, there is also a polynomial-time algorithm for the satisfiability problem of boolean formulas in conjunctive normal form with m clauses and $C \log^2 m$ variables. On the other hand, the problem can be reduced in polynomial time to a general satisfiability problem of length L with $O(\log^2 L)$ variables. Another relation between the satisfiability problem and the minimum dominating set in a tournament says that the former can be solved in $2^{O(\sqrt{v})}n^K$ time (where v is the number of variables, n is the length of the formula, and K is a constant) if and only if the latter has a polynomial-time algorithm.

1. Introduction

It is easy to pose restricted versions of NP-complete problems for which there exist algorithms which run in subexponential (yet superpolynomial) time. For

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example, consider the problem where one has to decide whether a given graph with n vertices has a clique of $\lceil \log_2 n \rceil$ vertices. Enumeration of all subgraphs induced by sets of $\lceil \log_2 n \rceil$ vertices reveals the answer. This procedure runs in $n^{O(\log n)}$ time. It would be very interesting to know whether this problem has a polynomial-time algorithm. The restriction of the clique problem to size $\lceil \log_2 n \rceil$ is somewhat unnatural. However, one would expect the complexity of the restricted problem to be similar if the size were restricted to $\lceil c \log_2 n \rceil$ for any positive constant c.

In this paper we consider a problem which is in a way more natural. This is the problem of finding a minimum dominating set in a tournament. It is in fact a restricted version of the minimum dominating set problem on a directed graph but there is no explicit restriction on the size of the set itself. However, it is easy to show (see Section 2 for definitions and proofs) that in any tournament with n vertices there exists a dominating set with $\lceil \log_2 n \rceil$ vertices and hence the problem has an $n^{O(\log n)}$ algorithm. We do not know whether it can be solved in polynomial time. However, in Section 3, we show that it is a hard problem in the following sense. We prove that if there exists a polynomial-time algorithm for the minimum dominating set problem in a tournament, then, for every constant C, there is also a polynomial-time algorithm for the class of satisfiability problems (of boolean formulas in conjunctive normal form) in which the number of variables is bounded by $C \log^2 m$ variables (where m is the number of clauses). On the other hand, the problem of minimum dominating set in a tournament can be reduced in polynomial time to a satisfiability problem (not necessarily in conjunctive normal form) with $O(\log^2 L)$ variables, where L is the length of the formula. It is not known to us whether general satisfiability problems of length L with $O(\log^2 L)$ variables can be reduced in polynomial time to satisfiability problems in conjunctive normal form keeping the number of variables $O(\log^2 L)$. However, in Section 4, we do prove a theorem which says that the satisfiability problem can be solved in time $2^{O(\sqrt{v})} n^K$ (where v is the number of variables, n is the length of the formula, and K is a constant) if and only if the minimum dominating set problem in a tournament can be solved in polynomial time. This equivalence sheds some light on the complexity of finding a minimum dominating set in a tournament. The naive exhaustive search takes $O(2^v n)$ time. There are only few NP-complete problems for which improvements over the naive exhaustive search are known. For instance, Tarjan and Trojanowski [7] give a search algorithm for finding a maximal independent set in a graph in time $O(2^{v/3})$, where v is the number of vertices. For more examples and discussion, see [2, Section 6].

2. Some properties of tournaments

Definition 2.1. A *tournament* is a directed graph $T = (V, E)$ where every two vertices $u, v \in V$ are connected by one edge, that is, $|E \cap \{(u, v), (v, u)\}| = 1$. If the edge is (u, v), then we say that *u dominates v*.

For more about tournaments, see [6].

Definition 2.2. A set $S \subset V$ is *dominating* in a tournament $T = (V, E)$ if for every $v \notin S$ there exists a $u \in S$ which dominates v.

We are concerned with the following computational problem:

Problem 2.3 (*Minimum dominating set*). Given a tournament $T = (V, E)$, find a dominating set $S \subset V$ of minimum cardinality.

Remark 2.4. Let $\mu(G)$ denote the cardinality of a minimum dominating set in G. In the literature on tournaments one finds a definition of property P_k of a tournament. A tournament T is said to have property P_k if for every set S of k vertices of T there exists a vertex v which dominates every vertex in S. Obviously, T has property P_k if and only if $\mu(T) > k$.

The following fact is attributed to Erdös [6, p. 28]. We include the simple proof for completeness.

Fact 2.5. *If T is a tournament with n vertices ($n \geq 2$), then $\mu(T) \leq \lceil \log_2 n \rceil$.*

Proof. Let $d(u)$ denote the number of vertices dominated by a vertex u. Obviously, $\sum_u d(u) = \frac{1}{2}n(n-1)$. It follows that there exists at least one vertex which dominates at least $\lceil \frac{1}{2}(n-1) \rceil$ vertices. Thus a dominating set can be found as follows. Pick a vertex u_1 which dominates at least $\lceil \frac{1}{2}(n-1) \rceil$ vertices. Remove u_1 and all the vertices dominated by u_1 together with the edges incident on these vertices, and continue recursively. Following the removal at the first recursive step, the remaining tournament has at most $\lceil \frac{1}{2}(n-1) \rceil$ vertices. This process finds a dominating set of no more than $\lceil \log_2 n \rceil$ vertices. □

Corollary 2.6. *A minimum dominating set in a tournament can be found in $n^{O(\log n)}$ time.*

Proof. In view of Fact 2.5, a minimum dominating set can be found by enumerating all subsets of V of cardinality not greater than $\lceil \log_2 n \rceil$. There are $\sum_{i=1}^{\lceil \log_2 n \rceil} \binom{n}{i}$ such subsets and this establishes the proof. □

Erdös [1] used the probabilistic method to prove the following fact.

Fact 2.7. *For every $\varepsilon > 0$ there is a number K such that for every $k \geq K$ there exists a tournament T_k with no more than $2^k k^2 \log(2+\varepsilon)$ vertices such that $\mu(T_k) > k$.*

Proof. Consider a random tournament T with n vertices; that is, for every pair of vertices, u, v, the direction of the edge connecting u and v is chosen to be (u, v) or

(v, u) with equal probability, independently of the directions of the other edges. Thus the probability that vertex u dominates vertex v is $\frac{1}{2}$. For every set S of k vertices and every vertex $u \notin S$, the probability that u dominates every vertex in S is 2^{-k}. The probability that S is dominating is hence $(1-2^{-k})^{n-k}$. The expected number of dominating sets of cardinality k is $\binom{n}{k}(1-2^{-k})^{n-k}$. If n is sufficiently large so that the latter is less than 1, then there exists a tournament T on n vertices so that $\mu(T) > k$. The claim follows by showing that if $n > 2^k k^2 \log(2+\varepsilon)$, then $\binom{n}{k}(1-2^{-k})^{n-k} < 1$. □

Corollary 2.8. *There exists a constant $c > 0$ such that for every n there exists a tournament T with n vertices such that $\mu(T) > c \log n$.*

Graham and Spencer [3] give an explicit construction of tournaments with similar properties as follows. For every k, if p is a prime such that $p > 2^{2k-2}k^2$ and $p \equiv 3 \pmod 4$, then the construction given in Definition 2.9 below gives a tournament T on p vertices so that $\mu(T) > k$.

Definition 2.9. Given a number k, let p denote the smallest prime such that $p > 2^{2k-2}k^2$ and $p \equiv 3 \pmod 4$. Define a tournament $T(k)$ with p vertices corresponding to the residues modulo p. For $u, v \in Z_p$ $(u \neq v)$ we say that u dominates v if and only if $u - v$ is a square, that is, $u - v \equiv a^2 \pmod p$ for some $a \in Z_p$.

Remark 2.10. We note that $T(k)$ is a tournament since in every prime field Z_p $(p \neq 2)$, half of the nonzero elements are squares. Moreover, if $p \equiv 3 \pmod 4$, then -1 is not a square. Thus, in this case $w \neq 0$ is a square if and only if $-w$ is not a square.

For our purposes it is important to know the rate of growth of the function $p^*(k)$ that assigns a prime p to each number k as in Definition 2.9. The question is of course related to the density of prime numbers in arithmetic progressions.

Proposition 2.11. $p^*(k) = O(2^{2k}k^2)$.

Proof. The following is taken from Huxley's book [4]. Let $\Lambda(j) = \log p$ if $j = p^i$ where p is prime and $i \geq 1$, and $\Lambda(j) = 0$ otherwise. Let

$$\psi(x; d, a) = \sum_{\substack{j \leq x \\ j \equiv a \pmod d}} \Lambda(j).$$

If a and d are relatively prime (in our case $a = 3$ and $d = 4$), then (see [4, p. 72])

$$\psi(x; d, a) = \frac{x}{\varphi(d)} + O(x\, e^{-c\sqrt{\log x}})$$

where $\varphi(j)$ denotes Euler's function (giving the number of integers $i < j$ relatively prime to j), and c is a certain constant. It follows that, for every a and d relatively prime and every $\alpha > 0$, there exists a K such that for all $x \geq K$, between x and $(1+\alpha)x$ there is at least one prime $p \equiv a \pmod d$. This implies our claim. □

Definition 2.12. Given a tournament $T=(V, E)$ and a positive integer r, we define another tournament $T^r=(V^r, E^r)$ as follows. Let $V^r=V\times\{1,\ldots,r\}$. For all i, j ($1\leq i, j\leq r$) and $u, v\in V$ ($u\neq v$), let (u, i) dominate (v, j) in T^r if and only if u dominates v in T. For $i\neq j$ and $u\in V$, let (u, i) dominate (u, j) if and only if $i<j$.

It is easy to verify that T^r is indeed a tournament.

Proposition 2.13. *For every tournament $T=(V, E)$, an integer r, and any i ($i=1,\ldots,r$), if S is a set of vertices of T^r (see Definition 2.12) which dominates all the vertices in $V\times\{i\}$, then the cardinality of S is at least $\mu(T)$.*

Proof. Let S be a set of vertices which dominates the set $V\times\{i\}$. Thus, for every $u\in V$, either there is a $j\leq i$ such that $(u, j)\in S$ or there is a $v\in V$ such that $(v, j)\in S$ for some j ($1\leq j\leq r$) and v dominates u in G. Let S' denote the projection of S on V, that is, S' is the set of vertices $v\in V$ such that $(v, j)\in S$ for some j. Obviously, S' is a dominating set in T. Since the cardinality of S' does not exceed that of S, our claim follows. □

3. Some classes between P and NP

The reader is referred to [2] for clarification of concepts related to reductions, the satisfiability problem, etc.

Definition 3.1. We define here the complexity classes $\mathrm{SAT}_{\log^k n}$, $\mathrm{SAT}^{\mathrm{CNF}}_{\log^k n}$ and $\mathrm{P}_{\log^k n}$ ($k>1$) (the definition of the latter is essentially the same as in [5]).

(i) A language L is in $\mathrm{SAT}_{\log^k n}$ if there exists a Turing machine M, a polynomial $p(n)$, and a constant C, such that for every string I of length n, M converts I in $p(n)$ time into a boolean formula ϕ_I (whose length is necessarily less than $p(n)$) with at most $C\log^k n$ variables, so that $I\in L$ if and only if ϕ_I is satisfiable.

(ii) The definition of $\mathrm{SAT}^{\mathrm{CNF}}_{\log^k n}$ is essentially the same as of $\mathrm{SAT}_{\log^k n}$ except that the formula ϕ_I is in conjunctive normal form.

(iii) A language L is in $\mathrm{P}_{\log^k n}$ if there exist a nondeterministic Turing machine M, a polynomial $p(n)$ and a constant C, such that M recognizes instances of length I in $p(n)$ time using no more than $C\log^k n$ nondeterministic steps.

It is easy to see that, for each $k\geq 1$,

$$\mathrm{P}\subset\mathrm{SAT}^{\mathrm{CNF}}_{\log^k n}\subset\mathrm{SAT}_{\log^k n}\subset\mathrm{P}_{\log^k n}\subset\mathrm{NP}.$$

Thus, if any of the inclusions in this chain is proper, then $\mathrm{P}\neq\mathrm{NP}$.

Problem 3.2 (DOMT). Given a tournament T with n vertices $v_1,\ldots,v_n$ and an integer K, recognize whether T contains a dominating set with no more than K vertices.

Proposition 3.3. *The problem* DOMT *is in the class* $\mathrm{SAT}_{\log^2 n}$.

Proof. Without loss of generality assume $K \leq \lceil \log_2 n \rceil$. Denote $l = \lceil \log_2 n \rceil + 1$. We use zero-one variables x_{ij} $(i = 1, \ldots, K, j = 1, \ldots, l)$ to describe an ordered set S of at most K vertices, which is our candidate for a dominating set. The variable x_{ij} signifies the jth digit in the binary representation of the index h (of a vertex v_h which is chosen as the ith member of a set S. Thus, the ith member is the vertex v_h where $h = \sum_{j=1}^{l} x_{ij} 2^{j-1}$. For any integer h, let $b_j(h)$ denote the jth binary digit of h, that is, $h = \sum_j b_j(h) 2^{j-1}$. The proposition: "The ith member of the set S is the vertex v_h" is expressed by the conjunction:

$$\phi_{ih} = \bigwedge_{j=1}^{l} \{x_{ij} = b_j(h)\}.$$

The proposition: "The vertex v_s is either in the set S or dominated by some member of S" is expressed by the disjunction:

$$\phi_s^* = \bigvee_{v_h \subset D_s} \bigvee_{i=1}^{K} \phi_{ih}$$

where D_s is the union of $\{v_s\}$ with the set of vertices which dominate v_s. Finally, the proposition: "S is a dominating set" is expressed by the conjunction:

$$\phi = \bigwedge_{s=1}^{n} \bigvee_{v_h \in D_s} \bigvee_{i=1}^{K} \bigwedge_{j=1}^{l} \{x_{ij} = b_j(h)\}.$$

Obviously, ϕ is satisfiable if and only if there exists a dominating set with no more than k vertices. The proof follows since ϕ has $O(\log^2 n)$ variables and its length is $O(n^3 \log^2 n)$. □

Theorem 3.4. *Every* $L \in \mathrm{SAT}^{\mathrm{CNF}}_{\log^2 n}$ *is reducible in polynomial time to* DOMT.

Proof. Let us denote by $\mathrm{SAT}(k, C)$ the set of instances of the satisfiability problem (in conjunctive normal form) with v variables and m clauses where $v \leq C(\log_2 m)^k$. Let $L \in \mathrm{SAT}^{\mathrm{CNF}}_{\log^2 n}$. Thus, there is a constant $C = C(L)$ and a polynomial $P_L(n)$ such that every $I \in L$ of length n can be reduced in $P(n)$ time to an instance of $\mathrm{SAT}(2, C)$. We now show that, for any C, there is a polynomial-time reduction from $\mathrm{SAT}(2, C)$ to the problem of minimum dominating set in a tournament. Let $\phi = E_1 \wedge \cdots \wedge E_m$ be a boolean formula where each E_i is a disjunction and the total number of distinct variables occurring in ϕ is not greater than $C \log^2 m$. Without loss of generality, assume the number of variables is precisely Cl where $l = \log_2 m$, and let us rename them for convenience with double indices: x_{ij} $(1 \leq i \leq Cl, 1 \leq j \leq l)$. We first construct a certain tournament T with $\mu(T) \geq Cl + 1$, and then prove that $\mu(T) = Cl + 1$ if and only if ϕ is satisfiable.

(i) The construction of the tournament T. The vertices of T are organized in groups as follows (see Fig. 1).

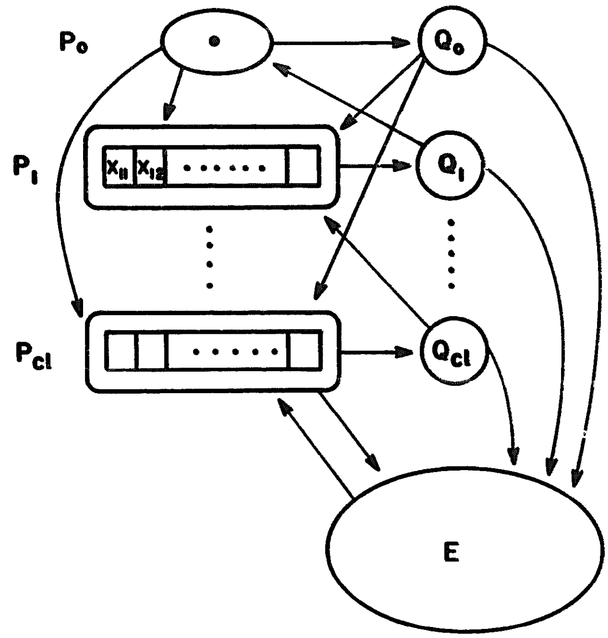

Fig. 1.

(1) For each clause E_τ ($\tau = 1, \dots, m$), assign a vertex with the same name in T. Denote $E = \{E_1, \dots, E_m\}$.

(2) For $i = 1, \dots, Cl$, let P_i be disjoint sets of m vertices each ($P_i \cap E = \emptyset$). Denote their vertices by p_{ih} ($i = 1, \dots, Cl$, $h = 0, \dots, m-1$) and let $P = \bigcup_{i=1}^{Cl} P_i$.

(3) Let $P_0 = \{u^*\}$ be the singleton set of an additional vertex.

(4) Let $k = Cl + 1$. Let $T(k)$ be the tournament constructed in Definition 2.9. Let $r = l + 1$ and let $(T(k))^r$ denote the tournament constructed in Definition 2.12. The vertices of $(T(k))^r$ will also be vertices of our tournament T. Denote this set by Q. Vertices in Q will be denoted by q_{ih} ($i = 0, \dots, Cl$, $h = 1, \dots, p^*(Cl+1)$), and the set of those with the same index i will be denoted by Q_i. (Recall that $p^*(Cl+1)$ is the number of vertices in $T(k)$).

We now define the edges of the tournament T.

(5) Within the set Q dominations are induced by the tournament $(T(k))^r$ as explained above (see Step (4)).

(6) Every vertex in Q dominates all the vertices in E.

(7) For every i ($i = 0, 1, \dots, Cl$), every vertex in P_i dominates all the vertices in Q_i. For $i \neq j$ ($0 \le i, j \le Cl$), every vertex in Q_i dominates all the vertices in P_j.

(8) The vertex u^* dominates all the vertices in P (as well as those in Q_0) and is dominated by all the vertices in E (as well as those in $\bigcup_{i=1}^{Cl} Q_i$).

(9) Within the set P and within the set E the dominations are set arbitrarily.

(10) The dominations between vertices in P and vertices in E are more complicated and depend on the structures of the clauses. These dominations are designed to establish the following connection with the assignment of truth values to the variables x_{ij}. Suppose there exists a dominating set for T which contains exactly one vertex P_{ih} ($0 \le h \le m-1$) for each i ($i = 0, \dots, Cl$). Let $b_j(h)$ denote the jth

binary digit of h; that is, $h=\sum_{j=1}^{l} b_j(h)2^{j-1}$. Then x_{ij} is true if $b_j(h)=1$ and false otherwise. Consider any vertex E_τ. The vertices p_{ih} which dominate E_τ are determined as follows. If the literal x_{ij} occurs in E_τ, then E_τ is dominated by all the vertices p_{ih} where $b_j(h)=1$. Analogously, if the literal $\bar{x}_{ij}$ occurs in E_τ, then E_τ is dominated by all the vertices p_{ih} where $b_j(h)=0$. In all other cases, E_τ dominates p_{ih}.

(ii) We claim that $\mu(T)\geq Cl+1$. To prove this fact, suppose, to the contrary, that S is a dominating set of vertices such that $|S|\leq Cl$. It follows that there exists an i $(0\leq i\leq Cl)$ such that $S\cap P_i=\emptyset$. Thus, to dominate the members of Q_i, the set S must use elements of Q. However, by the construction and from Proposition 2.13 it follows that S must contain at least $Cl+2$ elements of Q. The contradiction proves our claim.

(iii) Suppose ϕ has a satisfying assignment. We now show that in this case $\mu(T)=Cl+1$. Let $\xi_{ij}=1$ if x_{ij} is true and $\xi_{ij}=0$ otherwise $(i=1,\ldots,Cl, j=1,\ldots,l)$. Let S be the set of vertices consisting of u^* and all the vertices p_{ih} where $h=\sum_{j=1}^{l}\xi_{ij}2^{j-1}$ $(i=1,\ldots,Cl)$. Clearly, $|S|=Cl+1$. The set S is dominating since u^* dominates P, each set Q_i is dominated by the single member of P_i which is contained in S $(i=0,\ldots,Cl)$, and each E_τ is dominated by a member of S corresponding to a literal which makes E_τ true in the satisfying assignment.

(iv) Suppose $\mu(T)=Cl+1$. We now show that ϕ has a satisfying assignment. From part (ii) of this proof it follows that $S\cap P_i\neq\emptyset$ $(i=0,\ldots,Cl)$. Thus, S contains a unique element from each of the sets P_i. In other words, for each i $(i=1,\ldots,Cl)$ there is a unique $h=h(i)$ such that $p_{i,h(i)}\in S$. We now set x_{ij} to be true if $b_j(h(i))=1$ and false otherwise. It is easy to verify that each clause E_τ is satisfied since the corresponding vertex E_τ is dominated by some member of S.

(v) We finally argue that the reduction we have described runs in polynomial time. First, the reduction from L to SAT(2, C) takes $P(n)$ time. In the reduction from SAT(2, C) we construct a tournament T with $Cm\log m+1$ vertices in the sets P_i, $(C\log m+1)p^*(C\log m+2)$ vertices in Q, and m vertices in P. In view of Proposition 2.11, the function $p^*(C\log m+2)$ is polynomial in m and exponential in C. However, C is constant for a fixed L and hence for every L the size of T is polynomial in terms of the length of the instance of L. The same argument holds for the time it takes to construct the tournament $(T(C\log m+2))^{C\log m+1}$. Here we have to compute the prime number $p^*(C\log m+2)=O(m^C\log^2 m)$. This can obviously be done in polynomial time in terms of m in a brute-force way. Note that this prime number depends only on the numbers m and C and not on the particular instance. Altogether, the reduction is polynomial for every L in $\mathrm{SAT}^{\mathrm{CNF}}_{\log^2 n}$. □

4. The relation between dominating set and satisfiability

In this section we provide another interpretation to the results of Section 3. First, note that the complete enumeration algorithm for the satisfiability problem runs in $O(2^v n)$ time, where n is the length of the formula and v is the number of variables.

It is not known whether there exists an algorithm for the same problem which runs in $2^{O(\sqrt{v})}n^K$ time for any constant K. We claim that such an algorithm exists if and only if there exists a polynomial-time algorithm for the minimum dominating set problem in a tournament. This will follow from the following three propositions.

Proposition 4.1. *If there is a polynomial-time algorithm for* DOMT, *then there is an algorithm for the* (CNF)-*satisfiability problem with v variables and m clauses which runs in $2^{O(\sqrt{v})}m^K$ time for some constant K.*

Proof. We rely on the reduction described in the proof of Theorem 3.4. Let ϕ be a (CNF)-formula ϕ with v variables and m clauses. Without loss of generality, assume the number $2v$ is a square of an even number, $2v=(2\nu)^2$. We reduce the problem of deciding the satisfiability of ϕ to a problem of recognizing whether a tournament T with N vertices (the dependence of N on v and m will be described below) has a dominating set of cardinality $\nu+1$. Following the notation of the proof of Theorem 3.4, the tournament has the following structure. First, it has m vertices corresponding to the m clauses of ϕ. Second, it has disjoint sets of vertices $P_0, P_1, \ldots, P_\nu$ as follows. The set P_0 consists of a single vertex. Each set P_i $(i=1,\ldots,\nu)$ consists of $2^{2\nu}$ vertices. Thus, $\sum_{i=1}^{\nu}\log_2|P_i|=2\nu^2=v$. The vertices of the third type are organized in disjoint sets Q_i $(i=0,1,\ldots,\nu)$ where each set contains $p^*(\nu+1)=O(2^{2(\nu+1)}(\nu+1)^2)$ vertices. The total number of vertices is hence $m+2^{O(\sqrt{v})}$. If there is a polynomial-time algorithm for DOMT, then the satisfiability problem has an algorithm which runs in $2^{O(\sqrt{v})}m^K$ time where K is a constant. □

Proposition 4.2. *There is a linear-time reduction from the general satisfiability problem to the* (CNF)-*satisfiability problem.*

Proof. Let $\lambda(\phi)$ denote the total number of occurrences of variables in the formula ϕ. We call $\lambda(\phi)$ the *length* of ϕ. We actually prove the following claim: for every formula $\phi(x_1,\ldots,x_k)$ of length $\lambda(\phi)$ there is a (CNF)-formula $\phi'(x_1,\ldots,x_k,y)$ of length $\lambda(\phi')\leq 7\lambda(\phi)$ such that $\phi'(x_1,\ldots,x_k,y)$ is equivalent to the formula $\phi(x_1,\ldots,x_k)\equiv y$ (the formulas use only $\wedge$ and $\vee$; the connective $\equiv$ is used here for brevity). The proof goes by induction on $\lambda(\phi)$. The claim is trivial for formulas of length 1. Let ϕ be any formula of length greater than 1. Thus, ϕ is either the conjunction or the disjunction of two shorter formulas. Suppose

$$\phi(x_1,\ldots,x_k)=\phi_1(x_1,\ldots,x_k)\vee\phi_2(x_1,\ldots,x_k).$$

By the induction hypothesis applied to the formulas ϕ_1 and ϕ_2, there exist (CNF)-formulas $\phi_i'(x_1,\ldots,x_k,y_i)$ $(i=1,2)$ which are equivalent to $\phi_i(x_1,\ldots,x_k)\equiv y_i$ $(i=1,2)$, respectively. Obviously, $\lambda(\phi)=\lambda(\phi_1)+\lambda(\phi_2)$. Now, let $\phi'(x_1,\ldots,x_k,y_1,y_2,y_3)$ denote the following (CNF)-formula

$$\phi_1'(x_1,\ldots,x_k,y_1)\wedge\phi_2'(x_1,\ldots,x_k,y_2)\wedge(y_1\vee y_2\vee\bar{y}_3)\wedge(\bar{y}_1\vee y_3)\wedge(\bar{y}_2\vee y_3).$$

The conjunction of the last three terms in this formula is equivalent to $y_3 \equiv (y_1 \vee y_2)$. Thus, we have reduced ϕ to (CNF)-formula of length not greater than $7\lambda(\phi)$. The other possibility, $\phi = \phi_1 \wedge \phi_2$, is handled analogously. □

Proposition 4.3. *If there is an algorithm for the satisfiability problem which runs in $2^{O(\sqrt{v})}n^K$ time (where v is the number of variables, n is the length of the formula, and K is a constant), then there is a polynomial-time algorithm for* DOMT.

Proof. The proof follows easily from Proposition 3.3. The problem DOMT with N vertices can be reduced to a formula whose length is polynomial in N, and whose number of variables is $O(\log^2 N)$. This establishes the proof. □

We now have the following theorem.

Theorem 4.4. *The general satisfiability problem can be solved in $2^{O(\sqrt{v})}n^K$ time (where v is the number of variables, n is the length of the formula, and K is a constant) if and only if the minimum dominating set problem in a tournament has a polynomial-time algorithm.*

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