Lecture 6: Proofs

Secure Compilation Seminar

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Why Proofs?

- large systems
- require a lot of time
- building and planning focus on two different aspects
- proofs ensure that the building is doable

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- proofs ensure that the building is doable (also why we have design patterns for coding)

How to Prove?

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- IF we can assume something (P)
- THEN some other thing holds (Q)

Reduction ad Absurdum (or contradiction)

$$P \Rightarrow Q$$

- assume P
- assume $\neg Q$
- derive $\bot$ i.e., any contradiction (R and $\neg R$)

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- generally Q has an infinite universal quantification

Structural Induction

$$P \to Q$$

- generally done when Q has a (finite) structure
- e.g., reduction cases, typing cases, syntax

Contrapositive

$$P \Rightarrow Q$$

becomes

$$\neg Q \Rightarrow \neg P$$

and becomes oftentimes easier

What do we Prove?

- What are P and Q?

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$$\llbracket \cdot \rrbracket^{\mathsf{S}}_{\mathbf{T}} \text{ is FAC} \overset{\text{def}}{=} \forall \mathsf{P}_1, \mathsf{P}_2$$

$$\mathsf{P}_1 \simeq_{ctx} \mathsf{P}_2 \iff \llbracket \mathsf{P}_1 \rrbracket^{\mathsf{S}}_{\mathbf{T}} \simeq_{ctx} \llbracket \mathsf{P}_2 \rrbracket^{\mathsf{S}}_{\mathbf{T}}$$

Fully Abstract Compilation

- break the $\iff$:
  1. $\Rightarrow$: $\forall P_1, P_2.\ P_1 \simeq_{ctx} P_2 \Rightarrow [\![P_1]\!]^{S}_{\mathbf{T}} \simeq_{ctx} [\![P_2]\!]^{S}_{\mathbf{T}}$
  2. $\Leftarrow$: $\forall P_1, P_2.\ [\![P_1]\!]^{S}_{\mathbf{T}} \simeq_{ctx} [\![P_2]\!]^{S}_{\mathbf{T}} \Rightarrow P_1 \simeq_{ctx} P_2$
- point 2 (should) follow from compiler correctness

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    2. $\Leftarrow$: $\forall \mathsf{P}_1, \mathsf{P}_2.\ [\![\mathsf{P}_1]\!]^{\mathsf{S}}_{\mathbf{T}} \simeq_{ctx} [\![\mathsf{P}_2]\!]^{\mathsf{S}}_{\mathbf{T}} \Rightarrow \mathsf{P}_1 \simeq_{ctx} \mathsf{P}_2$
- point 2 (should) follow from compiler correctness
- point 1 is tricky, because of $\simeq_{ctx}$ and its $\forall \mathfrak{C}$

Trace Semantics

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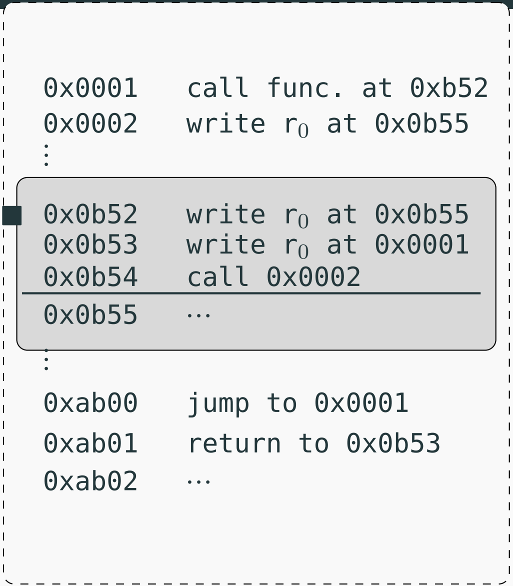
- a semantics that abstracts from the context (observer)
- and still describes the behaviour of a program precisely
- a trace semantics

Traces for PMA



0x0001 call func. at 0xb52

0x0002 write r_0 at 0x0b55

⋮

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0x0b54 call 0x0002

0x0b55 ⋯

⋮

0xab00 jump to 0x0001

0xab01 return to 0x0b53

0xab02 ⋯

- interest in the behaviour of the module

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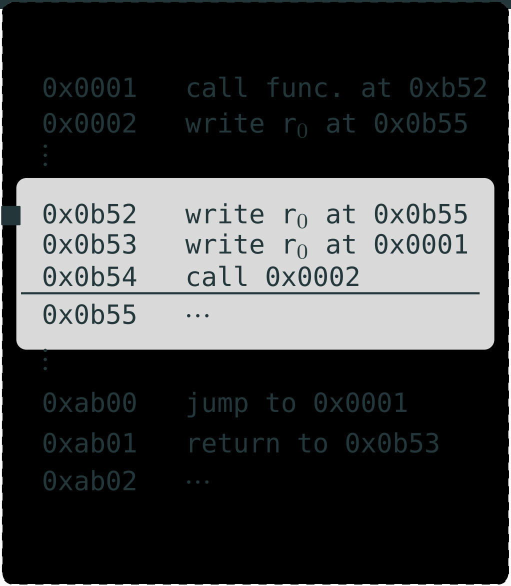

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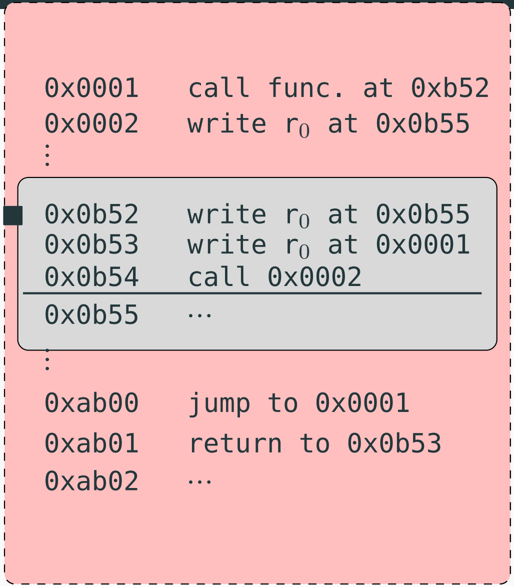

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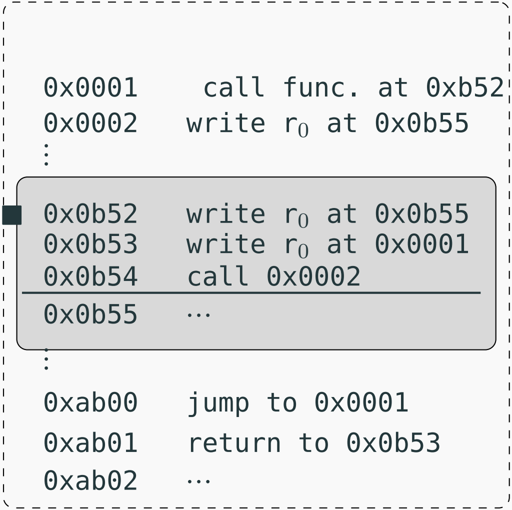

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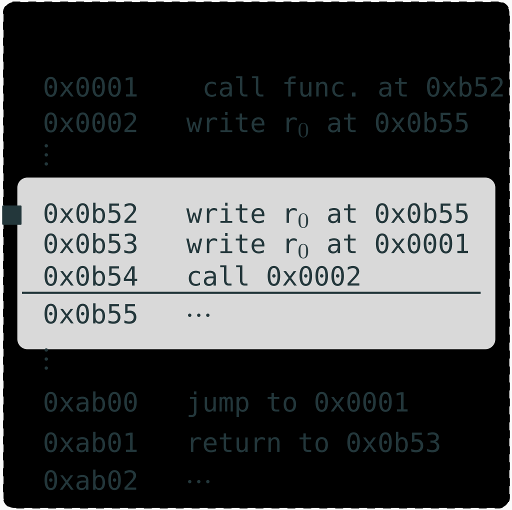

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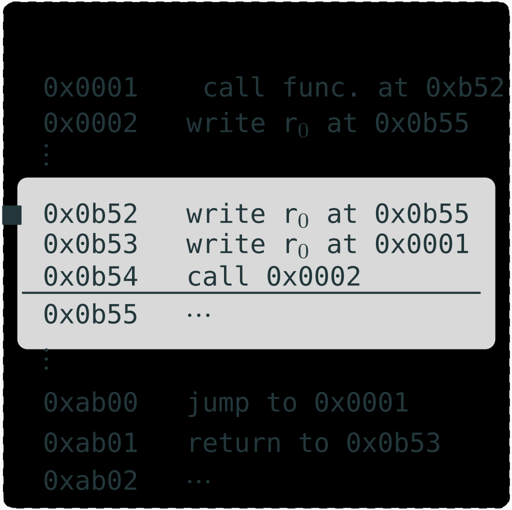

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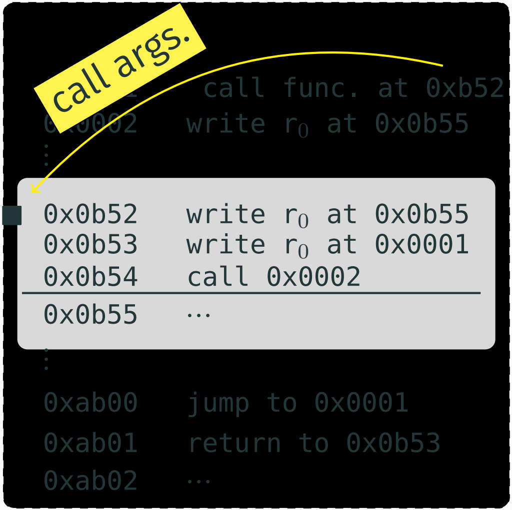

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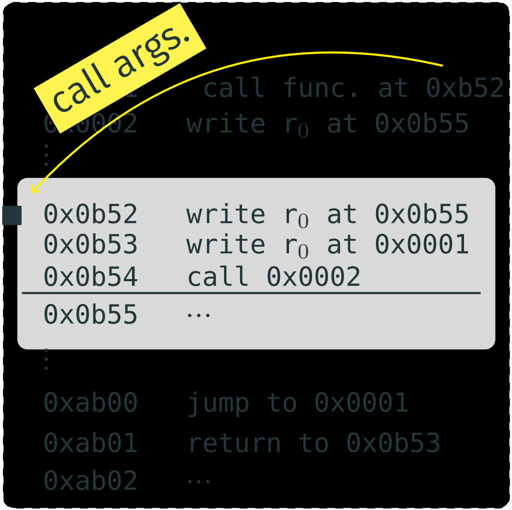

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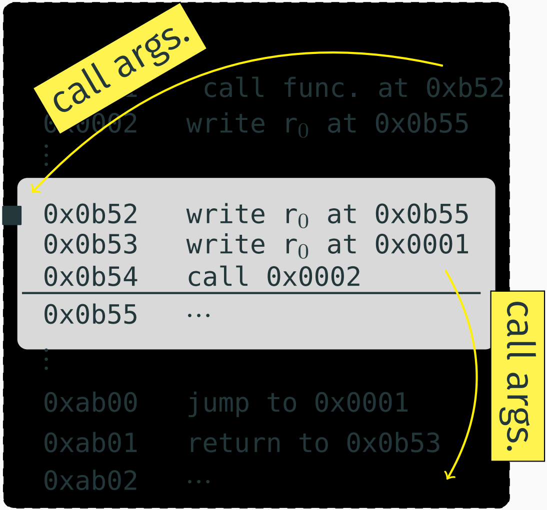

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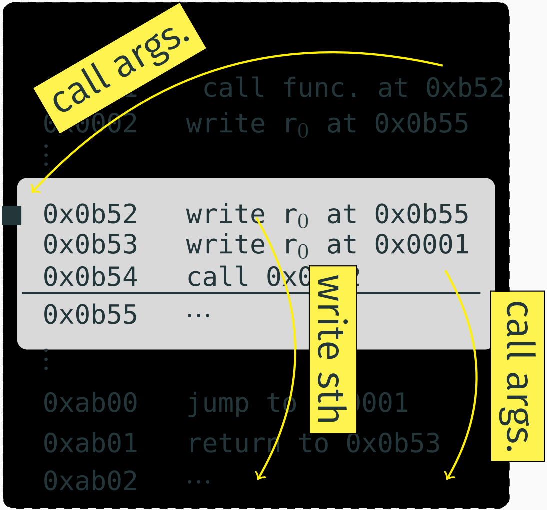

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- abstract the module behaviour from the rest perspective:
  1. call/return outside
  2. read/write

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- denotational: describes the behaviour of a component as sets of traces
- a trace is (typically) a sequence of actions that describe how a component interacts with an observer
- without needing to specify the observer
- indicated as $\mathsf{TR}(C) = \left\{ \overline{\alpha} \;\middle|\; C \overset{\overline{\alpha}}{\Longrightarrow} \_ \right\}$

Trace Actions

$$\begin{aligned}
\mathit{Labels} \quad & L ::= a \mid \epsilon \\
\mathit{Observable\ actions} \quad & \alpha ::= \surd \mid g? \mid g! \\
\mathit{Actions} \quad & g ::= \texttt{call}\ p\ (r) \mid \texttt{ret}\ p\ r(\texttt{r}_0)
\end{aligned}$$

Traces for PMA

We need to define:

- trace states (almost program states) Θ
- labels that make traces
- rules for generating labels and traces $\cdots$
- the traces of a component $\mathsf{TR}(C) = \cdots$

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Proofs about Trace Semantics

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  correct (⇐) and complete (⇒)

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- we have:
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- unfold $\simeq_{ctx}$
- backtranslation!

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- we need to prove
    - $\exists \alpha \in \mathsf{TR}(\mathbf{C_1}), \alpha \notin \mathsf{TR}(\mathbf{C_2}) \Rightarrow \exists \mathsf{C}.\mathsf{C}[\mathsf{C}_2] \not\simeq \mathsf{C}[\mathsf{C}_2]$
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Backtranslation at work

to the board