

CS 154 - Introduction to Automata and Complexity Theory

Spring Quarter, 2009

Assignment #6 - Due date: Tuesday, 5/26/09

Problem 1. [20 points] Consider the following two languages over the alphabet $\Sigma = \{0, 1\}$.

$$L_H = \{\langle M, w \rangle \mid \text{TM } M \text{ halts on input } w\}$$

$$L_{NC} = \{\langle \widehat{M} \rangle \mid L(\widehat{M}) \text{ is non-context-free}\}$$

The language L_H corresponds to the following decision problem called the *halting problem*: given a Turing machine M and an input string w , does M halt on input w ? Similarly, the language L_{NC} corresponds to the following decision problem: given a Turing machine \widehat{M} , is $L(\widehat{M})$ a non-context-free language?

Your goal is to show that L_{NC} is undecidable, assuming that L_H is undecidable. We propose the following reduction from L_H to L_{NC} .

Given a Turing machine M and input w , construct a Turing machine \widehat{M} which behaves as follows on being given input \widehat{w} .

1. \widehat{M} simulates the behavior of M on input w ;
2. if M halts on w , then \widehat{M} checks to see if its input \widehat{w} is of the form xx for some string x , halting in a final state if \widehat{w} is indeed xx and halting in a non-final state otherwise.

- (a). [5 points] In the case where M does not halt on w , what is $L(\widehat{M})$?
(b). [5 points] In the case where M does halt on w , what is $L(\widehat{M})$?
(c). [10 points] Show that L_{NC} is undecidable.

Problem 2. [15 points] Consider the following problem concerning Turing machines with a tape alphabet Γ .

Given a Turing machine M , input string w , and a symbol $X \in \Gamma$, decide whether M , when running on input w , will ever write the symbol X on its tape.

Show that this problem is undecidable. Is this problem recursively enumerable? (*Hint*: Can you give a reduction from the universal language L_U ?)

Problem 3. [15 points] Prove that the following language L is non-recursive:

$$L = \{\langle M_1, M_2 \rangle \mid L(M_1) \subseteq L(M_2)\}.$$

The strings in L encode two Turing machines M_1 and M_2 such that the language of M_1 is a subset of the language of M_2 .

To prove this result you may use the fact that the language L_{all} is not recursively enumerable:

$$L_{all} = \{\langle M \rangle \mid L(M) = \Sigma^*\}.$$

Problem 4. [15 points] Consider the following language:

$$L = \{ \langle M_1, M_2, M_3 \rangle \mid L(M_1) = L(M_2) \cdot L(M_3) \}$$

The strings in L encode three Turing machines with languages L_1 , L_2 and L_3 , such that $L_1 = L_2 \cdot L_3$. You are told that following language L_\emptyset is not recursively enumerable:

$$L_\emptyset = \{ \langle M \rangle \mid L(M) = \emptyset \}.$$

Give a *reduction* from L_\emptyset to L and prove that L is not recursively enumerable.

Problem 5. [25 points] Consider the following language involving the encodings of context-free grammars G_1 and G_2 .

$$L_t = \{ \langle G_1, G_2 \rangle \mid |L(G_1) \cap L(G_2)| \geq t \}.$$

Show that this language L_t is undecidable for all t , using a reduction from PCP (Post's Correspondence Problem). (*Hint:* See the proof of Theorem 9.22(a) on page 417 of the textbook.)

Problem 6. [10 points] Recall that Rice's Theorem states that every nontrivial property of the recursively enumerable languages is undecidable. For each of the languages given below, choose one of the following three statements which best describes the situation.

- A. The given language is decidable.
- B. The given language is undecidable and this follows from a *direct* application of Rice's Theorem.
- C. The given language is undecidable but we cannot directly apply Rice's Theorem to show this.

Indicate your choice by writing A, B, or C for each language description. Note that in all cases M refers to a Turing machine.

1. $L = \{ \langle M \rangle \mid L(M) \text{ is a context-free language} \}$.
2. $L = \{ \langle M \rangle \mid L(M) \text{ is recursively enumerable} \}$.
3. $L = \{ \langle M \rangle \mid \text{when started on a blank tape, Turing machine } M \text{ eventually halts in a final state} \}$.
4. $L = \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \}$.
5. $L = \{ \langle M, w \rangle \mid M \text{ accepts } w \text{ in less than 100 steps} \}$.

Reading Assignment: We are currently working on Chapter 9. We may touch upon Section 9.4 but will not cover it in any detail. Section 9.5 will be omitted. In the final two weeks of classes, we will be talking about Chapter 10.