Problem 1

(a) Exercise 3.2.6(c): The set of prefixes of strings in \( L \).

(b) Exercise 3.2.6(d): The set of substrings of strings in \( L \).

Problem 2

(a) \((10 + 0)^* (\epsilon + 1 + 11) (01 + 0)^*\) - the first and third sub-expressions ensure that there are no consecutive 1s, while the second allows for the presence of a single pair of consecutive 1s.

(b) \((1)^* (0 + 111)^* (1)^*\) - while the expression can begin and end with any number of 1s, a 0 must be followed by either zero 1s or at least two 1s before another 0.

Problem 3

(a) \[ R_{33}^{(0)} = a + b + c + \epsilon \]
\[ R_{12}^{(0)} = a \]
\[ R_{12}^{(4)} = a(ba)^* \]
\[ R_{11}^{(0)} = \epsilon \]
\[ R_{11}^{(4)} = (ab)^* \]

(b) \((ab)^* \epsilon\)

Problem 4

Assume \( L \) is regular and apply the Pumping Lemma.

By the P.L., there exists some \( n > 0 \).

We choose \( w = 0^n1^{2n} \) which is in \( L \) and also satisfies \( |w| \geq n \).

By the P.L., \( w = xyz \) such that \( |xy| \leq n \) and \( y \neq \epsilon \). Note that all \( x \) and \( y \) contain only 0s. Let \( |x| = a \) and \( |y| = b \), then \( 0 < b \leq n \).

We choose \( k = 2 : xy^2z \in L \) by P.L., but \( xy^2z = 0^{n+b}1^{2n} \) and \( b \) is at least 1. Since \( 2n < 2(n+b) \), this string cannot be in \( L \). Thus we have a contradiction, hence \( L \) cannot be regular. \( \square \)

Problem 5

Assume \( L \) is regular and apply the Pumping Lemma.

By the P.L., there exists some \( n > 0 \).

We choose \( w = 0^n \) which is in \( L \) and also satisfies \( |w| \geq n \) since \( n^3 > n \) for all \( n > 0 \).

By the P.L., \( w = xyz \) such that \( |xy| \leq n \) and \( y \neq \epsilon \). Note that all \( x \), \( y \) and \( z \) contain only 0s. Let
\[ |x| = a \text{ and } |y| = b, \text{ then } 0 < b \leq n. \]

We choose \( k = 2 : x y^2 z \in L \) by P.L., but \( xz = 0^{n^2 + b} \). But \( n^3 + b \) cannot be a perfect cube since the bounds on \( b \) imply the following:

\[
n^3 < n^3 + b \leq n^3 + n < n^3 + 3n^2 + 3n + 1 = (n + 1)^3
\]

Note that \( n < 3n^2 + 3n + 1 \) holds for all \( n \geq 0 \) because the quadratic \( 3n^2 + 2n + 1 \) has a negative discriminant. So \( n^3 + b \) is strictly between the values of two consecutive perfect cubes and cannot be a perfect cube itself. Thus we have a contradiction, hence \( L \) cannot be regular. \( \square \)

**Problem 6**

(a) \( b^i e^m : x = \epsilon, y = b^n \) and \( z = b^{-n} e^m \). It is easy to see that the first two conditions are satisfied. Since there are no \( a \), there are no constraints on the string, and condition 3 is satisfied as well.

\( a b^m e^m : x = \epsilon, y = a \) and \( z = b^m e^m \). It is easy to see that the first two conditions are satisfied. For \( i = 0 \), there are no constraints. For \( i = 1 \), we already have the same powers of \( b \) and \( c \). For all other \( i \), we have strings with no constraints because the power of \( a \) is greater than 1. Thus, condition 3 is also satisfied.

\( a^2 b^i e^m : x = \epsilon, y = a^2 \) and \( z = b^i e^m \). It is easy to see that the first two conditions are satisfied. For \( i = 0 \), there are no constraints. For all other \( i \), we have strings with no constraints because the power of \( a \) is greater than 1. Thus, condition 3 is also satisfied.

\( a^3 b^i e^m : x = \epsilon, y = a^3 \) and \( z = b^i e^m \). It is easy to see that the first two conditions are satisfied. For \( i = 0 \), there are no constraints. For all other \( i \), we have strings with no constraints because the power of \( a \) is greater than 1. Thus, condition 3 is also satisfied.

(b) In general, it is not true that if \( A \Rightarrow B \) then \( B \Rightarrow A \).

(c) **Solution 1:** Assume that \( L \) is regular. Consider the language \( L_1 = \{ab^*c^*\} \) — clearly, this is a regular language. Now, consider the language \( L_2 = L \cap L_1 = \{ab^m c^m | m \geq 0\} \). Since \( L_1 \) is regular, it follows that if \( L \) is regular then \( L_2 \) must also be regular by the closure of regular languages under intersection. But we can prove via the pumping lemma that \( L_2 \) is not regular, so it follows that \( L \) cannot be regular. The proof that \( L_2 \) is not regular is similar to the pumping lemma proof that \( \{0^n 1^n | n \geq 0\} \) is not regular and is omitted.

**Solution 2:** We know that if a language \( L \) is regular, then its reverse \( L^R \) must also be regular.

\[
L^R = \{e^m b^k a^l | m, l, k \geq 0, k = 1 \Rightarrow l = m\}
\]

Now assume \( L^R \) is regular and apply the Pumping Lemma. By the P.L., there exists some \( n > 0 \). We choose \( w = e^n b^n a \) which is in \( L^R \) and also satisfies \( |w| \geq n \). By the P.L., \( w = xyz \) such that \( |x| \leq n \) and \( y \neq \epsilon \). Note that all \( x \) and \( y \) contain only \( e's \). Let \( |y| = m \), then \( 0 < m \leq n \). We choose \( k = 2 : x y^2 z \in L \) by P.L., but \( x y^2 z = e^{n+m} b^n a \) and \( m \) is at least 1. Since \( n + m \neq n \), this string cannot be in \( L^R \). Thus we have a contradiction, hence \( L^R \) cannot be regular. \( \square \)