Some more challenges with algebraic manipulation

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1 Toolbox

Below is a general list of techniques that are useful in algebraic manipulations. Please note that algebraic manipulation is more of a tool and is not tailored to specific types of problems.

- Factoring
- Rationalizing or unrationalizing denominators
- Substitution
- Expanding, possibly with binomial theorem
- Finding and using symmetry of some sort
- Looking for patterns (small cases)
- Wishful thinking
- Partial fractions

2 Practice Problems

The problems are roughly arranged in increasing order of difficulty. Feel free to work with others.

1. Sams phone company charges him a per-minute charge as well as a connection fee (which is the same for every call) every time he makes a phone call. If Sam was charged \$4.88 for an 11-minute call and \$6.00 for a 19-minute call, how much would he be charged for a 15-minute call? (EMC2 2011)

2. $A = 10^9 - 987654321, B = \dfrac{123456789 + 1}{10}$, Compute $\sqrt{AB}$. (HMMT 2010)

3. Given that $x^2 + 5x + 6 = (x^2 + 5x + 4)^2$, find all possible real values of x.

4. Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3 \ldots$ have the same common ratio, with $a_1 = 27$,$b_1 = 99$, and $a_{15} = b_{11}$. Find a_9. (AIME 2012)

5. Find a pair of integers (a, b) for which $\dfrac{10^a}{a!} = \dfrac{10^b}{b!}$ and $a < b$. (EMC2 2012)

6. Find a positive real number r that satisfies

$$\frac{4+r^3}{9+r^6} = \frac{1}{5-r^3} - \frac{1}{9+r^6}.$$

(EMC2 2012)

7. The sequence $\{a_n\}$ is defined by

$$a_0 = 1, a_1 = 1, \text{ and } a_n = a_{n-1} + \frac{a_{n-1}^2}{a_{n-2}} \text{ for } n \geq 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0 = 1, b_1 = 3, \text{ and } b_n = b_{n-1} + \frac{b_{n-1}^2}{b_{n-2}} \text{ for } n \geq 2.$$

Find $\frac{b_{32}}{a_{32}}$. (AIME2 2008)

8. For reals a, b, c, show that

$$\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c} = \frac{(a-b)(b-c)(c-a)}{abc}$$

9. Given that a, b, c are pairwise distinct nonzero real numbers such that $a+b+c=0$, evaluate

$$\left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right)\left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right).$$

10. Solve in positive integers: $n^4 + 2n^3 + 2n^2 + 2n + 1 = m^2$.

11. Let k and m be constants such that for all triples (a, b, c) of positive real numbers,

$$\sqrt{\frac{4}{a^2} + \frac{36}{b^2} + \frac{9}{c^2} + \frac{k}{ab}} = \left|\frac{2}{a} + \frac{6}{b} + \frac{3}{c}\right| \text{ if and only if } am^2 + bm + c = 0$$

Find k. (EMC2 2011)

12. Determine the value of

$$26 + 36 + 998 + 26 \cdot 36 + 36 \cdot 998 + 998 \cdot 26 + 26 \cdot 36 \cdot 998.$$

(EMC2 2012)

13. Compute the value of $\left[\frac{1}{\sqrt[3]{10^3+1} - \sqrt[3]{10^3}}\right]$, where $[x]$ denotes the greatest integer less than or equal to x.

14. p, q, r are real numbers satisfying

$$\frac{(p+q)(q+r)(r+p)}{pqr} = 24$$

$$\frac{(p-2q)(q-2r)(r-2p)}{pqr} = 10.$$

Given that $\frac{p}{q} + \frac{q}{r} + \frac{r}{p}$ can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers, compute $m + n$. (OMO 2012)

15. The sequence $\{a_n\}$ satisfies $a_0 = 1, a_1 = 2011$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Let

$$S = \sum_{i=1}^{\infty} \frac{a_{i-1}}{a_i^2 - a_{i-1}^2}$$

What is $\frac{1}{S}$? (OMO 2012)

16. A sequence is defined as follows, $a_1 = a_2 = a_3 = 1$.

$$a_n = \frac{1 + a_{n-1}a_{n-3}}{a_{n-2}}$$

For all $n \geq 4$. If $\frac{a_{92}}{a_{100}} = \frac{m}{n}$ for relatively prime integers m, n, compute $n - m$. (WOOT Bonus AIME 2008)

17. A sequence of integers is defined as follows: $a_i = i$ for all $i \leq 5$, and $a_i = a_1 a_2 \cdots a_{i-1} - 1$ for all $i > 5$. Evaluate

$$a_1 a_2 \cdots a_{2011} - \sum_{i=1}^{2011} a_i^2.$$

(HMMT 2011)

18. If $(x + \sqrt{x^2+1})(y + \sqrt{y^2+1}) = 1$, for real numbers x, y, compute $x + y$. (Russia)

19. Heron's formula states that if a triangle has side lengths a, b, c, then its area is equal to

$$\text{Area of } ABC = \sqrt{\frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{16}}$$

Derive Heron's formula using law of cosines.

(Note: one useful formula for the area of a triangle is $\frac{1}{2}ab\sin C$)

20. Ptolemy's theorem states that in a cyclic quadrilateral with sides of length a, b, c, d in that order and diagonals of length e and f, we have

$$ac + bd = ef$$

Prove Ptolemy's Theorem using law of cosines. (Note: a quadrilateral is cyclic if and only if the opposite sides sum to $180°$.)