

Small Cases and Experimentation

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1 Problems

1. Let $f(x) = 4x + 3$, $g(x) = \frac{x+1}{4}$. Evaluate $g(f(g(f(g(f(g(f(42))))))))$ (Modified EMC2 2011)
2. Alice's favorite number has the following properties:
 - It has 8 distinct digits.
 - The digits are decreasing when read from left to right.
 - It is divisible by 180.

What is Alice's favorite number? (OMO 2012)

3. Find a pair of integers (a, b) for which $\frac{10^a}{a!} = \frac{10^b}{b!}$ and $a < b$. (EMC2 2012)
4. A sequence a_1, a_2, \dots is such that $a_1 = 1$, and for $n \geq 2$, a_n is defined as the sum of the terms before it. That is, $a_n = \sum_{i=1}^{n-1} a_i$. Determine a_{2012} .
5. Let a_0, a_1, a_2, \dots denote the sequence of real numbers such that $a_0 = 2$ and $a_{n+1} = \frac{a_n}{1 + a_n}$ for $n \geq 0$. Compute a_{2012} . (HMMT 2012)
6. A sequence of integers a_1, a_2, \dots is chosen so that $a_n = a_{n-1} - a_{n-2}$ for each $n \geq 3$. What is the sum of the first 2001 terms of this sequence if the sum of the first 1492 terms is 1985 and the sum of the first 1985 terms is 1492?
7. Al told Bob that he was thinking of 2011 distinct positive integers. He also told Bob the sum of those integers. From this information, Bob was able to determine all 2011 integers. How many possible sums could Al have told Bob? (OMO 2012)
8. Albert takes a 4×4 checkerboard and paints all the squares white. Afterward, he wants to paint some of the squares black, such that each square shares an edge with an odd number of black squares. Help him out by drawing one possible congruation in which this holds. (EMC2 2012)
9. Compute the sum of the greatest odd divisor of each of the numbers $2006, 2007, \dots, 4011$.
10. Let $a_1 = 1, a_n = \lfloor n^3/a_{n-1} \rfloor$, for $n > 1$. Determine the value of a_{999} . ($\lfloor x \rfloor$ is the greatest integer less than or equal to x .)

11. If

$$2011^{2011^{2012}} = x^x$$

for some positive integer x , how many positive integer factors does x have?

12. Let s_n be the number of solutions to $a_1 + a_2 + a_3 + a_4 + b_1 + b_2 = n$, where a_1, a_2, a_3 and a_4 are elements of the set $\{2, 3, 5, 7\}$ and b_1 and b_2 are elements of the set $\{1, 2, 3, 4\}$. Find the number of n for which s_n is odd. (OMO 2012)
13. For positive integers n , let $\nu_3(n)$ denote the largest integer k such that 3^k divides n . Find the number of subsets S (possibly containing 0 or 1 elements) of $\{1, 2, \dots, 81\}$ such that for any distinct $a, b \in S$, $\nu_3(a - b)$ is even.
14. Let f be a function such that

$$f(x) = \begin{cases} 2x & \text{if } x \leq \frac{1}{2} \\ 2 - 2x, & \text{if } x > \frac{1}{2} \end{cases}$$

What is the total length of the graph of $\underbrace{f(f(\dots f(x)\dots))}_{2012 \text{ f's}}$ on the interval between $x = 0$ and $x = 1$? (HMMT 2012)

15. For all real numbers x , let $f(x)$ denote the function,

$$f(x) = \frac{1}{\sqrt[2011]{1 - x^{2011}}}$$

Determine the value of $f(f(\dots f(2011)\dots))^{2011}$ where f is applied 2010 times. (HMMT 2011)

16. It's very easy to dissect a square into four smaller squares. For what n is it possible to dissect a square into n smaller squares (not necessarily congruent)?
17. Show that for every positive integer $n \geq 3$ there are distinct positive integers a_1, a_2, \dots, a_n with $a_1! a_2! \cdots a_{n-1}! = a_n!$.
18. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and the next number on the circle (that is, the numbers a, b, c, d are replaced by $|a - b|, |b - c|, |c - d|, |d - a|$) Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc - ad|, |ac - bd|, |ab - cd|$ are all primes? (IMO Shortlist 1996)
19. The numbers $1, 2, \dots, 2012$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers x and y , erases them, and writes the number $2x + 2y$ on the board. This continues until only one number N remains. Find the remainder when the maximum possible value of N is divided by 1000.
20. Shanille O' Keal is shooting free throws. She makes the first one, misses the second one, and for each shot afterwards, the probability that she makes it is equal to the number of baskets she has made so far divided by the total number of shots she has made. What is the probability that after 100 shots, she has made exactly 50 of them? (Putnam)

21. The sequence $\{a_i\}_{i \geq 0}$ satisfies $a_0 = 1$ and $a_n = \sum_{i=0}^{n-1} (n-i)a_i$ for $n \geq 1$. Evaluate $\sum_{k=0}^m \frac{a_k}{4^k}$.
22. Given a set of points in space, a *jump* consists of taking two points in the set, P and Q , removing P from the set, and replacing it with the reflection of P over Q . Find the smallest number n such that for any set of n lattice points in 10-dimensional-space, it is possible to perform a finite number of jumps so that some two points coincide. (OMO 2012)
23. Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime. (USAMO 2005)
24. An *animal* with n *cells* is a connected figure consisting of n equal-sized square cells[1].
 A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* if its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.
 [1]Animals are also called *polyominoes*. They can be defined inductively. Two cells are *adjacent* if they share a complete edge. A single cell is an animal, and given an animal with n cells, one with $n+1$ cells is obtained by adjoining a new cell by making it adjacent to one or more existing cells. (USAMO 2007)
25. Let $n \geq 2$ be an integer and $S = \{(p_1, b_1), (p_2, b_2), \dots, (p_n, b_n)\}$ be a set of n pairs of positive real numbers such that $p_i < 1$ for all i . Let $\{(p'_1, b'_1), (p'_2, b'_2), \dots, (p'_n, b'_n)\}$ be a permutation of the elements of S . Find an algorithm that determines the permutation for which

$$p'_1(b'_1 + p'_2(b'_2 + \dots + p'_{n-1}(b'_{n-1} + p'_n(b'_n)) \dots))$$

attains its maximum value. (Adapted USACO 2011)