Factoring

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1 Factoring Techniques

Below is a list of two common techniques. However, somes problems may not use these techniques, and may require more ad-hoc factoring tricks.

1.1 Difference of Squares

1. Compute $52 \cdot 48$.
2. Find the closest power of 2 to $(2^1+1)(2^2+1)(2^4+1)..(2^{2^{100}}+1)$
3. Show that $8x^2-2xy-3y^2$ can be written in the form A^2-B^2, where A and B are polynomials with integer coefficients.
4. Factor a^4+4b^4. (Hint: There is no difference of squares, so make one!)

1.2 Completing the Square

5. If $x^2+10xy+25y^2=0$, compute $\dfrac{x}{y}$.
6. Prove the quadratic formula for monic quadratics (leading coefficient is 1). That is, if x is a real number satisfying x^2+bx+c, then
$$x=\frac{-b\pm\sqrt{b^2-4c}}{2}$$
7. Prove the quadric formula for all quadratics. That is, if x is a number satisfying, ax^2+bx+c, then
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$
8. If a,b,c are positive real numbers satisfying, $a^2+b^2+c=a^2+b+c^2=a+b^2+c^2$, determine whether a,b,c are necesarily all equal.
9. If a,b,c are real numbers, with $a+b+c=2\sqrt{a+1}+4\sqrt{b+1}+6\sqrt{c-2}-14$, compute $a(b+c)+b(c+a)+c(a+b)$

2 Practice Problems

The problems below are arranged roughly in order of difficulty. Feel free to work with others.

2.1 Easier problems

1. Nick multiplies two consecutive numbers and obtains $4^5 - 2^5$. What is the smaller of the two numbers? (EMC2 2011)
2. Find $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots$
3. When n is a positive integer, $n!$ denotes the product of the first n positive integers; that is, $n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n$. Given that $7! = 5040$, compute $8! + 9! + 10!$. (EMC2 2011)
4. Compute $(13 + 37)^3 - 13^3 - 37^3$.
5. Two numbers have a product of 16 and a sum of 20. What is the sum of their recipricols? (MATHCOUNTS)
6. How many odd perfect squares are less than $8(1 + 2 + ... + 2011)$? (Adapted EMC2 2011)
7. How many positive integers between 1 and 100 can be expressed as the difference of two perfect squares? (EMC2 2011)
8. If $(a - c)|ab + cd$ for positive integers a, b, c, d, show that $(a - c)|ad + bc$
9. Find all nonnegative integer solutions to $ab + a + b = 20$.
10. Determine the largest prime factor of $64^3 - 36^3$.
11. The product N of three positive integers is 6 times their sum, and one of the integers is the sum of the other two. Find the sum of all possible values of N. (AIME2 2003)

2.2 Medium Problems

12. Let $N = 100^2 + 99^2 - 98^2 - 97^2 + 96^2 + \cdots + 4^2 + 3^2 - 2^2 - 1^2$, where the additions and subtractions alternate in pairs. Find the remainder when N is divided by 1000. (AIME2 2008)
13. Compute the largest two digit factor of $3^{2^{2^{2011}}} - 2^{2^{2^{2011}}}$. (EMC2 2011)
14. Let d be a number chosen at random from the set $\{142, 143, ..., 198\}$. What is the probability that the area of a rectangle with perimeter 400 and diagonal length d is an integer? (EMC2 2011)
15. $a\&b = ab + a + b$. What is $1\&(2\&(...98\&(99\&100)))$ (HMMT 2011)
16. Let n be a positive integer. Find the sum of all possible prime values of $n^4 - 49n^2 + 14n - 1$. (Mandelbrot)
17. Find a 5-digit prime factor of 104060405.
18. $A = a^2 - 2b + \frac{\pi}{2}, B = b^2 - 2c + \frac{\pi}{3}, C = c^2 - 2a + \frac{\pi}{6}$ for some positive reals a, b, c. Show that at least one of A, B, C is positive.

19. Show that

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}=\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{c+a}$$

20. The positive numbers a, b, c satisfy $4abc(a+b+c) = (a+b)^2(a+c)^2$. Prove that $a(a+b+c) = bc$. (NIMO)

21. Show that there are infinitely many positive integers a for which $x^4 - ax^2 + 100$ can be factored into two nonconstant polynomials with integer coefficients. (IDEAMATH Placement Test)

2.3 Hard Problems

22. If $4(\sqrt{x}+\sqrt{y-1}+\sqrt{z-2}) = x+y+z+9$, compute xyz

23. If the sides of $\triangle ABC$ are a, b, c, and $a^2+c^2+8b^2-4ab-4bc = 0$. Show ABC is degenerate.

24. Solve $x+y=\sqrt{4z-1}, y+z=\sqrt{4x-1}, x+z=\sqrt{4z-1}$. (Math Olympiad Challenges)

25. Determine the value of

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}.$$

26. Let a and b be positive real numbers satisfying,

$$\begin{aligned} a^4+a^2b^2+b^4 &= 900 \\ a^2+ab+b^2 &= 45 \end{aligned}$$

Determine the value of $2ab$. (OMO 2012)

27. Show that if n, a and b are positive integers such that $n|a-b$, then $n^2|a^n-b^n$.

28. Find all integers a, b, c, d, and e, such that

$$\begin{aligned} a^2 &= a+b-2c+2d+e-8, \\ b^2 &= -a-2b-c+2d+2e-6, \\ c^2 &= 3a+2b+c+2d+2e-31, \\ d^2 &= 2a+b+c+2d+2e-2, \\ e^2 &= a+2b+3c+2d+e-8. \end{aligned}$$

(USAMTS Year 23, Round 1)

2.4 Extra Tricky (For fun)

29. Show that $(5^{10}+2\cdot 6^6 7^7)^2+(6^{12}+2\cdot 5^5 7^7)^2+(7^{14}+2\cdot 5^5 6^6)^2$ can be expressed in the form x^2+2y^2, where x, y are positive integers. (Ray Li, Calvin Deng)

30. Find all positive integers x, y, z that satisfy

$$xy(x^2+y^2) = 2z^4.$$