My Favorite Problems

Ray Li

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1 Introduction

Here are some of the best (and worst) problems I saw as a high school student. I hope you enjoy them.

2 The Problems

2.1 ~~Not So~~ So Not Serious Problems

1. Find the sum of all distinct integers n such that $n|243$. (Mock AIME III 2010)

2. If $a^2 + b^2 = 2012^{2012}, ab = 2011^{2011}$, compute the sum of all possible values of $a + b$. (Alex Song)

3. Points A, B, C and D are given in space such that $AB = 4, BC = 2, CD = 5, DA = 7, DB = 5$, and $AC = 6$. Find the volume of tetrahedron $ABCD$. (Linus Hamilton)

4. Let ABC be a triangle with $AC = 5 - \sqrt{3}$, $BC = \sqrt{2}$. Let A', B' be points outside of triangle ABC such that $\angle A'BC = 15°, \angle A'CB = 75°, \angle ACB' = 60°, \angle B'AC = 30°$. Given that $A'B'$ can be expressed in the form $a\sqrt{b} + c$, where a, b, c are positive integers and b is not divisible by the square of any prime, find $a + b + c$. (Mock AIME III 2010)

5. Find all positive integers x, y, z that satisfy

$$xy(x^2 + y^2) = 2z^4.$$

   (Ray Li, Max Schindler)

6. The digits of the number 66134880 are permuted and the each digit is colored either yellow, blue, or red. In how many ways can this be done? (Linus Hamilton)

2.2 Actual Problems

7. There are 100 very small ants walking across a meter stick. Each one walks towards one end of the stick, independently chosen, at 1 cm/s. If two ants bump into each other, both immediately reverse direction and start walking the other way. If an ant reachs the end of the meter stick, it falls off. Determine whether all the ants will always eventually fall off the stick. If so, determine the longest possible amount of time before the last ant falls off.

8. Two trains 150 miles apart are traveling toward each other along the same track. The first train goes 60 miles per hour; the second train rushes along at 90 miles per hour. A fly is hovering just above the nose of the first train. It buzzes from the first train to the second train, turns around immediately, flies back to the first train, and turns around again. It goes on flying back and forth between the two trains until they collide. If the fly's speed is 120 miles per hour, how far will it travel?

9. In what context is the phrase "0 is odd" a true statement?

10. Vlad has a circular table radius R and wants to cut a hole in it of radius r so that he can put a circular umbrella through it. However, he is drunk so he cuts the hole of radius r a distance $L < R - r$ from the center of the table. Now he is sober again, and wants to cut the table in to two or more pieces so that he can glue them back together to form a table with a hole in the center. What is the minimum perimeter he must cut (in the worst case scenario)?

11. There is a rectangular castle surrounded by a rectangular moat, which is of uniform width 12 feet. You must get across the moat without using the drawbridge, but all you have are two planks with lengths 11 feet and 11 feet 9 inches. How do you get across?

12. One morning, a monk decides to go to the top of a very tall mountain. He starts at 8:00 am and reachs the top at noon. The monk does not necessarily walk at a constant rate. The next morning he starts at 8:00 am the top of the mountain and takes the exact same path(not necissarily at the same rate) down the mountain, arriving at the bottom at noon. Show that there is always some time such that the monk was at the same position at that time on both days. (The Art and Craft of Problem Solving)

13. There are several gas stations placed around a circular track. A car whose fuel tank is initially has some gas is placed at some point in the track. The car travels along the track, picking up fuel from gas stations as it passes it. Given that the combined total of the gas in the car and the stations gives the car exactly enough fuel to travel around the track once, determine if it is always possible to initially place the car so that it can make it all the way around the track without ever running out of fuel. Assume that when a car picks up gas from a station it does not lose any gas.

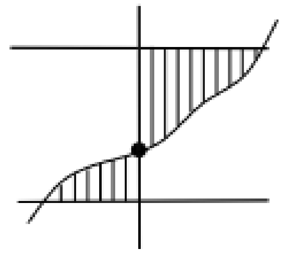
14. A and B play a game. They have the numbers $1, 2, \ldots, 9$ written on cards and lined up. Players alternate taking cards. The first player to have 3 cards whose sum is 15 wins. Determine which player, if any has a winning strategy.

15. At the national MATHCOUNTS competition, there are 256 students. (not really, but assume so for this problem) Al is watching the national countdown round on ESPN3, but has to go to class as soon as the final round begins. That is, Al knows the two finalists, but does not know the winner. While Al is in class, Bob turns on ESPN3, but is too late for the countdown round. He only sees the winner of the competition. If Al and Bob can only exchange text messages in the form of binary bits, what is the minimum number of binary bits that they must exchange so that Al can also know the winner.

16. Let a, b, c be positive real constants. Solve for x,

$$\sqrt{a+bx} + \sqrt{b+cx} + \sqrt{c+ax} = \sqrt{b-ax} + \sqrt{c-bx} + \sqrt{a-cx}.$$

17. The graph of a monotonically increasing function is cut off with two horizontal lines. Find a point on the graph between intersections such that the sum of the two areas bounded by the lines, the graph and the vertical line through this point is minimum.



18. Two players play a game by alternately placing equally-sized pennies on a circular table. In each turn, the player must place a penny so that its center lies on the table and so that it does not overlap with any other penny (though they can be tangent.) Which player, if any, has a winning strategy?

19. There are 2012 families in who each have just bought newly built houses. However, nobody is pleased with their current house, and each family has a dream house that is among the other 2011 houses. Conveniently, no two families have the same dream house. Each day, pairs of families may exchange their current houses, such that no family is part of more than one exchange that day. What is the minimum number of days needed until every person owns his or her dream house?

20. In a $100 * 100$ grid of squares, the numbers $1, 2, ..., 100^2$ are filled in row by row consecutively. That is, the first row contains the numbers $1, 2, ..., 100$ in that order, the second row contains $101, 102, ..., 200$, and so on. Each square in the grid is colored red or black such that each row and each column contains exactly 50 red squares and 50 black squares. Show that the sum of the values in red squares is equal to the sum of the values in black squares.

21. Consider an $m * n$ grid of squares. For each of the unit squares, draw exactly one of its diagonals. Show that there exists a path lying on only the drawn diagonals that travels either from the top border of the grid to the bottom border, or the left border of the grid to the right border.

22. Find all functions $F(x) : \mathbb{R} \to \mathbb{R}$ such that for any $a, b \in \mathbb{R}$,

$$f(a) - f(b) \leq (a - b)^2$$

23. Prove that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x^3 + y^3) = f(f(x) - y^3) + 2y^3.$$

24. There are 2012^2 people distributed throughout 2012 rooms. Each minute, someone moves from its room to a room with at least as many people in his or her current room. Show that eventually all the people are in one room.

25. A cameraman is taking some pictures of some students standing in a line. Each student has some designated position in the line. However, because the students are naughty, the cameraman needs to take multiple photos. Right before each photo is snapped, some subset of students will leave their position and sprint to a different position in the line. After the cameraman makes his shot, he replaces the students in the correct order. The students are lazy, so each student sprints during at most one photo. What is the minimum n such that if the cameraman takes n snaps, it is possible to determine from the order of the students in these n photos the designated order of the students. (Adapted USACO December 2012)

26. Let S and S' be two circles, with S' contained in S. Suppose that, for some circle S_1 tangent to both S and S', one can construct a ring of circles $S_1, S_2, \ldots, S_n$ such that S_i and S_{i+2} are tangent to S_{i+1} (indicies taken mod n) and all of $S_1, \ldots, S_n$ are tangent to S and S'. Show that if this sequence of circles exists for some choice of S_1, then for any choice of S_1 such a sequence exists. (Steiner's Porism)

27. Suppose that a rectangle R is exactly covered with non-overlapping rectangles with sides parallel to those of R, such that each of the inner rectangles has at least one side with integer length. Show that at least one side of R has integer length. (Russia)

28. Let n be a positive integer. There are n soldiers stationed on the verticies of a regular polygon. Each round, you pick a point inside the polygon, and all the soldiers simultaneously shoot in a straight line towards that point; if their shot hits another soldier, the hit soldier dies and no longer shoots during the next round. What is the minimum number of rounds, in terms of n, required to eliminate all the soldiers? (David Yang)

29. Prove that for every positive integer $n \geq 2$,

$$.785n^2 - n < \sqrt{n^2 - 1^2} + \sqrt{n^2 - 2^2} + \cdots + \sqrt{n^2 - (n-1)^2} < .79n^2$$

30. Let $n \geq 2$ be an integer and $S = \{(p_1, b_1), (p_2, b_2), \ldots, (p_n, b_n)\}$ be a set of n pairs of positive real numbers such that $p_i < 1$ for all i. Let $\{(p_1', b_1'), (p_2', b_2'), \ldots (p_n', b_n')\}$ be a permtuation of the elements of S. Find an algorithm that determines the permutation for which

$$p_1'(b_1' + p_2'(b_2' + \cdots p_{n-1}'(b_{n-1}' + p_n'(b_n'))\ldots))$$

attains its maximum value. (Adapted USACO 2011)

31. Al plays a game by himself on a rooted tree with n verticies. On a move, he chooses a vertex of the tree that has not been crossed off, and crosses it off, as well as all of its children. The game ends when he cannot make a move. Determine, as a function of the tree, the expected number of moves Al will make. (Codeforces 172 div 1)

2.3 Ridiculous Problems

32. Let $R_1, R_2, \ldots$ be the family of finite sequences of positive integers defined by the following rules: $R_1 = (1)$, and if $R_{n-1} = (x_1, x_2, \ldots, x_s)$, then

$$R_n = (1, 2, \ldots, x_1, 1, 2, \ldots, x_2, \ldots, 1, 2, \ldots, x_s, n)$$

For example, $R_2 = (1, 2), R_3 = (1, 1, 2, 3), R_4 = (1, 1, 1, 2, 1, 2, 3, 4)$. Prove that if $n > 1$, then the kth term from the left in R_n is equal to 1 if and only if the kth term from the right in R_n is different from 1.

33. Let n be an integer. There are several points $P_1, P_2, \ldots, P_n$ inside square $ABCD$ such that no three of these $n+4$ points are colinear. Points A and B are colored white, and C and D are colored black. Every point inside the square is colored black or white. Segments are drawn between some pairs points of the same color, such that for any two points of the same color, it is possible to move from one to the other traveling only on drawn segments. Show that, no matter how the interior points are colored, it is possible to draw segments with this property so that pairs of segments only intersect at $P_1, P_2, \ldots, P_n, A, B, C$ or D. (Adapted IOI 2005)

34. Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

    Type 1) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

    Type 2) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

    Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins. (IMO 2011)

35. Let ABC be a scalene triangle with altitudes AD, BE and CF. Suppose that M is the midpoint of BC, and X, Y are the midpoints of ME and MF. Let Z be a point on line XY such that $ZA||BC$. Show that $ZA = ZM$. (Iran)

36. Define the point $P_0 = (0, 0)$. Define the point P_n, where $n = 1, 2, ..., 8$, as the rotation of point P_{n-1} 45 degrees counter-clockwise around the point $(n, 0)$. Determine the coordinates of P_8. (Putnam)

37. Let $M = \{1, 2, \cdots, n\}$, each element of M is colored in either red, blue or yellow. Set

    $A = \{(x, y, z) \in M \times M \times M | x + y + z \equiv 0 \mod n,\ x, y, z \text{ are of same color}\}$,

    $B = \{(x, y, z) \in M \times M \times M | x + y + z \equiv 0 \mod n,\ x, y, z \text{ are of pairwise distinct color}\}$.

    Prove that $2|A| \geq |B|$ (China 2010)