Sponsored Search Auctions with Reserve Prices: Going Beyond Separability*

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Abstract. The original analysis of sponsored search auctions by Varian and independently by Aggarwal et al. did not take into account the notion of reserve prices, which are common across all major search engines. We investigate this further and show that the separability assumption derived by Aggarwal et al. is not sufficient for aligning the greedy allocation employed by GSP and the efficient allocation in the presence of reserve prices. We extend separability and derive the condition under which the greedy ranking allocation is an efficient truthful mechanism. We call this generalization the *extended separability condition*.

To complement the analysis of the extended separability condition we present an extension of the laddered auction in the presence of reserve prices, which we call the *bi-laddered auction*. We show that the bi-laddered auction is the unique truthful auction for advertisers that provides a price vector support for an *extended GSP* SNE scheme. Nevertheless the bi-laddered auction is shown to allow a budget deficit.

Building on our model of reserve prices we continue by depicting advertising networks as double sided sponsored search markets with advertisers on one side, syndicators on the other, and the search engine as the market maker. For the latter model we provide a truthful scheme for the seller and show that by assuming separability one can design a SNE, individually rational, and nearly efficient syndicated market that allows the market maker (search engine) to run the market with a surplus/budget balance. The uniqueness of our bi-laddered auction scheme implies that without the separability condition no truthful syndicated market can run without a deficit.

1 Introduction

Sponsored search auctions are the primary way that companies like Google and Yahoo! monetize their search engines. They allow advertisers to bid on particular queries, thereby ensuring the relevance of the advertisement to the user, and increasing the conversion rate. Sponsored search is a very large business,

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projected to grow to many billions of dollars in the next few years; it is not surprising that the analysis of the precise way the auctions are run has generated research interest in the past few years, (i.e, [8,4,19,9,3,11,12,7]).

The auctions have a very simple framework[1]. Each advertiser specifies the query she is looking to advertise on, and submits a bid, representing the maximum amount she is willing to pay. When a user enters a query, the system collects all of the advertisers bidding for the query, and runs a generalized second price auction to determine both the winners, and the prices that each would be charged. There are usually multiple winners, as there are multiple advertiser slots on the search result page, with higher slots being more valuable since they are seen by more users. Finally, the advertiser is charged *only* in the event of a user click on the ad, otherwise no money changes hands, the so-called *pay per click* scheme.

Separability. In one of the first analyses of these auctions, Aggarwal et al. [1] showed that, not withstanding the claims made by Google, the auctions were not truthful. The authors showed that the greedy ranking employed by Google agrees with the efficient allocation *only* when the clickthrough rates are separable, that is, they are the product of the function of the advertiser quality and the position in which the advertisement appeared. The separability property has since been used as a simplifying assumption in other work i.e., [8,11,12,7][2].

The separability assumption is also implicitly present in the work of Varian [20]. Varian assumes that each slot s has a click-through rate x_s but advertisers have a quality score of 1 for their ad, meaning that an advertiser a's click-through at slot s equals $1 \cdot x_s$. Varian's work presents a mechanism in an equilibrium state; following the revelation principle it is not surprising that his Symmetric Nash Equilibrium (SNE) supporting prices are essentially the Aggarwal et al.'s truthful prices.

GSP Enhancements. While the basic principles of the Generalized Second Price (GSP) auction are now well understood, the auctions that are run in practice have evolved beyond this bare-bones model. Some of the most pertinent extensions include advertiser budgets, exploration in learning advertisers click through rates, broad (as opposed to exact) matches of keywords and reserve prices. Not surprisingly, each engineering enhancement has unintended economic consequences, and may potentially wreak havoc on the equilibrium achieved by the players.

While these extensions are now widely acknowledged (see for instance the footnote in [7]) and used in practice, for the most part their precise effect on the equilibria has not been analyzed. In this work we tackle the notion of *slot* specific reserve prices, and detail the changes that this condition brings to the auctions and the equilibria. Independently of our work, Even-Dar et al. [7] have

[1] In practice the situation is much more complex - e.g. advertisers specify maximum daily budgets, there is fuzzy matching on the queries, etc. We do not consider these problems in this work.

[2] [8] used separability earlier than [1].

recently investigated the effect of *bidder* specific reserve prices. Not surprisingly, they show that many naïve modifications lead to non truthful behaviors.

1.1 Our Contribution

While the results of Aggarwal et al. and Varian provide an initial analysis of the sponsored search auctions, they fail to take into account reserve prices that search engines usually set for many of the queries. In this work we explore the effect of reserve prices, and show that the separability assumption derived by [1] is not sufficient for aligning the greedy allocation produced by the ranking function and the efficient allocation. Thus, if we are to follow the allocation produced by the ranking function, Clarke-Groves prices would not result in a truthful mechanism. Instead we present the *extended separability condition* which provides the necessary and sufficient condition for a ranking function to be truthful under the VCG prices and modify the laddered auction of [1] (which we call the *bi-laddered auction*) to derive a mechanism that is truthful in the presence of reserve prices. The bi-laddered scheme is shown to be a supporting price vector for a SNE created by the new *extended generalized second price* scheme that is used to replace the GSP in the presence of reserve prices.

We then turn our attention to the sellers. In the sponsored search auction where the ads are presented alongside the search results, it is the search engine that controls the placement and the reserve prices for each of the slots. Currently the sponsored search auction is evolving into an advertising network motivated by Google's acquisition of DoubleClick and Yahoo! buying RightMedia. In the advertising network there are advertisers and publishers and the search engines' role is that of market makers. In such a network it is easy to imagine a syndicate situation, where a publisher (e.g. LinkedIn) offers to place advertisements along its content pages, but has a set of reserve prices for these slots – as it encounters cost, such as user satisfaction decrease, in placing the ads. In this case, we describe a pricing scheme that is truthful for the *seller*. Motivated to design an advertising network in SNE we show that the greedy allocation without the separability assumption and with cost on the slots has a unique truthful pricing scheme that does not allow for budget balance or surplus for the marker maker (the search engine). Thus such a design would be unreasonable in practice.

We then tackle the problem of devising a market that would be budget balanced (or carry a budget surplus), while at the same time eliciting truthful behavior from both the buyers and the sellers simultaneously. It is easy to see that in this scenario we must relax one of the conditions to avoid the impossibility result of Myerson-Satterthwaite [18]. We present one such mechanism, that sacrifices some of the efficiency and prove its properties of maintaining SNE and budget balance/surplus under the separability condition.

Independently of our work Even-Dar et al. [7] analyzed the notion of bidder specific reserve prices. Our work differs in two major aspects. First, [7] explicitly assumes that the separability condition holds. While, this assumption has been made before, it has recently been called into question. See for example the results of [6] and the model introduced in [2,15]. In our work we showcase the *Extended*

Separability Condition and prove that the condition is necessary in order to align the greedy allocation with the efficient allocation. Second, the major contribution of [7] is the proof that GSP has an envy-free equilibrium with bidder specific reserve prices. We present an extended GSP pricing scheme for the slot specific reserve prices model which is shown to maintain SNE. Furthermore, while we provide a truthful auction in the presence of reserve prices, a major point of our work is the exploration of the strategies of the sellers (publishers in case of syndication), and the wider question of a SNE in double sided markets that carry a budget surplus.

Finally, let us define the notation that we will use for the rest of the exposition. In general there are n advertisers $= \{i_1, \ldots, i_n\}$, who bid for k slots $\{j_1, \ldots, j_k\}$. Denote by b_i the bid of advertiser (buyer i), by c_j the reserve price (cost) of slot j, and finally, by $\lambda_{i,j}$ the clickthrough rate of advertiser i when she is placed in slot j.

2 The Extended Separability Condition

In this section we present and analyze the extended separability condition that is required to guarantee the existence of a truthful efficient mechanism for the greedy allocation ranking auction of sponsored search with reserve prices.

Aggarwal et al. [1] showed that for sponsored search auctions with no reserve prices on slot, for all possible ranking functions $R = (w_1, \ldots, w_n)$ a truthful solution exists only if for every two buyers i, i' and slots $j, j+1$ the condition $\frac{\lambda_{i,j}}{\lambda_{i,j+1}} = \frac{\lambda_{i',j}}{\lambda_{i',j+1}}$ holds. Below we show that not only must the clickthrough rates be separable, but in any case where the same two buyers can be matched to two different slots, the costs of those slots must be equal. In other words, the only time when the allocation provided by the ranking function agrees with the socially efficient allocation (maximizing the total gain from trade), is when either such an allocation is straightforward, or many of the reserve costs are identical. We now state this formally:

Theorem 1. *For the sponsored search auction with reserve prices a truthful efficient mechanism for the greedy allocation ranking auction exists only if the following condition holds on the click through rates of the buyers and the cost of the slots. For every rank function $R = (w_1, \ldots, w_n)$ and any two buyers i and i' ranked to slots j and j' in R there exists a set of VCG[21,5,13] weights that always yield the same ranking as R only if:*

$$\frac{\lambda_{i,j}}{\lambda_{i,j+1}} = \frac{\lambda_{i',j}}{\lambda_{i',j+1}}$$

and either $c_j = c_{j+1}$ or $\lambda_{i,j+1} = \lambda_{i',j+1}$.

Proof. The proof follows the basic structure of the proof of the separability condition in [1]. See [14] for full details.

It is important to note that unlike the Aggarwal et al. model where every advertiser can potentially be allocated in every possible slot, our model might limit the allowed ranking functions as an advertiser i cannot be allocated to slot j where $b_i < c_j$. In other words if for some ranking function R i and i' are allocated to j and j' respectively it is not necessarily true that there exist a ranking function R' in which i and i' are allocated to j' and j respectively; and it is precisely in these situations that a seller can charge different reserve prices for different slots, and still have the allocation specified by R to be efficient.

3 Auctions with Reserve Prices

Aggarwal et al. [1] presented an auction that preserves and realizes an allocation consistent with a ranking function $R = (w_1, \ldots, w_n)$ with a laddered pricing scheme; and proved that the laddered price auction is truthful. Following a similar assumption of separability Varian proved that there exists a SNE in an auction that maintains a ranking function $R = (w_1, \ldots, w_n)$ where all $w_i = 1$. The laddered pricing scheme in Aggarwal et al. is the lower bound supporting price vector for the Varian SNE. Both authors' results do not take into account reserve prices (although they often occur in practice) and therefore do not consider equilibria in the presence of reserve prices. In the previous section we focused our attention on the condition required to maintain a truthful mechanism that is aligned with the efficient allocation in a ranking based auction with costs assigned to the slots. In this section we provide a pricing scheme to support a ranking based allocation with costs assigned to slots and prove that the pricing scheme is truthful for the advertisers. Our pricing scheme also provides a support price vector for a SNE in this extended model.

3.1 The Modified Laddered Auction

Recall, we are given n advertisers (buyers) with bid vector $b = (b_1, \ldots, b_n)$, a ranking function $R = (w_1, \ldots, w_n)$, and k slots (sellers) with associated costs $c = (c_1 \geq c_2 \geq \ldots \geq c_k)$. To assign buyers to sellers we first rank the buyers by the product $b_i w_i$. For the sake of exposition, reindex the buyers so that $b_1 w_1 \geq b_2 w_2 \geq \ldots \geq b_n w_n$. If the first buyer can afford the top slot, assign her to that slot, and repeat. Otherwise, leave slot the top slot unassigned, and recurse on the remaining slots. Observe that this allocation rule maintains the following two invariants:

1. If buyer i is assigned to a slot j and i' is assigned to slot j' with $j < j'$ then $b_i w_i \geq b_{i'} w_{i'}$.
2. If buyer i is assigned to slot j then $b_i \geq c_j$.

It remains to describe the prices charged. We proceed similar to the laddered auction [1] but add the effect of the reserve prices.

$$p_i = \sum_{j=i}^{k} \left(\frac{\lambda_{i,j} - \lambda_{i,j+1}}{\lambda_{i,i}} \right) \max \left(\frac{w_{j+1} b_{j+1}}{w_i}, c_j \right)$$

Theorem 2. *The auction presented above is truthful.*

Proof. Consider a buyer i, similar to the proof of [1], let x be the position of i in the allocation above, and r be the closest preferred position of i, holding everybody else's bids constant. We show that there exists a rank closer to than r to x, establishing a contradiction. Let p_y be the price charged to i if she ends up in in position y. If $r > x$, i.e. the merchant prefers to be lower, then the total change in her utility by moving to rank $r-1$ is:

$$\begin{aligned}\lambda_{i,r-1}(v_i - p_{r-1}) - \lambda_{i,r}(v_i - p_r) &= v_i(\lambda_{i,r-1} - \lambda_{i,r}) - (\lambda_{i,r-1}p_{r-1} - \lambda_{i,r}p_r) \\ &= v_i(\lambda_{i,r-1} - \lambda_{i,r}) - (\lambda_{i,r-1} - \lambda_{i,r}) \max(\frac{w_r b_r}{w_i}, c_j) \\ &= (\lambda_{i,r-1} - \lambda_{i,r})(v_i - \max(\frac{w_r b_r}{w_i}, c_j)) \\ &\geq 0,\end{aligned}$$

Where the last line follows from the two invariants we demonstrated above and the fact that click through rates decrease with position. Since the utility gain is non-negative, r cannot be the closest preferred rank. In the case that $r < x$ the proof is similar.

While the auction above is truthful, the reserve prices are not always met: it is easy to construct examples where the average per click price paid by the buyer is lower than the reserve price for that particular slot. In effect, the pricing scheme ensures that the buyer pays at least the reserve price for slot j *only* for those clicks that she is getting at j that she would not get at $j-1$. While a limitation, as the following theorem shows this is a direct consequence of the ranking used by this widely employed mechanism. (In Section 6 we will explore budget balanced/surplus mechanisms for this problem.)

Theorem 3. *The auction defined above is the unique truthful auction that ranks buyers according to decreasing $w_i b_i$.*

Proof. The proof parallels the uniqueness proof shown in [1]. We omit it here for space reasons.

4 The Symmetric Nash Equilibrium with Reserve Prices

In the previous sections we showed that the bi-laddered pricing scheme conducts a truthful sponsored search auction with reserve prices. In this section we present a new extended Generalized Second Price auction for sponsored search auction with reserve prices that conducts a Symmetric Nash Equilibrium. As expected a price vector that is shown to support the SNE presented is the bi-laddered prices.

For simplicity of presentation and similarly to Varian's paper [20] we will show the SNE of the sponsored search auction with reserve prices assuming that the weight w_i for all advertiser i is 1. Denote by p_j the price charged for a click at slot j.recall that we assume $\alpha \leq 1$.

Definition 1. *([20]) A symmetric Nash equilibrium set of prices satisfies*

$$(v_i - p_j)\lambda_{i,j} \geq (v_i - p'_j)\lambda_{i,j'}$$

for all i, j and j'.

Let $b_{m(j+1)}$ be the bid of the advertiser in slot $j+1$, where $m(j+1)$ is the index of the advertiser placed in slot $j+1$.

The GSP pricing scheme in [20] is defined to be $p_j = b_{m(j+1)}$. Consider the following extended GSP scheme for sponsored search auction with reserve prices as $p_j = \max(b_{m(j+1)}, c_j)$. Though our greedy ranking scheme may not maintain efficiency (some slots may be left unallocated) our allocation still maintains the same key properties as were shown in [20] that allow for a SNE to exist with the presence of reserve prices. The allocation is individual rational (it has non-negative surplus), monotone in values and prices, the SNE is included in the NE (SNE $\subset$ NE) and a local SNE implies global SNE. These facts allow us to provide an explicit characterization of equilibrium prices and bids. For the proof below we make one more technical assumption, namely that in the greedy ranking among all advertisers that can afford a particular slot, the one with higher click-through rate will be ranked higher. Formally, we assume that $\frac{\lambda_{m(j+1),j}}{\lambda_{m(j),j}} = \alpha \leq 1$.

Since advertiser i in slot j (indexed $m(j)$) does not want to move down one slot it follows that

$$(v_i - p_j)\lambda_{m(j),j} \geq (v_i - p_{j+1})\lambda_{m(j),j+1}$$

or, equivalently

$$v_{m(j)}(\lambda_{m(j),j} - \lambda_{m(j),j+1}) + p_{j+1}\lambda_{m(j),j+1} \geq p_j\lambda_{m(j),j}.$$

Similarly since advertiser z in slot $j+1$ does not want to move up one slot it follows that

$$(v_z - p_{j+1})\lambda_{m(j+1),j+1} \geq (v_z - p_j)\lambda_{m(j+1),j}$$

or that

$$p_j\lambda_{m(j+1),j} \geq v_{m(j+1)}(\lambda_{m(j+1),j} - \lambda_{m(j+1),j+1}) + p_{j+1}\lambda_{m(j+1),j+1}.$$

As we have assumed that $\frac{\lambda_{m(j+1),j}}{\lambda_{m(j),j}} = \alpha \leq 1$, it follows that when combining the above two formulas we get:

$$\begin{aligned} v_{m(j)}(\lambda_{m(j),j} - \lambda_{m(j),j+1}) + p_{j+1}\lambda_{m(j),j+1} \geq p_j\lambda_{m(j),j} \geq \\ \geq p_j\lambda_{m(j+1),j} \geq v_{m(j+1)}(\lambda_{m(j+1),j} - \lambda_{m(j+1),j+1}) + p_{j+1}\lambda_{m(j+1),j+1}. \end{aligned}$$

Since $p_j = \max(b_{m(j+1)}, c_j)$ it follows that

$$\begin{aligned} v_{m(j-1)}(\lambda_{m(j-1),j-1} - \lambda_{m(j-1),j}) + \max(b_{m(j+1)}, c_j)\lambda_{m(j-1),j} \geq \\ \geq \max(b_{m(j)}, c_{j-1})\lambda_{m(j-1),j-1} \\ \geq \max(b_{m(j)}, c_{j-1})\lambda_{m(j),j-1} \\ \geq v_{m(j)}(\lambda_{m(j),j-1} - \lambda_{m(j),j}) + \max(b_{m(j+1)}, c_j)\lambda_{m(j),j} \end{aligned}$$

We can then write down the upper and lower bounds on the bids:

$$\max(b^U_{m(j)}, c_{j-1})\lambda_{m(j-1),j-1} = v_{m(j-1)}(\lambda_{m(j-1),j-1} - \lambda_{m(j-1),j}) + \max(b_{m(j+1)}, c_j)\lambda_{m(j-1),j}$$

$$\max(b^L_{m(j)}, c_{j-1})\lambda_{m(j-1),j-1} = v_{m(j)}(\lambda_{m(j),j-1} - \lambda_{m(j),j}) + \max(b_{m(j+1)}, c_j)\lambda_{m(j),j}$$

The solution to the recursions is:

$$b^L_{m(j)}\lambda_{m(j-1),j-1} = \sum_{t \geq j} v_{m(t)}(\lambda_{m(t),t-1} - \lambda_{m(t),t})$$

$$b^U_{m(j)}\lambda_{m(j-1),j-1} = \sum_{t \geq j} v_{m(t-1)}(\lambda_{m(t-1),t-1} - \lambda_{m(t-1),t})$$

Note that since $c_{j-1} \leq v_{m(j-1)}$ even in the slots where $\max(b_{m(j)}, c_{j-1}) = c_{j-1}$ our bi-laddered scheme is bounded by $b^U_{m(j)}$ from above. Since $\max(b_{m(j)}, c_{j-1}) \geq b_{m(j)}$ our bi-laddered scheme is bounded by $b^L_{m(j)}$ from below.

5 A Truthful Scheme for the Seller

In the previous section we discussed the truthful scheme for the advertisers, i.e., the buyers. In this section we will present a pricing scheme for the seller of k slots and prove that the scheme is truthful for the seller.

One can imagine that the advertising space might be managed by a third party such as a syndicator (i.e., LinkedIn) and therefore we would like to design a pricing scheme that motivates the slot seller to report his true costs for the advertising slots.

First let us consider the sponsored search setting in which the syndicator, i.e., the seller is interested in selling k slots of advertising. Each slot of advertising it has an associated cost, stemming from the impact of advertising on users, opportunity cost of utilizing the space for other results (or content), etc. We will assume that the higher the advertising slot, the higher the cost inflicted on the publisher.

The seller has to determine a set of reserve prices, reporting a cost c_j for every slot j to the market maker (search engine in this case). By our assumption, the costs are decreasing in j, i.e. $c_j > c_{j+1}$.

The pricing scheme for the seller is the double-sided auction extension of the laddered scheme provided in Aggarwal et al. [1] for the buyer. Unlike the buyers' side, the seller's side of our double-sided auction has only a single seller. The above fact simplifies significantly the formula for the seller's pricing scheme. Let A be the set of all the allocated slots' indexes in the optimal efficient allocation, and denote by $m(j)$ the buyer assigned to slot j. Then the pricing scheme is as follows: For all $j \in A$, set the price per click paid to the seller for allocated slot j, to $b_{m(j)}\lambda_{m(j),j}$. So overall,

$$p_{seller} = \sum_{j \in A} \frac{b_{m(j)}\lambda_{m(j),j}}{\lambda_{j,j}} \quad (1)$$

The intuition behind the pricing scheme is simple as there is only a single seller, in every slot there will be no trade without him.

Theorem 4. *The pricing scheme in equation 1 is truthful with respect to the seller.*

Proof. We omit the proof here for space reasons. The full proof can be found in [14].

6 The Syndicated Sponsored Search

In the previous sections we showed two pricing schemes one for the seller and one for the buyers. One scheme allows the buyers to be truthful and the other allows the seller to be truthful. As the sponsored search market evolves into networks of advertisers and syndicators a natural question arises: is it possible to conduct a market where all parties, i.e., the buyers and the seller are motivated to tell the truth simultaneously. We can consider the results from the previous sections. We already saw that the bi-laddered auction, while truthful for the buyers, leads us to charge prices that may be below the reserve prices for particular slots (and this is inevitable given the ranking function). A similar problem plagues us in the seller's case - since the allocation is efficient, following the Myerson-Satterthwaite result [18], the market maker will potentially sustain a budget deficit. As the market maker in the desired syndicated market is the search engine it is unreasonable to expect the market maker to carry a loss. One way to overcome the impossibility of Myerson-Satterthwaite is to give up some of the efficiency and maintain the other properties of individual rationality (no player losses by participating in the market), truthfulness and budget balance/surplus.

Of course other properties, e.g. truthfulness, can be sacrificed in order to avoid the budget deficit. Nevertheless as we investigate the design of a market in equilibrium (such as the SNE) if the requirement for a truthful market is relaxed then following the revelation principle there will not exist a SNE in the designed market.

The question that this section tries to answer is under what condition is it possible to create a syndicated sponsored search market that operates without a loss (i.e. budget balanced/surplus), while maintaining the desired properties of SNE, individual rationality and minimal loss of efficiency.

Interestingly the question of maintaining budget balance in a truthful syndicated sponsored search market ties back to the separability condition and the extended separability condition. While under separability it is possible to create a truthful budget balanced/surplus syndicated sponsored search market, and a SNE budget balanced/surplus syndicated sponsored search market, we could not show a similar result for the case were separability does not hold. Moreover the uniqueness of the bi-laddered truthful pricing scheme and the fact that it does not maintain budget balance indicates that no truthful budget balanced/surplus syndicated sponsored search market exists for the general case.

6.1 Separable Budget Balanced Syndicated Market

Consider a syndicator sponsored search double sided auction with buyers (advertisers) and seller of multiple slots (syndicator) and denote that auction by S. We assume that S is separable: for every buyer i and slot j, $\lambda_{i,j} = x_i \cdot y_j$ where x_i is advertiser i's click through rate and y_j is slot j's click through rate. Now let D be a double sided auction where every buyer with a valuation v_i in S is represented by a buyer with valuation $v_i \cdot x_i$ in D; and every slot with cost c_j in S is a slot with cost $c_j \cdot y_j$ in D. The efficient allocation that maximizes gain from trade in D is the following standard double sided auction allocation: order the buyers according to decreasing valuations and the slots according to increasing costs and consider for allocation all the buyer-slot pairs that have positive gain from trade. The difference between the efficient allocation of S and the efficient allocation of D is that after determining which buyers and slots made it into the allocation(D's or S's), S orders the slots in the allocation in descending order and matches to the buyers that made it into the allocation also ordered in descending order.

Since D is a standard double sided auction it is well known (e.g. [17,10]) that there exist a budget balanced solution which is truthful, individually rational, and gives up at most one trade of the efficient allocation. Thus it is reasonable to expect that there should exist a mechanism that gives up at most one buyer and one slot, such that S is budget balanced, has a SNE and individually rational. And indeed the following mechanism produces the desirable properties:

The BB SS Syndicator double-sided auction mechanism:

1. Let A be the efficient allocation for a Syndicator double-sided auction.
2. remove from A buyer $i \in A$ such that $v_i \cdot x_i$ is minimal
3. unmatch in A slot $j \in A$ such that $c_j \cdot y_j$ is maximal
4. rematch buyers and slots that remain in A according to buyers ordered in descending order and slots ordered in descending order.
5. charge buyer i $p_i = \max\{b_{i+1}, c_{e(i)}\}$ where $c_{e(i)}$ is the cost of the slot matched with buyer i in the efficient allocation.
6. charge the seller for slot j $p_j = b_{e(j)}$ where $b_{e(j)}$ is the bid of the buyer matched with slot j in the efficient allocation.

Lemma 1. *The BB SS Syndicator double-sided auction mechanism is budget balanced, SNE, and individually rational.*

Proof. The mechanism is individually rational for every buyer as his price is always less than the buyer's bid, i.e., $\max\{b_{i+1}, c_{e(i)}\} \leq b_i$. It is also individually rational for the seller as $b_{e(j)} \geq c_j$ for every j.

The mechanism is a SNE as the pricing scheme for every allocated buyer is identical to the efficient mechanism pricing scheme in SNE (see section 4) and since all of the allocated buyers after the trade reduction did not change their relative ranking, no one has a new incentive to move in the rankings. Similarly, for every slot that is allocated, the seller is payed the truthful price from the efficient allocation. Thus he also has no incentive to deviate from the truthful strategy.

The mechanism is budget balanced as the trade reduced allocation shifted every buyer by one slot down and therefore for every buyer i that is matched with seller j in the efficient allocation, buyer i's price in the trade reduced allocation is $p_i = \max\{b_{i+1}, c_j\}$ and the seller expects to be payed for every allocated slot j in the trade reduced allocation $p_j = b_{i+1}$. Thus $p_i \leq p_{j+1}$, ensuring budget balance.

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References

1. Aggarwal, G., Goel, A., Motwani, R.: Truthful Auctions for Pricing Search Keywords. In: Proceeding of EC 2006 (2006)
2. Aggarwal, G., Feldman, J., Muthukrishnan, S., Pal, M.: Sponsored Search Auctions with Markovian Users. In: Ad Auctions Workshop (2008); In: WINE short papers 2008 (to appear)
3. Abrams, Z., Mendelevitch, O., Tomlin, J.: Optimal Delivery of Sponsored Search Advertisements Subject to Budget Constraints. In: Proc. EC 2007 (2007)
4. Borgs, C., Chayes, J., Immorlica, N., Mahdian, M., Saberi, A.: Multi-unit auctions with budget-constrained bidders. In: Proc. 6th ACM Conference on Electronic Commerce, pp. 44–51 (2005)
5. Clarke, E.H.: Multipart Pricing of Public Goods. Journal Public Choice 2, 17–33 (1971)
6. Craswell, N., Zoeter, O., Taylor, M., Ramsey, B.: An experimental comparison of click position-bias models. In: ACM WSDM 2008 (2008)
7. Even-Dar, E., Feldman, J., Mansour, Y., Muthukrishnan, S.: Position Auctions with Bidder Specific Minimum Prices. In: Ad Auctions Workshop (2008); In: WINE short papers 2008 (to appear)
8. Edelman, B., Ostrovsky, M., Schwarz, M.: Internet Advertising and the Generalized Second Price Auction: Selling Billions of Dollars Worth of Keywords. American Economic Review 97 (2007)
9. Feng, J., Bhargava, H., Pennock, D.: Implementing sponsored search in web search engines:Computational evaluation of alternative mechanisms. Informs Journal on Computing
10. Gonen, M., Gonen, R., Pavlov, E.: Generalized Trade Reductions: The Role of Competition in Designing Budget Balanced Mechanisms. In: Proc. of ACM EC 2007 (2007)
11. Gonen, R., Pavlov, E.: An Incentive-Compatible Multi Armed Bandit Mechanism. In: Third Workshop on Sponsored Search Auctions WWW 2007, PODC 2007 (2007)
12. Gonen, R., Pavlov, E.: An Adaptive Sponsored Search Mechanism δ-Gain Truthful in Valuation, Time, and Budget. In: Deng, X., Graham, F.C. (eds.) WINE 2007. LNCS, vol. 4858, pp. 341–346. Springer, Heidelberg (2007)
13. Groves, T.: Incentives in teams. Journal Econometrica 41, 617–631 (1973)

14. Gonen, R., Vassilvitskii, S.: Sponsored Search Auctions With Reserve Prices: Going Beyond Separability. Yahoo! Research Technical Report 2008-004, http://www.ricagonen.com/ http://theory.stanford.edu/~sergei/index.html
15. Kempe, D., Mahdian, M.: A Cascade Model for Externalities in Sponsored Search. In: Ad Auctions Workshop (2008); In: WINE 2008 (to appear)
16. Lahaie, S.: An Analysis of Alternative Slot Auction Designs for Sponsored Search. In: Proceedings of the 7th ACM Conference on Electronic Commerce. ACM Press, New York (2006)
17. McAffee, P.: A Dominant Strategy Double Auction. Journal of Economic Theory 56, 434–450
18. Myerson, R., Satterthwaite, M.: Efficient Mechanisms for Bilateral Trading. Journal of Economic Theory 29, 265–281
19. Mehta, A., Saberi, A., Vazirani, U., Vazirani, V.: Adwords and the Generalized Bipartite Matching Problem. In: Proceedings of the Symposium on the Foundations of Computer Science, pp. 264–273 (2005)
20. Varian, H.: Position Auctions. International Journal of Industrial Organization (October 2006)
21. Vickrey, W.: Counterspeculation, Auctions and Competitive Sealed Tenders. Journal of Finance 16, 8–37 (1961)