CS364A: Problem Set #2

Due to the TAs by noon on Friday, October 25, 2013

Instructions:

(0) We'll grade this assignment out of a total of 75 points; if you earn more than 75 points on it, the extra points will be treated as extra credit.

(1) Form a group of at most 3 students and solve as many of the following problems as you can. You should turn in only one write-up for your entire group.

(2) Turn in your solutions directly to one of the TAs (Kostas or Okke). Please type your solutions if possible and feel free to use the LaTeX template provided on the course home page. Email your solutions to `cs364a-aut1314-submissions@cs.stanford.edu`. If you prefer to hand-write your solutions, you can give it to one of the TAs in person.

(3) If you don't solve a problem to completion, write up what you've got: partial proofs, lemmas, high-level ideas, counterexamples, and so on.

(4) Except where otherwise noted, you may refer to your course notes, and to the textbooks and research papers listed on the course Web page *only*. You cannot refer to textbooks, handouts, or research papers that are not listed on the course home page. If you do use any approved sources, make you sure you cite them appropriately, and make sure that all your words are your own.

(5) You can discuss the problems verbally at a high level with other groups. And of course, you are encouraged to contact the course staff (via Piazza or office hours) for additional help.

(6) No late assignments will be accepted.

Problem 7

This exercise derives an interesting interpretation of a virtual valuation $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ and the regularity condition. Consider a strictly increasing distribution function F with a strictly positive density f on the interval $[0, v_{\max}]$ (with $v_{\max} < +\infty$).

For $q \in [0,1]$, define $V(q) = F^{-1}(1-q)$ as the posted price resulting in a probability q of a sale (for a single bidder with valuation drawn from F). Define $R(q) = q \cdot V(q)$ as the expected revenue obtained when (for a single bidder) the probability of a sale is q. The function $R(q)$, for $q \in [0,1]$, is often called the "revenue curve" of a distribution F. Note that $R(0) = R(1) = 0$.

(a) (2 points) What is the revenue curve for the uniform distribution on $[0,1]$?

(b) (4 points) Prove that the slope of the revenue curve at q (i.e., $R'(q)$) is precisely $\varphi(V(q))$, where φ is the virtual valuation function for F.

(c) (3 points) Prove that a distribution is regular if and only if its revenue curve is concave.

(d) (6 points) Prove that, for a regular distribution, the median price $V(\frac{1}{2})$ is a $\frac{1}{2}$-approximation of the optimal posted price. That is, prove that $R(\frac{1}{2}) \geq \frac{1}{2} \cdot \max_{q \in [0,1]} R(q)$.

[Hint: use (c).]

Problem 8

Recall the Prophet Inequality from Lecture 6.

(a) (6 points) Extend the Prophet Inequality to the case where there is no threshold t with $q(t) = \frac{1}{2}$, where $q(t)$ is the probability that no prize meets the threshold.

[Hint: define t such that $\Pr[\pi_i > t \text{ for all } i] \leq \frac{1}{2} \leq \Pr[\pi_i \geq t \text{ for all } i]$, and show that one of the two corresponding strategies ("first prize $\geq t$" or "first prize $> t$") satisfies the Prophet Inequality.]

(b) (4 points) Show that the $\frac{1}{2}$-approximation in the Prophet Inequality cannot be improved: for every constant $c > \frac{1}{2}$, there are distributions $G_1, \ldots, G_n$ such that *every* strategy, threshold or otherwise, has expected value less than $c \cdot \mathbf{E}_{\pi \sim \mathbf{G}}[\max_i \pi_i]$.

(c) (5 points extra credit) Can the Prophet Inequality be improved for i.i.d. distributions, where $G_1 = G_2 = \cdots = G_n$?

Problem 9

This problem considers variations on the Bulow-Klemperer theorem (Lecture 6).

(a) (3 points) Consider selling $k \geq 1$ identical items (with at most one given to each bidder) to bidders with valuations drawn i.i.d. from F. Prove that for every $n \geq k$, the expected revenue of the Vickrey auction (with no reserve) with $n + k$ bidders is at least that of the Bayesian-optimal auction for F with n bidders.

(c) (7 points) In the same setup as (a), assume that $n \geq 2$ and consider the following alternative mechanism: pick one of the n bidders uniformly at random, say with bid r, and run the Vickrey auction with reserve r on the other $n - 1$ bidders. Prove that the expected revenue of this auction is at least $(n-1)/2n$ times that of the Bayesian-optimal auction (with n bidders).

Problem 10

This problem steps through a reasonably general "simple near-optimal auction" result. Consider a single-parameter environment for which every feasible vector $(x_1, \ldots, x_n) \in X$ is 0-1 — that is, in every outcome, each bidder either "wins" or "loses". Suppose also that the feasible set is *downward-closed*, meaning that if S is feasible set of winning bidders and $T \subseteq S$, then T is also a feasible set of winning bidders. Assume further that, for every bidder i, its valuation distribution satisfies the *monotone hazard rate (MHR)* condition, meaning that $\frac{f_i(v_i)}{1-F_i(v_i)}$ is nondecreasing in v_i.[1]

(a) (2 points) Prove that a MHR distribution is regular.

(b) (2 points) Prove that uniform and exponential distributions satisfy the MHR condition..

(c) (3 points) Let r_i be the monopoly price for the (MHR) distribution F_i — recall that this is $\operatorname{argmax}_{r \geq 0}\{r \cdot (1 - F_i(r))\}$. Let φ_i be its virtual valuation function. Prove that, for every $v_i \geq r_i$, $r_i + \varphi_i(v_i) \geq v_i$.

(d) (0 points) Consider the following allocation rule $\mathbf{x}$:

(i) Let S denote the bidders i that satisfy $v_i \geq r_i$.

(ii) Maximize surplus among bidders in S, subject to feasibility. That is, compute

$$W = \operatorname*{argmax}_{T \subseteq S \,:\, T \text{feasible}} \sum_{i \in T} v_i.$$

[1] For intuition behind the MHR condition, think about how long it takes for a light bulb to burn out. Given that it hasn't burned out yet, the chance that it burns out right now is increasing in the amount of time that has elapsed.

(iii) Define $x_i(\mathbf{v}) = 1$ if $i \in W$ and 0 otherwise.

Convince yourself that this allocation rule is monotone and always outputs a feasible outcome. Recall that with a 0-1 feasible set, the payment of a winning bidder is its critical bid — the smallest bid it could make and continue to win.

(e) (5 points) For the rest of this problem, let $\mathcal{M}$ denote the DSIC mechanism induced by the allocation rule in (d), and let $\mathcal{M}^*$ denote the mechanism that maximizes expected revenue in the given environment.

Prove that the expected surplus of $\mathcal{M}$ is at least that of $\mathcal{M}^*$. [Hint: use the downward-closure assumption to reason about the outcome selected by $\mathcal{M}^*$.]

(f) (5 points) Prove that the expected revenue of $\mathcal{M}$ is at least half of its expected surplus. [Hint: use (c).]

(g) (3 points) Prove that the expected revenue of $\mathcal{M}$ is at least half the expected revenue of the optimal mechanism $\mathcal{M}^*$.

Problem 11

Consider the following pricing problem. There is one consumer who wants at most one of n non-identical goods. Assume that the consumer's private valuations $v_1, \ldots, v_n$ for the n goods are drawn from known independent regular distributions $F_1, \ldots, F_n$. Our goal is to set prices $p_1, \ldots, p_n$ for the n goods (which can depend on the F_i's but not the actual v_i's) to maximize expected revenue, assuming that the consumer responds to prices by picking the good that maximizes $v_i - p_i$ (or picking no good if $p_i > v_i$ for every i).

(a) (7 points) Prove that the maximum-achievable expected revenue is bounded above by the expected revenue of an optimal single-item auction with n bidders with valuations drawn i.i.d. from distributions $F_1, \ldots, F_n$.

(b) (8 points) Use the Prophet Inequality to exhibit a set of prices $p_1, \ldots, p_n$ for the single-consumer pricing problem that achieves expected revenue at least half that of the upper bound identified in (a).

[Warning: The proof is a little tricky. The equivalence of expected revenue and expected virtual surplus holds only for auctions in single-parameter environments. The single-consumer pricing problem studied here has one agent with multiple private valuations, and thus is not a single-parameter environment.]

Problem 12

(a) (5 points) Consider a general mechanism design problem, with a set Ω of outcomes, and n players, where player i has a private real-valued valuation $v_i(\omega)$ for each outcome $\omega \in \Omega$. Suppose the function $f : \Omega \to \mathcal{R}$ has the form

$$f(\omega) = c(\omega) + \sum_{i=1}^{n} w_i v_i(\omega),$$

where c is a publicly known function of the outcome, and where each w_i is a nonnegative, public, player-specific weight. Such a function is called an *affine maximizer.*

Show that for every affine maximizer objective function f and every subset $\Omega' \subseteq \Omega$ of the outcomes, there is a DSIC mechanism that maximizes f over Ω'.

[Hint: modify the VCG mechanism. Don't worry about individual rationality.]

(b) (3 points) For the rest of this problem, consider a combinatorial auction with a set S of m goods and n bidders. Assume that the valuation $v_i(\cdot)$ of bidder i depends only on its bundle T_i of goods; that it is nondecreasing (so $T_1 \subseteq T_2$ implies that $v_i(T_1) \leq v_i(T_2)$); that $v_i(\emptyset) = 0$; and that v_i is *subadditive*, meaning that $v_i(T_1) + v_i(T_2) \geq v_i(T_1 \cup T_2)$ for every pair T_1, T_2 of disjoint subsets of goods.

In this and the next two parts, we ignore incentive constraints and focus on polynomial-time approximate surplus maximization, assuming that valuations are known. Given S and $v_1, \ldots, v_n$, call the surplus maximization problem *lopsided* if there is an optimal allocation of goods in which at least half of the total surplus of the allocation is due to players that were allocated a bundle with at least $\sqrt{m}$ goods. (I.e., if $2\sum_{i\in A} v_i(T_i^*) \geq \sum_{i=1}^n v_i(T_i^*)$, where $\{T_i^*\}$ is the optimal allocation and A is the subset of bidders i with $|T_i^*| \geq \sqrt{m}$.)

Show that in a lopsided problem, there is an allocation that gives all of the goods to a single player and achieves an $\Omega(1/\sqrt{m})$ fraction of the maximum-possible surplus.

(c) (4 points) Show that in a problem that is not lopsided, there is an allocation that gives at most one good to each player and achieves an $\Omega(1/\sqrt{m})$ fraction of the maximum-possible surplus.

[Hint: use subadditivity.]

(d) (4 points) Give a polynomial-time $\Omega(1/\sqrt{m})$-approximate surplus maximization algorithm for subadditive valuations.

[Hint: make use of a graph matching algorithm.]

(e) (4 points) Give a polynomial-time, $\Omega(1/\sqrt{m})$-approximate, DSIC combinatorial auction for bidders with subadditive valuations.

[Hint: use part (a).]

Problem 13

In a *procurement auction*, the tables are turned: there is a single buyer and multiple sellers. Each seller i has an intrinsic and private value for their good (which we will call a *cost* c_i), and the buyer wants to obtain a desired subset of goods. In the simplest case, the buyer needs to acquire one good — and, subject to this hard constraint, wants to pay as little as possible. The analog of the second-price auction is to award the buyer the cheapest good and charge the buyer the second-lowest reported cost.

We consider a stylized but educational more complex example. Let $G = (V, E)$ be an undirected graph. Each edge e is a seller and has a private nonnegative cost c_e, while the buyer is required to buy a spanning tree. Assume that G is 2-edge-connected (i.e., at least two edges cross every cut). You can also assume that all reported edge costs are distinct.

(a) (5 points) We consider only the cost-minimizing allocation rule, which in this context picks the MST of G (according to the reported costs of the edges). Prove that the DSIC payments (from the buyer to the edges) that always pay zero to unpicked edges are the following: for every edge e of the chosen MST T, let f denote the cheapest edge other than e that crosses the (unique) cut induced by the two connected components of $T - \{e\}$; then the payment from the buyer to edge e is c_f.

(b) (7 points) Fix distinct edge costs for G, let T_1 be the corresponding MST, and T_2 the cheapest spanning tree edge-disjoint from T_2. (Assume that such a tree exists.) Construct a bipartite graph $H = (U, W)$ in which vertices of U and W correspond to the edges of T_1 and T_2, respectively. Include edge (e_1, e_2) in H if and only if $T_1 - \{e_1\} \cup \{e_2\}$ is a connected subgraph (and hence a spanning tree) of G. Prove that H has a perfect matching.

[Hint: You should assume and use *Hall's Theorem*, which states that a bipartite graph $H = (U, W)$ with $|U| = |W|$ has a perfect matching if and only if for every subset $S \subseteq U$ with neighbors $\Gamma(S) \subseteq W$ on the other side, $|S| \leq |\Gamma(S)|$. (The "only if" direction is trivial; the "if" direction is not.)]

(c) (3 points) Using parts (a) and (b), prove that the total payment charged by the mechanism in (a) to the buyer is at most that of the cheapest spanning tree that is edge-disjoint from the MST.

(d) (5 points) In contrast, suppose a buyer is required to buy a path from vertex s to vertex t in a directed graph G with n vertices (again in which edges are sellers with private costs). Prove that the (unique) mechanism that implements the optimal allocation (i.e., that always selects a shortest path P) might charge the buyer $\Omega(n)$ times the length of the shortest path that is disjoint from the chosen path P.