CS269I: Incentives in Computer Science
Lecture #20: Fair Division*

Tim Roughgarden†

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1 Cake Cutting

1.1 Properties of the Cut and Choose Protocol

For our last lecture we embark on a nostalgia trip, and continue the theme of the course's first week, where we revisit familiar procedures and study their incentive properties. (In the first week, we talked about the Draw and serial dictatorship mechanisms, and college admissions and the Gale-Shapley deferred acceptance algorithm.) Unlike the examples in the first week, we focus primarily on fairness properties. (In the first week we focused on strategyproofness and Pareto optimality, neither of which says anything about the "fairness" of the solution.)

Suppose two people need to split a heterogeneous and divisible good. The usual euphemism in the fair division literature is that of cutting a cake. More practically, the good could be an estate (e.g., in a divorce settlement) or processing time on a computer cluster (perhaps with some times of the day more valuable than others).

Why not just split the good 50/50? This makes sense when the good is homogeneous, but its not clear what this means with a heterogeneous good. A player may value a part of the good much more than another, and different players can have different opinions about which parts are the most valuable.

We all know a reasonable protocol for two-person cake-cutting—it is mentioned already in the Bible, and is reinvented every year by siblings around the world.

†Department of Computer Science, Stanford University, 474 Gates Building, 353 Serra Mall, Stanford, CA 94305. Email: tim@cs.stanford.edu.

The Cut and Choose Protocol

1. Player 1 splits the good into two pieces A and B, such that the player's value for each is exactly half that of the entire good.
2. Player 2 picks whichever of A, B she likes better.

The description above is the intended behavior of the players in the protocol—we'll talk shortly about whether or not they are incentivized to follow this behavior. We've been led all our lives to believe that this is a "fair" protocol. But is it? How would we formally argue one way or the other?

Here's the formal model. The good, or "cake," is the unit interval $[0, 1]$. (Yes, it's a weird-looking cake.) Each player i has a valuation function v_i, which specifies the value $v_i(S)$ to i of a given subset S of the cake. We'll make the following two assumptions about each valuation function v_i:[1]

1. v_i is normalized, with $v_i([0, 1]) = 1$. This is more or less without loss of generality, by scaling.
2. v_i is additive on disjoint subsets. That is, if $A, B \subseteq [0, 1]$ are disjoint, then

$$v_i(A) + v_i(B) = v_i(A \cup B).$$

First question: is the cut and choose protocol strategyproof? Let's start with the second player. Since she can't affect the split of the cake into A and B, and is supposed to choose the piece she likes better, she has no incentive to deviate. But if the first player knows something about the second player's valuation function, she might want to deviate from the protocol. For example, suppose the good is a hot fudge sundae, the first player likes all parts of the sundae equally, while the second player likes ice cream but really cares about the cherry. The first player could split the sundae into the cherry and the rest, knowing that the second player would take the cherry, leaving a very valuable piece for the first player. If the first player doesn't know anything about what the second player wants, and assumes that the second player will always leave the piece that is worse for the first player, then the first player is incentivized to follow the protocol (to guarantee herself a piece with value $\frac{1}{2}$). In any case, the cut and choose protocol is not strategyproof in the same sense as the Draw (where you should always follow the protocol, no matter what other people want and are doing).

Second question: is the cut and choose protocol guaranteed to produce a Pareto optimal solution (assuming both players behave as intended)? A little thought shows that the answer is again "no." Consider the cake in Figure 1, where the first player only wants the first and third quarters of the cake, while the second player only wants the second and fourth quarters

[1]Actually, we also need a "continuity" assumption for everything to make sense—e.g., in the cut and choose protocol, it's important that there exists a cut that makes player 1 indifferent between the two pieces. We omit further discussion of this assumption.

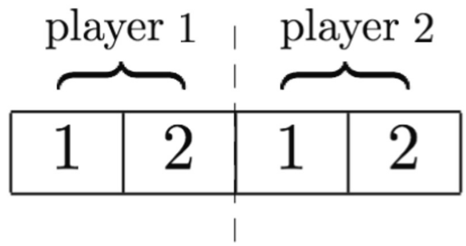

Figure 1: The solution produced by the cut and choose protocol may not be Pareto Optimal.

of the cake. One way the first player might split the cake would be into its first and second halves, resulting in both players getting a piece valued at $\frac{1}{2}$. But allocating each piece of cake to the only player who wants it results in both players having value 1.

Maybe these problems are not specific to the cut and choose protocol, and there is a fundamental impossibility result? Not if we consider only strategyproofness and Pareto optimality: for example, always giving the entire cake to the first player is clearly strategyproof and (under weak assumptions on the v_i's) Pareto optimal. So hopefully the cut and choose protocol has some other nice property, to compensate for not being strategyproof or Pareto optimal.

What about "fairness?" One possible definition of fairness would be that both players wind up equally happy. But this property is also not satisfied by the cut and choose protocol: the first player is guaranteed to get a piece that she values at $\frac{1}{2}$, while the second player might well end up with a piece that she values at greater than $\frac{1}{2}$.

One definition of "fairness" is that each player receives at least her fair share, at least from her perspective.

Definition 1.1 An allocation $A_1, A_2, \ldots, A_n$ of cake to n players is *proportional* if

$$v_i(A_i) \geq \frac{1}{n}$$

for every player i.

The cut and choose protocol satisfies proportionality—the first player gets a piece that she values at $\frac{1}{2}$ and the second player does at least as well. Obviously, the dictator protocol does not satisfy proportionality.

A second definition is that no player wants to trade places with any other player.

Definition 1.2 An allocation $A_1, A_2, \ldots, A_n$ of cake to n players is *envy-free* if

$$v_i(A_i) \geq v_i(A_j)$$

for every pair i, j of players.

This means that while player j might like her piece more than i likes her own, to player i's own tastes, her piece is better than that of player j's.

The second definition is more stringent than the first. To see this, note that for every i, $\sum_{j=1}^{n} v_i(A_j) = v([0,1]) = 1$ (by our assumptions on the v_i's). So if player i likes A_i better than every other piece (as dictated by envy-freeness), it must be that $v_i(A_i) \geq \frac{1}{n}$.

The converse also holds for the special case of $n = 2$ (if you get a piece of cake that you value at least $\frac{1}{2}$, then swapping would net you a piece that you value at most $\frac{1}{2}$). In particular, the cut and choose protocol is envy-free. (This is also easy to see directly: the first player is indifferent between the two pieces, while the second player gets her favorite one.) For $n \geq 3$, there can be proportional allocations that are not envy-free (exercise).

1.2 Beyond Two Players

The obvious next question is to ask about analogs of the cut and choose protocol—meaning envy-free protocols—that work with 3 or more players. This turns out to be a tricky question, and there is now a small cottage industry around envy-free protocols. And the biggest breakthrough in decades occurred just earlier this year! We focus on envy-freeness and ignore strategyproofness issues.

First, let's consider the case of 3 players. Selfridge and Conway (around 1960) independently designed the same envy-free protocol for this case. (Neither saw fit to publish it, so it came to light only a few years later.) One good exposition is [5]; here's the rough idea. In the first step, the third player cuts the cake into three pieces such that she has value $\frac{1}{3}$ for each of them. The first and second players then select their favorite pieces. If they select different pieces, then we are done (why?). So the interesting case is when the first and second players both prefer, say, the first piece. Then, the second player is asked to trim off a subpiece of the first piece such that she is indifferent between the trimmed piece and her second-favorite of the original three pieces. The protocol then recurses on the trimmings with the roles of the players swapped (with the second player cutting the trimmings into three pieces of equal value to her). The protocol does not need to recurse further, because of the special structure of the recursively defined subproblem. In the worst case, the Selfridge-Conway protocol makes 5 cuts (the first two cuts by the third player, the trim by the second player, and two more cuts in the recursive call).

The problem of coming up with an envy-free protocol for four or more players was open for decades. The next breakthrough came in 1995, by Brams and Taylor [3], who gave a finite protocol for computing an envy-free allocation with any number of players. The only drawback is that the protocol is *unbounded*: while for any fixed valuation functions $v_1, \ldots, v_n$ it halts in a finite number of steps, for every $n \geq 4$ and T, there is a choice of $v_1, \ldots, v_n$ such that the protocol requires more than T steps to terminate.

The obvious next goal was to come up with a *bounded* envy-free protocol, where the maximum number $f(n)$ of steps can depend on the number of players n (by necessity) but not on the v_i's. This was a big open question for a couple of decades, and many experts believed that no such protocol existed. But earlier this year, Aziz and Mackenzie [2] gave a four-player protocol that is guaranteed to produce an envy-free allocation after at most 203 cuts. (So as you can imagine, the protocol is pretty complicated.) Aziz and Mackenzie [1] followed up with a paper, published less than two months ago, that extends the result to the general n-player case. The bound on the number of cuts? Currently, it's a tower of 6 n's, meaning $n^{n^{n^{n^{n^{n}}}}}$.[2] As for lower bounds, $n-1$ is obvious (why?) and the best known is $\Omega(n^2)$ [6]. Now there's a gap that's in need of narrowing! In particular, is a bound polynomial in n possible?

2 Rent Division: Fair Division in Practice

One place where fair division protocols are used in practice is on `spliddit.org`, which has been used by tens of thousands of people. One of the problems that spliddit solves is the *rent division* problem, where there are n people, n rooms, and a rent of R. The goal is to assign people and rents to rooms, with one person per room and with the sum of rents equal to R, in the "best" way possible.

We assume that each person i has a value v_{ij} for each room j, and normalize these values so that $\sum_j v_{ij} = R$. (In effect, we force each player to acknowledge the constraint that the entire rent must get paid.) We use a quasi-linear utility function (like in our lectures on auctions), meaning that we assume that each player i wants to maximize v_{ij} minus the rent paid for her room j. This is a more specific assumption than we made for the abstract cake cutting problem, but it enables envy-free solutions and is reasonable in this context.[3]

A solution to a rent division problem is *envy-free* if

$$v_{if(i)} - r_{f(i)} \geq v_{if(j)} - r_{f(j)} \tag{1}$$

for every pair i, j of players, where $f(i)$ denotes the room to which i is assigned and r_j denotes the rent assigned to the room j. That is, no one wants to trade places with anyone else (where trading places means swapping both rooms *and* rents).

The good news is that an envy-free solution is guaranteed to exist, and that one can be computed efficiently.[4] The bad news is that there can be many envy-free solutions, and not all of them are reasonable. For example, suppose there are two players and two rooms, that the total rent R is 1000 (so clearly not in the Bay Area. . .), and that the first player only wants the first room ($v_{11} = 1000$ and $v_{12} = 0$) while the second player only wants the second room ($v_{21} = 0$ and $v_{22} = 1000$). The only reasonable room assignment is to give each person the room that they want. Intuitively, by symmetry, each person should pay 500 in rent. But every division of the rent is envy-free! Even if you make the first person pay almost 1000 for her room, she still doesn't want to swap with the other person.

The upshot is that we need a method for selecting one out of the many envy-free solutions. One can imagine several ways of doing this; here's what happens on spliddit (given v_{ij}'s and R):

1. Choose the room assignment f to maximize $\sum_i v_{if(i)}$.

[2] I'm not kidding. Hey, at least it's not a tower of n n's!

[3] Rent division isn't really a special case of cake cutting, since the rooms are indivisible.

[4] Both these facts follow from the theory of maximum-weight bipartite matchings, as studied in CS261.

2. Set the room rents so that envy-freeness (1) holds, and subject to this, maximize the minimum utility:

$$\max_{\mathbf{r}} \left(\min_{i=1}^{n} (v_{if(i)} - r_{f(i)}) \right).$$

Algorithmically, the first step can be done by computing a maximum-weight bipartite matching and the second step is easily encoded as a linear program, for which good off-the-shelf solvers exist (see CS261). One could imagine selecting an envy-free solution in a different way, for example by optimizing a different objective, but the method above seems successful empirically, according to user studies. See [4] for further details.

References

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