# Restoring Pure Equilibria to Weighted Congestion Games

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**Abstract.** Congestion games model several interesting applications, including routing and network formation games, and also possess attractive theoretical properties, including the existence of and convergence of natural dynamics to a pure Nash equilibrium. Weighted variants of congestion games that rely on sharing costs proportional to players' weights do not generally have pure-strategy Nash equilibria. We propose a new way of assigning costs to players with weights in congestion games that recovers the important properties of the unweighted model. This method is derived from the Shapley value, and it always induces a game with a (weighted) potential function. For the special cases of weighted network cost-sharing and atomic selfish routing games (with Shapley value-based cost shares), we prove tight bounds on the price of stability and price of anarchy, respectively.

**Keywords:** congestion games, network design, Shapley value

## 1 Introduction

Congestion games are a well-studied generalization of several game-theoretic models, including some fundamental network formation games and routing games. In the standard model [19], there is a ground set of resources, and each player has a set of allowable strategies, each of which is a subset of resources. For example, the strategies of a player could correspond to the paths of a network with a given source and sink. Each resource has a per-user cost that depends on the number of players that use it, and the goal of each player is to minimize the sum of the resources' costs in its strategy, given the strategies chosen by the other players. In atomic selfish routing games [20, 22], strategies correspond to paths and resource cost functions $c_e(\cdot)$ are nondecreasing. In network cost-sharing games [2], strategies correspond to paths and the (decreasing) cost functions have the form $c_e(f_e) = \gamma_e/f_e$, where $\gamma_e$ is the fixed installation cost of an edge $e$ and $f_e$ is the number of players that share it.

A *pure Nash equilibrium (PNE)* is a strategy profile such that no player can decrease its cost via a unilateral deviation. Many games, such as "Rock-Paper-Scissors", have no PNE. Rosenthal [19] used a potential function argument to show that every congestion game — and thus every atomic selfish routing and network cost-sharing game — has at least one PNE. Moreover, several natural dynamics are guaranteed to converge to a PNE in congestion games [14, 15].

Every player of a congestion game imposes the same load on a resource. There are many motivations for relaxing this assumption and allowing non-uniform resource consumption: for example, in a network context, players could have different durations of resource usage, different bandwidth requirements, or different contracts with the network provider. Almost all research to date has modeled non-uniform players in congestion-type games through *proportional cost sharing* [1–3, 5, 8, 10, 13, 15]. The first assumption in proportional cost sharing is that each player $i$ has a positive weight $w_i$, with larger weights indicating larger resource usage. To explain the second assumption in a general way, let $C_e(S_e)$ denote the joint cost incurred by the subset $S_e$ of users of the resource $e$. For example, in a network cost-sharing game, $C_e(S_e)$ is the fixed cost $\gamma_e$ provided $S_e$ is non-empty (and is 0 otherwise). In (weighted) atomic selfish routing, $C_e(S_e)$ is $f_e \cdot c_e(f_e)$, where $c_e(\cdot)$ is the per-flow unit resource cost function and $f_e = \sum_{i \in S_e} w_i$ is the total weight of the players using $e$. Proportional cost sharing dictates that each player $i \in S_e$ pays a $w_i / \sum_{j \in S_e} w_j$ fraction of $C_e(S_e)$ for the resource $e$.

Unfortunately, most of the attractive theoretical properties of congestion games do not carry over to their weighted counterparts with proportional cost sharing. Network cost-sharing games with at least three players need not have a PNE [5]. Even when PNE do exist in such games, they can be much costlier (relative to an optimal solution) than in the unweighted case [2, 5]. Atomic selfish routing games with weighted players do not generally have PNE [9, 10, 20], except when all cost functions are affine [7] and in some other isolated special cases [10].

Guaranteed existence of PNE is an important property. There are, of course, more general equilibrium concepts like the mixed-strategy Nash equilibrium that are guaranteed to exist in every finite game, but randomized solution concepts suffer from well-known drawbacks in interpretation and implementation (see e.g. [18, §3.2]). Particularly when designing or influencing the game being played — the standard interpretation of network cost-sharing games [2, 5, 6] and also a well-motivated one for routing games [16, 17, 24] — there is good reason to make design decisions (e.g., queueing policies [24]) that guarantee the existence of and convergence of natural dynamics to a PNE.

We propose a new way of assigning costs to players with weights in congestion-type games, which is derived from the Shapley value. We call the resulting class of games *SV weighted congestion games*. Extending work of Hart and Mas-Colell [11], we show that every SV weighted congestion game admits a (weighted) potential function. The existence of and convergence of natural dynamics to a PNE in every such game follow immediately.

For example, for the special case of atomic selfish routing games, we derive the cost shares for the users $S_e$ of edge $e$ by applying the standard Shapley value (defined in the next section) to the cost function $C_e$ above with the player set $S_e$. Since the incremental effect of a player on the joint cost is increasing in its weight, so is its cost share. These Shapley value-based cost shares coincide with proportional shares when all per-user cost functions are affine, but not otherwise. These observations explain the previously mysterious fact that the traditional proportional cost shares always yield a potential game if and only if all cost functions are affine [7, 10].

For the special case of network cost-sharing games, the joint cost function $C_e$ is insensitive to players' weights. To introduce weight-dependent cost shares, we use the *weighted* Shapley value [12, 23], which averages over orderings of the players in a non-uniform way (see the next section for a definition). The resulting cost shares are increasing in weight, and coincide with proportional shares (for all weight vectors) if and only if there are at most two players. Again, these facts demystify the previously observed phase transition between the cases of two players [2] and three or more players [5] for proportional cost shares.

We also provide tight bounds on the inefficiency of equilibria in SV weighted network cost-sharing and atomic selfish routing games. For network cost-sharing games our results are tight in a strong sense: for every number $k$ of players and *every* weight vector $w$, we characterize the worst-case price of stability (POS) — the ratio between the cost of the best PNE and of an optimal solution — in games with player weight vector $w$. For each $w$, we give an explicit lower bound and prove a matching upper bound for all networks. The special case of $w = (1, 1, \ldots, 1)$ — where the worst-case POS is the $k$th Harmonic number — is one of the main results in Anshelevich et al. [2]. Similarly, for atomic selfish routing games, we give tight bounds on the worst-case price of anarchy (POA) — the ratio between the cost of the worst PNE and of an optimal outcome — with respect to every set of convex cost functions (for a worst-case set of player weights). This worst-case POA is larger, but only slightly, than the one in weighted congestion games with proportional cost sharing that have PNE.

## 2 Preliminaries

We first explain the weighted Shapley value in general [12, 23]. Consider a set $S$ of players and a cost function $C : 2^S \to \mathbb{R}$. (For us, $S$ is the users of a resource and $C(T)$ is the joint cost that would be incurred if the players of $T \subseteq S$ were its sole users.) For a given ordering $\pi$ of the players, let $\Delta_i(\pi)$ denote $C(S^i(\pi) \cup \{i\}) - C(S^i(\pi))$, where $S^i(\pi)$ denotes the players preceding $i$ in $\pi$.

Each player has a weight $w_i$ and a sampling probability $\lambda_i$. Traditionally, [12, 23] $\lambda_i$ is set to $1/w_i$. We use the $\lambda_i$'s to define a distribution over orderings of players, as follows. (When all $\lambda_i$'s are equal, we get the uniform distribution and recover the usual Shapley value.) We first choose the final player in the ordering, with probabilities proportional to the $\lambda_i$'s; given this choice, we choose the penultimate player randomly from the remaining ones, again with probabilities proportional to the $\lambda_i$'s; and so on. The weighted Shapley value of player $i$ is defined as the expected value of $\Delta_i(\pi)$ with respect to this distribution over orderings $\pi$.

### 2.1 Weighted Network Cost-Sharing

In weighted network cost-sharing, each player $i = 1, 2, \ldots, k$ has a weight $w_i \geq 1$ and a sampling weight $\lambda_i = 1/w_i$. Player $i$ aims to construct a path $P_i$ from a given node $s_i$ to a given node $t_i$ in a directed graph $G = (V, E)$, where every $e \in E$ has a fixed nonnegative cost $\gamma_e$. We next give a probabilistic representation of weighted Shapley cost shares and the corresponding potential function, in terms of independent random variables $X_1, \ldots, X_k$, where $X_i$ is exponentially distributed with rate $\lambda_i$. With $S_e$ the players on edge $e$, the weighted Shapley value of player $i \in S_e$ is defined as $\xi_i(S_e) \cdot \gamma_e$, where $\xi_i(S_e)$ is the probability that $X_i \geq X_j$ for every $j \in S_e$. Some thought shows that this definition is equivalent to the one given at the start of the section (for our joint cost function that equals 1 for every non-empty set).

A network cost-sharing game with these cost shares admits the following (weighted) potential function. For every path vector $P$, we define $\Phi_e(P) = \gamma_e \cdot \mathbf{E}[\max_{j \in S_e} X_j]$, where $S_e$ is the set of players that use $e$ in the profile $P$, and $\Phi(P) = \sum_{e \in E} \Phi_e(P)$. The fact that $\Phi$ is a weighted potential function can be derived easily assuming results of Hart and Mas-Colell [11] about cooperative games; but for the present cost function we can give a direct proof.

**Proposition 1.** *Given a path vector $P$, suppose player $i$ deviates from $P_i$ to $P_i'$. Let $\Delta\Phi$ be the resulting change in the potential function and $\Delta C_i$ the resulting change in the cost of player $i$. Then $\Delta\Phi = w_i \cdot \Delta C_i$.*

*Proof.* We prove the equality for each $e \in E$. By symmetry we can assume that $e \in P_i' \setminus P_i$. The cost share of player $i$ on edge $e$ increases from 0 to $\gamma_e \cdot \xi_i(S_e \cup \{i\})$. Let $\Delta\Phi_e$ be the change in the potential on edge $e$. We have

$$\Delta\Phi_e = \gamma_e \cdot \mathbf{E}\left[\left(X_i - \max_{j \in S_e} X_j\right) \middle| X_i \geq \max_{j \in S_e} X_j\right] \cdot \mathbf{Pr}\left[X_i \geq \max_{j \in S_e} X_j\right].$$

From the facts that the exponential distribution is memoryless and $w_i = 1/\lambda_i$ we get that the above is equal to $\gamma_e \cdot w_i \cdot \xi_i(S_e \cup \{i\})$. ☐

### 2.2 Atomic Selfish Routing

In atomic selfish routing, each player $i = 1, 2, \ldots, k$ has weight $w_i$ and selects a path $P_i$ from a node $s_i$ to a node $t_i$ in a given graph $G = (V, E)$. For every $e \in E$, the cost function $c_e(\cdot)$ is nonnegative and nondecreasing and the players in $S_e$ have to pay $C_e(S_e) = f_e \cdot c_e(f_e)$, where $f_e$ is their total weight. In atomic selfish routing, the usual (unweighted) Shapley value already gives good weight-dependent cost shares, because the joint cost function $C_e$ is asymmetric. Let $X_{i,e}$ be a random variable with value equal to the total weight of those that appear before $i$ in a uniformly random ordering of the players on edge $e$. Then the Shapley value of $i$ for $e$ is $c_{i,e}(S_e) = \mathbf{E}[(X_{i,e} + w_i) \cdot c_e(X_{i,e} + w_i) - X_{i,e} \cdot c_e(X_{i,e})]$.

We now present the potential function of the game. With $P$ the selected path vector and $\pi$ an arbitrary ordering of the players on $e$, the edge potential is $\Phi_e(P) = \sum_{i \in S_e} c_{i,e}(S_e^i(\pi))$, where $S_e^i(\pi)$ is the set of players preceding $i$ in $\pi$. The (exact) potential function is $\Phi(P) = \sum_{e \in E} \Phi_e(P)$. The fact that this is a potential function can be derived from Hart and Mas-Colell [11].

## 3 Weighted Network Cost-Sharing

### 3.1 Lower Bounding the Price of Stability

Given the weight vector $w$, which we assume is sorted in nondecreasing order, we construct the following network. Each player $i = 1, 2, \ldots, k$ starts from node $s_i$ and they all have a common terminal node $t$. The network offers two possible paths to each $i$. The first path consists of the edge $\{s_i, v\}$ with $\gamma_{\{s_i,v\}} = 0$ and the edge $\{v, t\}$ with $\gamma_{\{v,t\}} = 1 + \epsilon$ for $\epsilon > 0$ a very small number. The second path consists of a single edge from $s_i$ to $t$ with $\gamma_{\{s_i,t\}} = \xi_i(\{1, 2, \ldots, i\})$. The largest player that uses its two-hop path always has an incentive to deviate to its one-hop path. Hence, in the unique PNE all players are using their one-hop paths. This implies that the worst-case POS in all games with weight vector $w$ is lower bounded by $\sum_i \xi_i(\{1, 2, \ldots, i\})$. If we take $w = (1, 1, ..., 1)$, we recover the well-known unweighted lower bound example of Anshelevich et al. [2].

### 3.2 Upper Bounding the Price of Stability

We prove that for *every* weight vector $w$, the example of Section 3.1 is the worst case among all possible networks.

**Theorem 1.** *Given $k$ players with weights $w_1 \leq w_2 \leq \ldots \leq w_k$, the worst case POS among all games with these player weights is $\sum_i \xi_i(\{1, 2, \ldots, i\})$.*

Consider a game with player set $S = \{1, 2, \ldots, k\}$ and weight vector $w$ on $G = (V, E)$. Let $P^*$ be the path vector that minimizes the total cost and let $P$ be the vector of paths that minimizes the potential function and is, therefore, a PNE. We observe the changes in the total cost as players deviate one by one from their paths in $P^*$ to their paths in $P$, in the order $k, k-1, \ldots, 1$. Our proof follows the main idea and steps outlined below.

*Main idea and steps of the proof.* As player $i$ deviates from path $P_i^*$ to $P_i$ the cost of $i$ changes, as does the cost of each $j$ that was or is sharing edges with $i$. We will prove that the sum of these changes over the $k$ deviations will be at most $C^* \cdot (\sum_i \xi_i(\{1, 2, \ldots, i\}) - 1)$, where $C^*$ is the total cost of path vector $P^*$.

The first step focuses on the impact that the deviation of each $i$ has on every $j \neq i$. The worst case for these changes is when all deviating players depart all edges shared with others and join new unused edges. We show this in a reverse order, starting from player 1 (the last to deviate) and going to player $k$.

The second step also focuses on the same cost changes as the first step. We show that in the worst case, every edge used in $P^*$ is shared by all players. By the end of this step, we get that the total cost increase of non-deviating players is at most $C^* \cdot (\sum_i \xi_i(\{1, 2, \ldots, i\}) - 1)$.

The third and final step focuses on the change in the cost of the deviating player during each round. We prove that the sum over all $k$ rounds is nonpositive.

*Proof.* We define the following sets of edges in the graph. For every $S_e \subseteq S$ we will write $\kappa(S_e)$ for the edges that initially (i.e., in $P^*$) are shared by the players in $S_e$. For every $S_e \subseteq S$ and for every $i \in S_e$, we will write $\delta_i(S_e)$ for the edges that are shared by the players in $S_e$ after the deviation of $i-1$ and by the players in $S_e \setminus \{i\}$ after the deviation of $i$ (these are the edges that $i$ departs from). Similarly, for every $S_e \subseteq S$ and for every $i \in S_e$, we will write $\sigma_i(S_e)$ for the edges that are shared by the players in $S_e \setminus \{i\}$ after the deviation of $i-1$ and by the players in $S_e$ after the deviation of $i$ (these are the edges that $i$ joins). We will write $\gamma(F)$ for the total cost of the edges in $F \subseteq E$. We will also denote the change in the potential function due to the deviation of player $i$ as $\Delta\Phi_i$.

Consider the deviation of player $i$. The change in $i$'s cost is tracked by the change in the potential function scaled by $1/w_i$, while the change that this deviation causes to the costs of other players is tracked by the portion of $\gamma(\delta_i(S_e))$ that $i$ pays (the other players have to pay for it after $i$ departs) and by the portion of $\gamma(\sigma_i(S_e))$ that $i$ pays (the other players no longer have to pay for it after $i$ joins the edges) for all $S_e \subseteq S$. So the difference between the optimal cost and the total cost of the potential function minimizer can be written as

$$\Delta C = \sum_i \frac{\Delta\Phi_i}{w_i} + \sum_i \sum_{S_e \ni i, |S_e| \geq 2} \gamma(\delta_i(S_e)) \cdot \xi_i(S_e) - \gamma(\sigma_i(S_e)) \cdot \xi_i(S_e). \quad (1)$$

At the moment $i$ is about to deviate, the set of edges that are shared by the players in $S_e \subseteq S$ has been shaped by the deviations of $k, k-1, k-2, \ldots, i+1$ and is

$$\tau_i(S_e) = \left[\kappa(S_e) \cup \left(\bigcup_{j \in S_e, j > i} \sigma_j(S_e)\right) \cup \left(\bigcup_{j \notin S_e, j > i} \delta_j(S_e \cup \{j\})\right)\right] \setminus \left[\left(\bigcup_{j \in S_e, j > i} \delta_j(S_e)\right) \cup \left(\bigcup_{j \notin S_e, j > i} \sigma_j(S_e \cup \{j\})\right)\right]. \quad (2)$$

It is clear that $\delta_i(S_e) \subseteq \tau_i(S_e)$. We proceed with the following proposition.

**Proposition 2.** *We can obtain an upper bound for the expression in (1) by setting $\delta_i(S_e) = \tau_i(S_e)$ and $\sigma_i(S_e) = \emptyset$ for every $S_e \subseteq S$ and every $i \in S$.*

*Proof.* We will prove this by induction. Our base case is player 1, the last one to deviate from $P_1^*$ to $P_1$. It is obvious that once everyone else has deviated, the worst case for the expression in (1) is if for every $S_e \subseteq S$ with $|S_e| \geq 2$, $\delta_1(S_e)$ is as large as possible, which means equal to $\tau_1(S_e)$, while $\sigma_1(S_e)$ is empty. Our inductive hypothesis assumes the worst case is when every player $j = 1, \ldots, n$ sets for every $S_e \subseteq S$ with $|S_e| \geq 2$, $\delta_i(S_e) = \tau_i(S_e)$ and $\sigma_i(S_e) = \emptyset$. We will prove that in the worst case, for every $S_e \subseteq S$ with $|S_e| \geq 2$, we have $\delta_{n+1}(S_e) = \tau_{n+1}(S_e)$ and $\sigma_{n+1}(S_e) = \emptyset$. After plugging the worst case deviations for players $1, 2, \ldots, n$ into (1) using (2), we look at the coefficient of $\gamma(\delta_{n+1}(S_e))$ and the coefficient of $\gamma(\sigma_{n+1}(S_e))$, for which we write $\alpha(\delta_{n+1}(S_e))$ and $\alpha(\sigma_{n+1}(S_e))$ respectively. So, from (1), (2), and the inductive hypothesis, we get that for every $S_e \subseteq S$,

$$\begin{aligned}\alpha(\delta_{n+1}(S_e)) &= -\alpha(\sigma_{n+1}(S_e))\\ &= \xi_{n+1}(S_e) + \sum_{j\in S_e, j<n+1} \alpha(\delta_j(S_e \setminus \{n+1\})) - \alpha(\delta_j(S_e)).\end{aligned}$$

To complete the proof of the proposition, it suffices to show that this is nonnegative. Assume $S_e = \{i_1, i_2, \ldots, i_m\}$ in nondecreasing weight order. The solution of this recurrence is as follows.

$$\alpha(\delta_{i_m}(S_e)) = \sum_{j=2}^{m} \xi_j(\{i_1, \ldots, i_j\}) - \sum_{j=1}^{m-1} \xi_j(\{i_1, \ldots, i_j, i_m\}) \geq 0, \text{ and for } l < m,$$

$$\alpha(\delta_{i_l}(S_e)) = \sum_{j=1}^{l} \xi_j(\{i_1, \ldots, i_j, i_{l+1}, \ldots, i_m\}) - \sum_{j=1}^{l-1} \xi_j(\{i_1, \ldots, i_j, i_l, \ldots, i_m\}) \geq 0.$$

□

From (1) and Proposition 2, we get that

$$\Delta C \leq \sum_i \frac{\Delta\Phi_i}{w_i} + \sum_i \sum_{S_e \ni i, |S_e|\geq 2} \gamma(\tau_i(S)) \cdot \xi_i(S).$$

We order the players in $S_e \subseteq S$ in nondecreasing weight order as $i_1, i_2, \ldots, i_m$ and we denote $\Xi(S_e) = \sum_{j=1}^{m} \xi_j(\{i_1, i_2, \ldots, i_j\})$. Then, we can see that the inequality above becomes

$$\Delta C \leq \sum_i \frac{\Delta\Phi_i}{w_i} + \sum_{S_e\subseteq S} \gamma(\kappa(S_e)) \cdot [\Xi(S_e) - 1]. \tag{3}$$

We can see this as follows. The cost $\gamma(\kappa(S_e))$ appears with coefficient $\xi_m(S_e)$ when $i_m$ departs from all these edges, then with coefficient $\xi_{m-1}(S_e \setminus \{i_m\})$ when $i_{m-1}$ departs from all these edges, etc. We proceed with the following proposition.

**Proposition 3.** *For any set of players $S$ we have* $\max_{S_e\subseteq S} \Xi(S_e) = \Xi(S)$.

*Proof.* It suffices to show that for any player $j \in S$ it is the case that $\Xi(S) \geq \Xi(S\setminus\{j\})$. Recall that $X_i$ is an exponential random variable with parameter $\lambda_i = 1/w_i$ and that the players in $S$ are such that $1 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. We will write $B_i(S_e)$ for the event that $X_i$ is larger than $X_j$ for every $j \in S_e$. We can see that $\Xi(S) = 1 + \sum_{i=2}^{k} \mathbf{Pr}[B_i(\{1, 2, \ldots, i\})]$ and

$$\Xi(S \setminus \{j\}) = 1 + \sum_{i=2}^{j-1} \mathbf{Pr}[B_i(\{1, 2, \ldots, i-1\})]$$

$$+\sum_{i=j+1}^{k}\mathbf{Pr}[B_i(\{1,2,\ldots,i-1\}\setminus\{j\})].$$

Hence, we get

$$\begin{aligned}
&\Xi(S)-\Xi(S\setminus\{j\})\\
&=\mathbf{Pr}[B_j(\{1,2,\ldots,j-1\})]-\sum_{i=j+1}^{k}\mathbf{Pr}[B_i(\{1,2,\ldots,i-1\}\setminus\{j\})\wedge B_j(\{i\})]\\
&=\int_0^\infty \lambda_j e^{-\lambda_j x}\prod_{l=1}^{j-1}(1-e^{-\lambda_l x})-\sum_{i=j+1}^{k}\lambda_i e^{-\lambda_i x}e^{-\lambda_j x}\prod_{l=1,l\neq j}^{i-1}(1-e^{-\lambda_l x})dx\\
&\geq\int_0^\infty \lambda_j e^{-\lambda_j x}\prod_{l=1}^{j-1}(1-e^{-\lambda_l x})-\sum_{i=j+1}^{k}\lambda_j e^{-\lambda_j x}e^{-\lambda_j x}\prod_{l=1,l\neq j}^{i-1}(1-e^{-\lambda_l x})dx\\
&\geq\int_0^\infty \lambda_j e^{-\lambda_j x}\prod_{l=1}^{j-1}(1-e^{-\lambda_l x})\left[1-\sum_{i=j+1}^{k}e^{-\lambda_j x}(1-e^{-\lambda_l x})^{i-j-1}\right]dx\geq 0.
\end{aligned}$$

□

From (3) and Proposition 3, we get

$$\Delta C\leq\sum_i\frac{\Delta\Phi_i}{w_i}+C^*\cdot[\Xi(S)-1], \tag{4}$$

with $C^*$ the optimal total cost. At this point, to complete the proof of the theorem, we only need to show that the following proposition holds.

**Proposition 4.** *It is the case that* $\sum_{i=1}^{k}\Delta\Phi_i/w_i\leq 0$.

*Proof.* We will prove by induction that for every $j=2,3,\ldots,k-1$ it is the case that

$$\sum_{i=1}^{j}\frac{\Delta\Phi_i}{w_i}\leq\frac{\sum_{i=1}^{j}\Delta\Phi_i}{w_j}.$$

Our base is the case with $j=2$. Note that $\Delta\Phi_1$ is the difference in the potential function when moving from some path vector to the potential function minimizer, so it is nonpositive. We get

$$\frac{\Delta\Phi_1}{w_1}+\frac{\Delta\Phi_1}{w_2}\leq\frac{\Delta\Phi_1+\Delta\Phi_2}{w_2}.$$

Our inductive hypothesis assumes the statement is true for $j=n$ and we will prove it is true for $j=n+1$. We have

$$\sum_{i=1}^{n+1}\frac{\Delta\Phi_i}{w_i}\leq\frac{\Delta\Phi_{n+1}}{w_{n+1}}+\frac{\sum_{i=1}^{n}\Delta\Phi_i}{w_n}\leq\frac{\sum_{i=1}^{n+1}\Delta\Phi_i}{w_{n+1}}.$$

The last step follows from the fact that $\sum_{i=1}^{n} \Delta\Phi_i$ is the change in the value of the potential function while moving from some path vector to the potential function minimizer and is, therefore, nonpositive. Now we have

$$\sum_{i=1}^{k} \frac{\Delta\Phi_i}{w_i} \leq \frac{\sum_{\mathrm{i=1}}^{\mathrm{k}} \Delta\Phi_i}{w_k} \leq 0,$$

which proves the proposition. □

Combining Proposition 4 with (4) proves the theorem. □

This result implies that as long as all players' weights lie in a bounded range, the price of stability remains $O(\log k)$ (unlike with proportional cost sharing [5]).

### 3.3 Further Analysis of the Price of Stability

In this subsection we focus on the expression $\sum_i \xi_i(\{1, 2, \ldots, i\})$, which is the worst case POS for a game with weight vector $w$. We first prove the following useful lemma.

**Lemma 1.** *The value $\xi_i(\{1, 2, \ldots, i\})$ does not decrease if we replace the weights of players $1, 2, \ldots, i-1$ with their average.*

*Proof.* We can assume that $w_{i-1} > w_1$, and show that the cost share of player $i$ increases if we replace $w_{i-1}$ with $w_{i-1} - \epsilon$ and $w_1$ with $w_1 + \epsilon$ for a very small $\epsilon$. Recall from Section 2 our probabilistic representation of the weighted Shapley cost shares. Suppose that the maximum of $X_2, X_3, \ldots, X_{i-2}$ is $y$. Consider the following experiment to determine the values of $X_1, X_{i-1}$. Numbers $\eta_1$ and $\eta_{i-1}$ are picked uniformly and independently at random from $[0, 1]$. The value of $X_1$ is $F_{\lambda_1}^{-1}(\eta_1) = -w_1 \ln(1 - \eta_1)$, which is the inverse of the cumulative distribution function of the exponential distribution with parameter $\lambda_1 = 1/w_1$. This does not change the distribution of $X_1$. Similarly the value of $X_{i-1}$ is $-w_{i-1} \ln(1 - \eta_{i-1})$.

For the same $\eta_1$ and $\eta_{i-1}$, after we have changed the weights, the values become $-(w_1+\epsilon) \ln(1-\eta_1)$ and $-(w_{i-1}-\epsilon) \ln(1-\eta_1)$. We know that $\xi_i(\{1, 2, \ldots, i\})$ is the probability that $X_i$ is larger than all $X_j$ for $j = 1, 2, \ldots, i-1$. Hence, to prove the lemma, it suffices to show that this probability increases with the updated weights. The $(\eta_1, \eta_{i-1})$ pairs which make it harder for $X_i$ to be the largest with the updated weights compared to the original weights, are the ones where $X_1$ is larger than $X_{i-1}$ and also larger than $y$. We assume $\epsilon$ is small enough to keep the relative ordering of $X_1, X_2, \ldots, X_{i-1}$ intact even after the change of the weights. It is clear that such $\epsilon$ exists. We prove that for every pair $(\eta, \eta')$ that decreases the probability of $X_i$ being the largest by $\Delta p$, the pair $(\eta', \eta)$ increases the same probability by $\Delta p' \geq \Delta p$. We have

$$\Delta p = e^{\lambda_i w_1 \ln(1-\eta)} - e^{\lambda_i (w_1+\epsilon) \ln(1-\eta)} = \left[(1-\eta)^{\lambda_i}\right]^{w_1} - \left[(1-\eta)^{\lambda_i}\right]^{w_1+\epsilon}.$$

Since $X_1$, which has a distribution with larger rate, is bigger than $X_{i-1}$ and $y$ when the pair $(\eta, \eta')$ is picked, it follows that $X_{i-1}$ is larger than $X_1$ and $y$ when the pair $(\eta', \eta)$ is picked. We then have

$$\Delta p' = e^{\lambda_i (w_{i-1}-\epsilon)\ln(1-\eta)} - e^{\lambda_i w_{i-1}\ln(1-\eta)} = \left[(1-\eta)^{\lambda_i}\right]^{w_{i-1}-\epsilon} - \left[(1-\eta)^{\lambda_i}\right]^{w_{i-1}} .$$

Since $\left[(1-\eta)^{\lambda_i}\right]^x$ is decreasing and convex for $x > 0$, we get $\Delta p' \geq \Delta p$, which completes the proof. □

We can now prove the following upper bound for $\xi_i(\{1, 2, \ldots, i\})$.

**Lemma 2.** *Let $S_e = \{1, 2, \ldots, i\}$. It is the case that*

$$\xi_i(S_e) \leq \left(\frac{w_i' + 1}{w_i' i + 1}\right)^{1/w_i'}$$

*where $w_i'$ is the ratio of $w_i$ to the average weight of payers $1, 2, \ldots, i-1$.*

*Proof.* From Lemma 1 we get that we can obtain an upper bound on $\xi_i(S_e)$ by substituting every $w_i$ for $i = 1, 2, \ldots, i-1$ with their average, which we denote $\bar{w}_{-i}$. This would be equivalent to having a weight $w_i' = w_i/\bar{w}_{-i}$ for player $i$ and weight 1 for all other $i-1$ players. Then using the definition of the weighted Shapley value from the start of Section 2, we get

$$\begin{aligned}\xi_i(S_e) &\leq \prod_{j=1}^{i-1} \frac{j w_i'}{j w_i' + 1} = \prod_{j=1}^{i-1}\left(1 - \frac{1}{j w_i' + 1}\right) \leq \prod_{j=1}^{i-1} e^{-1/(j w_i' + 1)} \\ &= \exp\left(-\sum_{j=1}^{i-1} \frac{1}{j w_i' + 1}\right) \leq \exp\left(-\int_1^i \frac{dx}{x w_i' + 1}\right) = \left(\frac{w_i' + 1}{w_i' i + 1}\right)^{1/w_i'} .\end{aligned}$$

This completes the proof. □

Combining Lemma 2 with Theorem 1, provides an expression that upper bounds the POS for a given weight vector. An example is the case with equal weights, where we get the tight $H_k = O(\log(k))$ bound shown in [2]. Another interesting case is that with weight $w_i = i$ for $i = 1, 2, \ldots, k$, for which we get that the POS is $O(\sqrt{k})$.

## 4 Atomic Selfish Routing

Let $\mathcal{C}$ be a set of nonnegative nondecreasing cost functions and suppose all resource cost functions of our atomic selfish routing game are picked from $\mathcal{C}$. We make the following assumptions for every $c \in \mathcal{C}$ and $w \geq 0$. It is the case that $(x+w)\cdot c(x+w) - x \cdot c(x)$ is a convex and nondecreasing function of $x$ (this is true, for example, for twice differentiable functions with nondecreasing first and second derivatives). Also, if $\bar{c}(x) = w \cdot c(x)$, then $\bar{c} \in \mathcal{C}$ (i.e., $\mathcal{C}$ is closed under scaling). Finally, if $\bar{c}(x) = c(w \cdot x)$, then $\bar{c} \in \mathcal{C}$ (i.e., $\mathcal{C}$ is closed under dilation).

### 4.1 Upper Bounding the Price of Anarchy

We define

$$\mathcal{A}(\mathcal{C}) = \Bigg\{(\lambda, \mu) : \mu < 1, \quad \text{for all} \quad x, x^* \geq 0 \quad \text{and for all} \quad c \in \mathcal{C}$$
$$\left(\lambda - \frac{1}{2}\right) x^* \cdot c(x^*) + \left(\mu + \frac{1}{2}\right) x \cdot c(x) - \frac{1}{2}(x + x^*) \cdot c(x + x^*) \geq 0\Bigg\}.$$

The following proposition gives an upper bound on the POA.

**Proposition 5.** *If $(\lambda, \mu) \in \mathcal{A}(\mathcal{C})$, then the POA of an atomic selfish routing game with Shapley cost shares and cost functions from $\mathcal{C}$ is at most $\lambda/(1-\mu)$.*

Using the above proposition we derive the upper bound $\zeta(\mathcal{C}) = \inf\{\lambda/(1-\mu) : (\lambda, \mu) \in \mathcal{A}(\mathcal{C})\}$. This bound is robust in the sense of [21], and thus applies also to more general equilibrium concepts, like coarse correlated equilibria. If $\mathcal{A}(\mathcal{C})$ is empty, then we define $\zeta(\mathcal{C}) = \infty$.

**Upper Bound for Polynomials.** Suppose $\mathcal{C}$ is the class of polynomials with nonnegative coefficients and maximum degree $d$. Also let $\chi_d$ be the number that satisfies $3\chi_d^{d+1} = 1 + (\chi_d + 1)^{d+1}$. We get the following theorem.

**Theorem 2.** *If $\mathcal{C}$ is the set of polynomials with nonnegative coefficients and maximum degree $d$, then the POA of an atomic selfish routing game with Shapley cost shares and cost functions in $\mathcal{C}$ is at most $\chi_d^{d+1} = (\Theta(d))^{d+1}$.*

For comparison, the worst-case POA with proportional sharing, in such games that happen to possess PNE, is the slightly smaller quantity $\Theta((d/\ln d)^{d+1})$.

### 4.2 Lower Bounding the Price of Anarchy

The upper bounds presented in Subsection 4.1 are tight. The analysis that proves this is similar to the one for weighted congestion games with proportional cost sharing [4].

**Lower Bound for Polynomials.** We now present the lower bound construction for the case when $\mathcal{C}$ is the set of polynomials with nonnegative coefficients and maximum degree $d$. Consider a game with the set of players being $S = \{1, 2, \ldots, k\}$ and the set of edges being $E = \{1, \ldots, k+1\}$. The weight of player $i \in S$ is $w_i = \chi_d^i$, the cost function of edge $k+1$ is $c_{k+1}(x) = x^d$, and the cost function of every other $e \in E$ is $c_e(x) = \chi_d^{(d+1)(k-e)} x^d$. Each player $i$ has to select between using edge $i$ and edge $i+1$. The fact that the outcome where every player $i$ picks edge $i$ is a PNE gives the following theorem.

**Theorem 3.** *The POA of an atomic selfish routing game with Shapley cost shares and cost functions that are polynomials with nonnegative coefficients and maximum degree $d$ can be arbitrarily close to $\chi_d^{d+1}$.*

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