Computing Equilibria in Multi-Player Games

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Abstract

We initiate the systematic study of algorithmic issues involved in finding equilibria (Nash and correlated) in games with a large number of players; such games, in order to be computationally meaningful, must be presented in some succinct, game-specific way. We develop a general framework for obtaining polynomial-time algorithms for optimizing over correlated equilibria in such settings, and show how it can be applied successfully to symmetric games (for which we actually find an exact polytopal characterization), graphical games, and congestion games, among others. We also present complexity results implying that such algorithms are not possible in certain other such games. Finally, we present a polynomial-time algorithm, based on quantifier elimination, for finding a Nash equilibrium in symmetric games when the number of strategies is relatively small.

1 Introduction

The Complexity of Equilibria A fundamental problem on the increasingly active interface of game theory and theoretical computer science is determining the computational complexity of computing equilibria. For example, the most popular equilibrium concept in noncooperative game theory is the *Nash equilibrium*—a "stable point" among n strategic players from which no player has a unilateral incentive to deviate. A polynomial-time algorithm for computing a Nash equilibrium is arguably the "holy grail" in this research area (see [32]), and much progress has been made on this [5, 10, 20, 26, 27, 36] and related problems [7, 8, 9, 18, 28] by the theoretical computer science community in the last few years.

Multi-player Games In this paper, we will study the complexity of computing equilibria in games with many players. While two-player games are the most classical [39] and well-studied type of game, and are possibly the most tractable from a complexity-theoretic perspective, we nevertheless believe that multi-player games demand immediate study. Indeed, much of the current research on game theory in theoretical computer science is motivated by large networks, such as the Internet, where games are obviously being played by a large number of players. While multi-player games have been extensively studied in the game theory literature (see [22] and the references therein), and their importance has long been recognized by the artificial intelligence community (see e.g. [37]), even less is known about computing equilibria in multi-player games than in the (still mysterious) special case of two-player games.

There is an immediate obstacle to discussing complexity results for general n-player games: massive input complexity. For example, to specify a general game in which n players each have to make a binary decision, $n2^n$ numbers are required—for each of the 2^n possible outcomes of the game, a payoff to each player. This exponential input complexity is worrisome in two respects. Most obviously, it threatens to render all positive algorithmic results moot—who cares about a polynomial-time algorithm, when the input size is already exponential in "natural" parameters, such as the number of players? Secondly, it illustrates a potential disconnect between our complexity theory of games and the games that we actually want to study—rare is the game that models an application of interest and yet lacks sufficient structure to be specified with a reasonable number of parameters.

Compact Representations The exponential input complexity of general multi-player games motivates an important research direction: the complexity of computing equilibria in multi-player games that admit a compact representation. Many recent papers in the theoretical computer science literature are in essence applying this philosophy to concrete applications, such as load-balancing (see [6, 12] and the references therein),

network routing (see [35]), facility location [38], and congestion games [11]. In this paper, we are aiming for a more systematic investigation—what properties of a compact representation permit polynomial-time algorithms for computing equilibria? To illustrate our results, we will focus on the following three broad classes of structured multi-player games.

Symmetric Games In a symmetric game, all players are identical and indistinguishable. They have the same strategy sets, their utility functions are the same function of their own strategy and the other players' actions, and this function is symmetric in the other players' actions. This function thus depends only on the *number* of players choosing each strategy, and on the player's strategy. Symmetric games have been widely studied since the dawn of game theory; together with zero-sum games, they form one of the most classical subclasses of games. For example, Nash [30] proved that every symmetric game must have a *symmetric* equilibrium—an equilibrium in which all players play the same strategy. Several of the papers in the first volume of *Contributions to the Theory of Games* [24] are devoted to symmetric games. Symmetric games' long tenure in the spotlight is due in large part to the famous examples they have provided: the Prisoner's Dilemma, Chicken, coordination games, and so on. More recently, symmetric games have played a central role in evolutionary game theory (see e.g. [40]). Despite this long history, little is known about the complexity of computing equilibria of symmetric games.

Multi-player symmetric games admit a compact representation. Specifically, a symmetric game can, by definition, be specified by giving the payoff of each strategy, given how many players choose each strategy. If there are n players who choose among k strategies, then there are $\binom{n+k-1}{k-1}$ distinct distributions of n players among k strategies (equivalently, ordered partitions of n into k parts), and the game can be summarized with only $k\binom{n+k-1}{k-1}$ numbers. This is always smaller than the nk^n numbers required for the standard representation, exponentially so if $k = O(n)$.

Graphical Games Graphical Games were first proposed by Kearns, Littman, and Singh [20]. (See also Koller and Milch [23] and the references therein for related concepts.) In a graphical game, the players are the vertices of a graph, and the payoff of each player only depends on its strategy and those of its neighbors. Algorithms that run in time polynomial in the obvious compact representation have recently been developed for computing Nash [20, 27] and correlated equilibria [19] for graphical games defined on trees.

Congestion Games Congestion games are an abstraction of network routing games and were first defined by Rosenthal [33, 34]. In a congestion game, there is a ground set of elements, and players choose a strategy from a prescribed collection of subsets of the ground set. The cost of an element is a function of the number players that select a strategy that contains it, but this cost is independent of the identities of these players. The cost (negative payoff) to a player is then the sum of the costs of the elements in its strategy.

Congestion games enjoy a flexibility useful for modeling diverse applications, as well as enough structure to allow non-trivial theoretical analyses, and for these reasons have been extensively studied in the last 10 years; see [35, §4.4] for a survey and [11] for recent results concerning pure-strategy Nash equilibria.

Correlated Equilibria While we also give algorithms for computing Nash equilibria, our widest-ranging theory concerns *correlated equilibria*. Correlated equilibria were first defined by Aumann [1], and we will describe them in detail in Section 2. For now, suffice it to say that every Nash equilibrium is a correlated equilibrium and that the set of all correlated equilibria of a game can be described by a system of linear inequalities whose size is polynomial in the length of the game's standard description. Unfortunately, the size of this system is generally exponential in that of the compact representations of all of the games mentioned above. Hence correlated equilibria, currently the only known tractable solution concept in game theory, long appeared beyond the power of polynomial-time computation for these fundamental classes of games.

Our Results For computing correlated equilibria in multi-player games with a compact representation, we prove the following.

- For symmetric games we explicitly describe the set of all correlated equilibria with a linear system that has size polynomial in the natural compact representation of the game. A correlated equilibrium— in fact, one that optimizes an arbitrary linear function, such as the expected sum of player payoffs— can thus be found efficiently.

- We present a general framework, based on dimensionality reduction of natural linear programming formulations, for optimizing over the correlated equilibria of a game in time polynomial in the size of a compact representation. In addition to the class above, this framework applies to certain congestion games and to graphical games defined on trees (or more generally, graphs of bounded treewidth).

- For two important classes of games not covered by our general framework—general congestion games and general graphical games—we prove that there is no algorithm that optimizes over the set of correlated equilibria in time polynomial in the size of the natural compact representation (assuming $P \neq NP$).

None of these results were previously known, with the exception of the tractability of optimizing over the correlated equilibria of a graphical game defined on a tree, which was first proved by Kakade et al. [19]. Here, we rederive this result from a much more general perspective, and also give the first complexity-theoretic justification of restricting the topology of graphical games.

Finally, we present a polynomial-time algorithm, based on the theory of real closed fields, for finding a Nash equilibrium in n-player, k-strategy symmetric games with $k = O(\log n / \log \log n)$.

2 Preliminaries

Games A *normal form game*, or simply a *game*, is a collection $S_1, \ldots, S_n$ of finite strategy sets and a collection $u_1, \ldots, u_n$ of real-valued utility functions, each defined on $S_1 \times \cdots \times S_n$. We identify a strategy set S_i and utility function u_i with *player i*. A element s of $S_1 \times \cdots \times S_n$ is called a *strategy profile*. The set of all strategy profiles is the *state space* of the game. For a strategy profile s, s_i is the strategy of player i, s_{-i} denotes the $(n-1)$-vector of strategies of players other than i, and the value $u_i(s)$ will be called the *payoff to player i*.

Symmetric, Congestion, and Graphical Games A game is *symmetric* if $S_1 = \cdots = S_n$, $u_i(s)$ depends only on s_i and the other players' strategies (but not on i), and $u_i(s)$ is a symmetric function of s_{-i}. In other words, the payoff to a player depends only on its strategy and on the number of players choosing each of the different strategies. A symmetric game can be specified by giving, for each ordered partition of the number of players, the payoffs of each player—$k\binom{n+k-1}{k-1}$ numbers, where k is the number of strategies. This expression is $\Theta(n^{k-1})$ when $k = O(1)$, polynomial in n^k when $k = O(\log n / \log \log n)$, and is super-polynomially smaller than k^n unless $k = \Omega(n^{1+\epsilon})$ for some $\epsilon > 0$.

A *graphical game* is compactly described by an undirected graph $G = (V, E)$, where each vertex is a player with an arbitrary strategy set. The payoffs to a player are an arbitrary function of its strategy and the strategies of the adjacent players. The number of parameters needed to specify the payoffs of a graphical game is therefore exponential in the maximum degree but polynomial in the number of players.

Finally, in a *congestion game* there is a ground set E of elements, k collections $S_1, \ldots, S_k$ of subsets of E, and, for $i = 1, \ldots, k$, a positive integer number n_i of players with strategy set S_i. Each element $e \in E$ has a real-valued cost function c_e, defined on the positive integers, which describes its cost given the number of players that select strategies that include it. The cost (negative payoff) to a player is the sum of the costs of the elements in its strategy. In a congestion game with $n = \sum_i n_i$ players and $m = |E|$ ground elements, payoffs can thus be completely summarized with only nm real numbers.

Nash Equilibria Let $G = (\{S_i\}, \{u_i\})$ be an n-player game and $p_1, \ldots, p_n$ a collection of probability distributions on the strategy sets. Distributions $p_1, \ldots, p_n$ are a *Nash equilibrium* if, for each player i, picking a strategy from S_i according to the distribution p_i maximizes i's expected payoff, assuming that each player $j \neq i$ picks a strategy according to the distribution p_j. Nash [29] showed that every game admits a Nash equilibrium. In a subsequent paper [30], he showed that every symmetric game admits a *symmetric* Nash equilibrium, meaning a Nash equilibrium in which $p_1 = \cdots = p_n$. (A symmetric game can also have other Nash equilibria.)

Correlated Equilibria Let $G = (\{S_i\}, \{u_i\})$ be an n-person game. Let q be a probability distribution on $S_1 \times \cdots \times S_n$. Distribution q is a *correlated equilibrium* if for each player i and each pair ℓ, ℓ' of strategies in S_i,

$$(2.1) \qquad \sum_{s \,:\, s_i = \ell} q(s) u_i(s) \geq \sum_{s \,:\, s_i = \ell} q(s) u_i(s'),$$

where s' is obtained from s by reassigning i's strategy to be ℓ'. One interpretation of a correlated equilibrium is as follows. A trusted authority picks a strategy profile s at random according to q, and "recommends" strategy s_i to each player i. Each player i is assumed to know only its recommended strategy, and not those for other players. Player i then compares the conditional expected payoffs of its strategies, assuming that the other players follow their recommendations (conditioning on the strategy recommended to i). Inequality (2.1) states that this conditional expectation should be maximized by the recommended strategy. If this holds for all players, then no player has a unilateral incentive to deviate from the trusted authority's recommendation. The inequalities (2.1) evidently describe the set of all correlated equilibria, and this linear system has size polynomial in the normal form description of the game. Since every Nash equilibrium, viewed as a product distribu-

tion, is a correlated equilibrium, Nash's theorem [29] implies that this system is always feasible. A popular concrete example of a correlated equilibrium is a traffic signal that recommends "red" (stop) or "green" (go) to drivers (see e.g. [31]). For more applications and interpretations of correlated equilibria, see [2, 13, 14].

3 Explicit Descriptions of Correlated Equilibria

In this section, our ambition will be to explicitly describe the correlated equilibria of a game that is represented compactly. Put differently, we will aim for a characterization that is equally powerful and complete as the classical one, while at the same time demanding that the linear system be just as economical as the game's compact description. We will accomplish this goal for the class of symmetric games. As a consequence, every linear function can be efficiently optimized over the set of correlated equilibria—and in particular, one can be found.

For simplicity, we will work primarily in the $k = 2$ case; this suffices to illustrate most of our proof techniques. In Subsection 3.2 we make a few comments about what is required to extend the analysis to arbitrary k-strategy symmetric games.

3.1 Symmetric Games with Two Strategies
Let $G = (S = \{1, 2\}, u_1, \ldots, u_n)$ be an n-player, 2-strategy symmetric game. An explicitly represented correlated equilibrium of G must specify a probability $q(s)$ for each of the 2^n strategy profiles s. The variables in our compact representation of the correlated equilibria of G will be of the form $p_i(j)$ (*basic* variables) and $p(j)$ (*auxiliary* variables) for $i \in \{1, 2, \ldots, n\}$ and $j \in \{0, 1, \ldots, n\}$. We intend the basic variable $p_i(j)$ to represent the aggregate probability assigned to the strategy profiles $S_i(j)$ in which exactly j players, including player i, choose strategy 1. Similarly, $p(j)$ represents the total probability of the strategy profiles $S(j)$ in which exactly j players choose strategy 1. We will sometimes refer to subsets of S^n of the form $S_i(j)$ and $S(j)$ as *basic* and *auxiliary* sets, respectively. The constraints are as follows:

$$(3.2) \qquad \sum_{j=0}^{n} p_i(j)u_i(j, 1) \geq \sum_{j=0}^{n} p_i(j)u_i(j-1, 2)$$

for all $i \in \{1, 2, \ldots, n\}$;

$$(3.3) \qquad \sum_{j=0}^{n} [p(j) - p_i(j)] \, u_i(j, 2) \geq$$
$$\sum_{j=0}^{n} [p(j) - p_i(j)] \, u_i(j+1, 1)$$

for all $i \in \{1, 2, \ldots, n\}$;

$$(3.4) \qquad \sum_{j=0}^{n} p(j) = 1;$$

$$(3.5) \qquad \sum_{i=1}^{n} p_i(j) = j \cdot p(j)$$

for all $j \in \{0, 1, \ldots, n\}$; and

$$(3.6) \qquad 0 \leq p_i(j) \leq p(j) \leq 1$$

for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{0, 1, \ldots, n\}$. In the above constraints, $u_i(j, \ell)$ denotes the payoff to player i in a strategy profile in which player i chooses strategy ℓ and a total of j players choose strategy 1. This payoff is well defined—i.e., independent of the strategy profile meeting the above criteria—by the definition of a symmetric game.

With respect to an n-player, 2-strategy game $G = (\{1, 2\}, u_1, \ldots, u_n)$, we will call the equations and inequalities (3.2)–(3.6) the *basic linear system* of G. The size of this system is polynomial in that of the compact representation of G. We will sometimes refer to equations (3.5) as the *covering equations*. Since constraints (3.2)–(3.3) are effectively aggregated versions of the correlated equilibrium constraints (2.1), every correlated equilibrium of an n-player, 2-strategy symmetric game G (defined on all of S^n) induces a solution to G's basic linear system via the intended aggregations of probability.

The interesting direction is the converse. Let p be a solution to the basic linear system of a 2-strategy symmetric game $G = (S, u_1, \ldots, u_n)$. We say that p *extends to* S^n if there is a function $q : S^n \to \mathcal{R}^+$ with $\sum_{s \in S_i(j)} q(s) = p_i(j)$ and $\sum_{s \in S(j)} q(s) = p(j)$ for all i and j. It is easy to check that if p extends to S^n, then the extension is a correlated equilibrium of G. It is not at all obvious, however, that such an extension must exist; this is our main result in this section.

Theorem 3.1 *Let G be a 2-strategy symmetric game. Then every solution to G's basic linear system can be extended to a correlated equilibrium of G.*

We will prove Theorem 3.1 in two parts. The glue that holds the two parts together is the notion of a uniform solution to a game's basic linear system.

Definition 3.2 *Let $G = (S = \{1, 2\}, u_1, \ldots, u_n)$ be a 2-strategy symmetric game. Let $\{S_i(j)\}$ denote the basic sets and $\{S(j)\}$ the auxiliary sets of G.*

(a) A *j-basic cover* is a function $x :$ $\{S_1(j), \ldots, S_n(j)\} \to \mathcal{R}^+$ with

$$\sum_{S_i(j)\,:\,s \in S_i(j)} x_i(j) \geq j,$$

for all $s \in S(j)$, where we have written $x_i(j)$ for $x(S_i(j))$.

(b) A solution p to G's basic linear system is *uniform* if for all $j \in \{0, 1, \ldots, n\}$,

$$\sum_{i=1}^{n} p_i(j) x_i(j) \geq \sum_{i=1}^{n} p_i(j) = j \cdot p(j)$$

for every j-basic cover x. (The equality is the jth covering equation (3.5).)

Definition 3.2 is justified by the following two lemmas, which immediately imply Theorem 3.1.

Lemma 3.3 *Let G be a 2-strategy symmetric game. Then every uniform solution to G's basic linear system can be extended to a correlated equilibrium of G.*

Lemma 3.4 *Let G be a 2-strategy symmetric game. Then every solution to G's basic linear system is uniform.*

Lemma 3.3 is essentially a consequence of strong linear programming duality, and we omit its proof. Before proving Lemma 3.4, we establish a preliminary lemma. In its statement, we will use the notation $[x]^+$ to denote $\max\{0, x\}$ for a real number x.

Lemma 3.5 *Let $G = (S = \{1, 2\}, u_1, \ldots, u_n)$ be a 2-strategy symmetric game, and p a solution to G's basic linear system.*

(a) *If $j \in \{0, 1, \ldots, n\}$, $\mathcal{C}$ is a collection of $\ell \leq j$ distinct j-basic sets, and $\mathcal{C}'$ is a collection of $r \leq n - \ell$ distinct j-basic sets not in $\mathcal{C}$, then some element in $\cap_{\mathcal{C}} S_i(j)$ lies in only $[r + j - n]^+$ sets of $\mathcal{C}'$.*

(b) *If $\mathcal{C}$ is a collection of r distinct j-basic sets, then $\sum_{S_i(j) \in \mathcal{C}} p_i(j) \geq [r + j - n]^+ \cdot p(j)$.*

Proof. Part (b) follows immediately from constraints (3.5) and (3.6). To prove part (a), relabel the players so that $\mathcal{C} = \{S_1(j), \ldots, S_\ell(j)\}$ and $\mathcal{C}' = \{S_{n-r+1}(j), \ldots, S_n(j)\}$. Let s be the strategy profile in which the first j players choose strategy 1 and the last $n - j$ players choose strategy 2. The profile s then lies in all sets of $\mathcal{C}$ but only in $[r + j - n]^+$ sets of $\mathcal{C}'$.

Proof of Lemma 3.4: Let $G = (S = \{1, 2\}, u_1, \ldots, u_n)$ by a symmetric game and p a solution to G's basic linear system. We need to show that p is uniform in the sense of Definition 3.2.

For $j \in \{0, 1, \ldots, n\}$ and a function $x :$ $\{S_1(j), \ldots, S_n(j)\} \to \mathcal{R}$, define the function C_j by $C_j(x) = \sum_{i=1}^{n} p_i(j) x_i(j)$ (as usual, $x_i(j)$ is shorthand for $x(S_i(j))$). For every $j \in \{0, 1, \ldots, n\}$, setting $x_i(j) = 1$ for all $i \in \{1, 2, \ldots, n\}$ yields a j-basic cover with $C_j(x) = \sum_{i=1}^{n} p_i(j)$. We call this cover the *uniform j-cover*. Proving that p is uniform is tantamount to showing that, for each $j \in \{0, 1, \ldots, n\}$, the uniform j-cover minimizes $C_j(x)$ over all j-basic covers x.

Toward this end, let x be a non-uniform j-basic cover for some $j \in \{0, 1, \ldots, n\}$. Let U denote the indices of the sets underused by x ($x_i(j) < 1$) and O the indices of the sets overused by x ($x_i(j) > 1$). We can assume that U is non-empty (else clearly $C_j(x)$ is no smaller than in the uniform solution). Since x is a feasible j-basic cover, O is then non-empty as well.

We first claim that the number $|U|$ of underused sets is at most $j - 1$. To see why, note that every j basic sets of the form $S_i(j)$ have exactly one point in common—the strategy profile s in which $s_i = 1$ if and only if $S_i(j)$ is one of the sets—and this point is contained in no other basic set. Thus if s is in the common intersection of j sets $S_i(j)$ with $i \in U$, then $\sum_{i\,:\,s \in S_i(j)} x_i(j) < j$, and x is not a j-basic cover.

Without loss of generality, $x_1(j) \geq x_2(j) \geq \cdots \geq x_n(j)$. Let $O = \{1, 2, \ldots, m\}$ and $U = \{t, \ldots, n\}$ for $1 \leq m < t \leq n$. The contribution of underused sets to the sum $\sum_{i\,:\,s \in S_i(j)} x_i(j)$ for elements s in their (non-empty) common intersection is $c \equiv \sum_{i=t}^{n} (1 - x_i(j))$ less than in the uniform solution. Since x is a j-basic cover, the extra contribution from the overused sets, relative to the uniform solution, must be at least c for all such elements.

Let $z_m(j) = x_m(j) - 1$ and $z_r(j) = x_r(j) - x_{r-1}(j)$ for $r \in \{1, 2, \ldots, m - 1\}$. The z-variables should be regarded as a decomposition of the x-values that permits the application of Lemma 3.5. We can express the previous inequality in terms of the z-variables as follows:

$$\sum_{(r,i)\,:\,s \in S_i(j), i \leq r \leq m} z_r(j) = \sum_{i\,:\,s \in S_i(j), i \leq m} \sum_{r=i}^{m} z_r(j)$$

$$= \sum_{i\,:\,s \in S_i(j), i \leq m} [x_i(j) - 1]$$

(3.7) $$\geq c.$$

By Lemma 3.5(a), for every $r \in \{1, 2, \ldots, m\}$ there

is an element s in all underused sets for which

$$(3.8) \qquad \sum_{i \le r \,:\, s \in S_i(j)} z_r(j) \le [r + j - n]^+ \cdot z_r(j).$$

The proof also shows that, since the set $\{i \,:\, i \le r\}$ is increasing in r, there is a single strategy profile s in all underused sets for which (3.8) holds simultaneously for all $r \in \{1, 2, \ldots, m\}$. Summing over all r and combining with (3.7), we find that

$$(3.9) \qquad \sum_{r=1}^{m} z_r(j) \cdot [r + j - n]^+ \ge c.$$

We can now complete the proof. Let u be the uniform solution. Write

$$
\begin{aligned}
C_j(x) \;=\;& C_j(u) + \sum_{i=1}^{m} [x_i - 1] p_i(j) \\
(3.10) &- \sum_{i=t}^{n} [1 - x_i] p_i(j).
\end{aligned}
$$

Since $p_i(j) \le p(j)$ for all i, the last term is at most $p(j) \sum_{i=t}^{n} (1 - x_i) = c \cdot p(j)$. To lower bound the second term on the right-hand side of (3.10), use Lemma 3.5(b) and (3.9) to write

$$
\begin{aligned}
\sum_{i=1}^{m} [x_i - 1] p_i(j) \;=\;& \sum_{r=1}^{m} z_r(j) \sum_{i=1}^{r} p_i(j) \\
\ge\;& \sum_{r=1}^{m} z_r(j) [r + j - n]^+ p(j) \\
(3.11) \qquad \ge\;& c \cdot p(j).
\end{aligned}
$$

The inequality $C_j(x) \ge C_j(u)$ now follows from (3.10) and (3.11), and the proof is complete. ∎

3.2 Symmetric Games with Many Strategies

It is straightforward to extend the definition of a basic linear system to k-strategy symmetric games. There are variables $p_i(j, \ell)$ and $p(j)$, where j is now an ordered partition of n into k non-negative integers, and $\ell \in S = \{1, 2, \ldots, k\}$, with $\sum_\ell p_i(j, \ell) = p(j)$ for every player i. Analogs of constraints (3.2)–(3.6) are straightforward to describe. As in the previous subsection, we have the following theorem.

Theorem 3.6 *Let G be a symmetric game. Then p is a solution to G's basic linear system if and only if it can be extended to a correlated equilibrium of G.*

The proof of Theorem 3.6 proceeds as for the 2-strategy case, hinging on an extension of the notion of uniformity (Definition 3.2) to k-strategy symmetric games. For a fixed ordered partition j, the j-basic sets are now indexed by both a player i and a strategy ℓ. This causes no difficulty for extending Lemma 3.3, but extending Lemma 3.4 to these richer collections of basic sets requires more sophisticated combinatorial arguments. We omit further details.

Remark 3.7 Nowhere in the proofs above did we use the fact that the utility functions are equal. Thus our results apply more generally to symmetric-like games where different players have different (but symmetric w.r.t. other players) utility functions.

4 Finding Correlated Equilibria of General Compact Games

In this section, we continue to devise algorithms for finding and optimizing over correlated equilibria that run in time polynomial in the size of a game's compact representation. We will, however, relax our previous ambition of explicitly describing the set of correlated equilibria. As our reward, we will be able to work in a very general setting, with essentially arbitrary compact representations. In Subsection 4.1, we will present a general result that shows that the tractability of optimizing over the correlated equilibria of a game in time polynomial in a compact representation is controlled by an optimization problem related to the representation. In Subsection 4.2, we will see that for all of the classes of games studied in this paper, their natural compact representations give rise to combinatorial optimization problems with easily determined computational complexity. As a result, we will be able to derive numerous positive and negative results with minimal effort.

4.1 A General Framework

At the highest level the algorithmic approach of this section will be similar to that of the previous one. We will formulate a linear program where the number of variables is comparable to the size of the given compact representation, and will then hope that solutions to the linear program can be extended to correlated equilibria defined explicitly on the set of all strategy profiles. At the bare minimum, to implement this idea we will require equilibrium constraints analogous to (3.2)–(3.3). In turn, the essentially minimal assumptions needed to define such constraints are given in the next definition.

Definition 4.1 Let $G = (S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a game in normal form. For $i = 1, 2, \ldots, n$, let $P_i = \{P_i^1, \ldots, P_i^{m_i}\}$ be a partition of S_{-i} into m_i classes, where S_{-i} denotes the $(n-1)$-fold product of strategy sets other than S_i.

(a) For a player i, two strategy profiles s and s' are *i-equivalent* if $s_i = s'_i$, and both s_{-i} and s'_{-i} belong to the same class of the partition P_i.

(b) The set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of partitions is a *compact representation of G* if $u_i(s) = u_i(s')$ whenever s and s' are i-equivalent.

The motivation of Definition 4.1 is that it permits a reasonable definition of the correlated equilibrium constraints. To see this, let $\mathcal{P} = \{P_i^j\}$ be a compact representation for a game G. For a player i, a class P_i^j in player i's partition, and a strategy $\ell \in S_i$, let $u_i(j, \ell)$ denote the payoff to player i in a strategy profile s with $s_i = \ell$ and $s_{-i} \in P_i^j$; this is well defined by Definition 4.1. We can then write the correlated equilibrium constraints as $\sum_{j=1}^{m_i} p_i(j, \ell) u_i(j, \ell) \geq \sum_{j=1}^{m_i} p_i(j, \ell) u_i(j, \ell')$ for all i and all $\ell, \ell' \in S_i$, where $p_i(j, \ell)$ is the aggregate probability assigned to strategy profiles s with $s_i = \ell$ and $s_{-i} \in P_i^j$.

In Definition 4.1, there is one partition of (most of) the state space for each player. In some applications, there will be an obvious partition of the state space that cuts across player types. For example, the state space of an n-player, k-strategy symmetric game admits an obvious global partition, with one class of the partition for each ordered partition of n into k parts. Such a global partition easily defines a compact representation in the sense of Definition 4.1 that has comparable size. In the symmetric game example, the partition corresponding to player i has one class for each ordered partition of $n - 1$ (the other players) into k parts (the distribution of their strategies).

We need one further definition. As our main result in this section, we will show that the tractability of optimizing over the correlated equilibria of a compactly represented game is controlled by the computational complexity of a related optimization problem. As we will see in Subsection 4.2, this general reduction will have immediate consequences for symmetric, congestion, and graphical games. We next define the relevant optimization problem corresponding to a compact representation.

Definition 4.2 Let $\mathcal{P} = \{P_i^j\}$ be a compact representation of a game G. The *separation problem for $\mathcal{P}$* is the following algorithmic problem:

Given rational numbers $y_i(j, \ell)$ for all i, j, and $\ell \in S_i$, is there a strategy profile s with $\sum_{(i,j,\ell)\,:\,s_i = \ell, s_{-i} \in P_i^j} y_i(j, \ell) < 0$?

We will see several concrete examples of such separation problems in Subsection 4.2. We now conclude this subsection by proving that a tractable separation problem is all that is required for the efficient computation of a correlated equilibrium. We will state this result in terms of the *size* of a compact representation, which is defined in the obvious way (total number of classes in its partitions, plus the number of bits needed to describe player payoffs).

Theorem 4.3 *Let $\mathcal{P}$ be a compact representation of a game. If the separation problem for $\mathcal{P}$ can be solved in polynomial time, then a correlated equilibrium of G can be computed in time polynomial in the size of $\mathcal{P}$.*

Proof. Let $G = (S_1, \ldots, S_n, u, \ldots, u_n)$ be a game and $\mathcal{P}$ a compact representation such that the separation problem for $\mathcal{P}$ is solvable in polynomial time. Define $u_i(j, \ell)$ as in the discussion following Definition 4.1, and consider the following system of equations and inequalities:

$$(4.12) \qquad \sum_{j=1}^{m_i} p_i(j, \ell) u_i(j, \ell) \geq \sum_{j=1}^{m_i} p_i(j, \ell) u_i(j, \ell')$$

for all $i, \ell, \ell' \in S_i$;

$$(4.13) \qquad \sum_{j=1}^{m_i} \sum_{\ell \in S_i} p_i(j, \ell) = 1$$

for all i; and

$$(4.14) \qquad p_i(j, \ell) \geq 0$$

for all $i, j \in \{1, 2, \ldots, m_i\}, \ell \in S_i$. Every correlated equilibrium naturally induces a feasible solution to this linear system. In contrast to Theorem 3.1, the converse need not hold unless the system is augmented by additional inequalities. We explore this idea next.

By Farkas's Lemma, there is a matrix A, with columns indexed by the exponentially many strategy profiles, so that a solution p to (4.12)–(4.14) can be extended to a correlated equilibrium of G if and only if for every y with $y^T A \geq 0$, $y^T p \geq 0$. Note that the vector y is indexed by the variables in (4.12)–(4.14).

This observation suggests extra inequalities to add to (4.12)–(4.14): for every vector y with $y^T A \geq 0$, include the inequality $y^T p \geq 0$. Every such inequality is *valid* in the sense that every correlated equilibrium of G induces a solution to (4.12)–(4.14) that also satisfies this extra inequality. Naively, there are infinitely many such extra inequalities to add. Fortunately, we need only include those inequalities that can arise as an optimal solution to the following problem:

(*) Given p satisfying (4.12)–(4.14), minimize $y^T p$ subject to $y^T A \geq 0$.

Since problem (*) is a linear program, the minimum is always attained by one of the finitely many basic solutions [4]. Moreover, in all such basic solutions, the vector y can be described with a number of bits polynomial in $\mathcal{P}$ [17, §6.2].

We have therefore defined a linear system, which we will call the *full linear system* for $\mathcal{P}$, so that p is a solution to the full linear system if and only if p can be extended to a correlated equilibrium of G. While this full linear system has many inequalities, each inequality has size polynomial in $\mathcal{P}$. We can therefore efficiently compute a solution to the full linear system via the ellipsoid algorithm [17, 21], provided we can define a polynomial-time *separation oracle*—an algorithm that takes as input a candidate solution and, if the solution is not feasible, produces a violated constraint.

Such a separation oracle is tantamount to a polynomial-time algorithm for problem (*), a linear program with exponentially many constraints (indexed by strategy profiles). We can solve problem (*) with a second application of the ellipsoid method. Here, the separation oracle required is precisely the separation problem for $\mathcal{P}$ (Definition 4.2) which, by assumption, admits a polynomial-time algorithm. The proof is therefore complete.

More generally, the proof of Theorem 4.3 shows that every linear function can be efficiently optimized over the set of correlated equilibria of such a game.

4.2 Applications We now demonstrate the power of Theorem 4.3. We begin by revisiting symmetric games, and then proceed to congestion and graphical games.

Symmetric Games and Extensions We begin by reconsidering symmetric games. This will illustrate the definitions and results of Subsection 4.1 in a familiar setting. As we have noted, n-player, k-strategy symmetric games admit a natural compact representation $\mathcal{P} = \{P_i^j\}$ in the sense of Definition 4.1, where the classes of $\mathcal{P}$ are indexed by a player i and an ordered partition j of $n-1$ into k parts corresponding to a distribution of the other $n-1$ players among the k available strategies. The separation problem for $\mathcal{P}$ is then: given rational numbers $y_i(j, \ell)$ for each player i, each ordered partition j of $n-1$ into k parts, and each choice ℓ for player i's strategy, is there a strategy profile s with $\sum_{(i,j,\ell)\,:\,s_i=\ell,s_{-i}\in P_i^j} y_i(j, \ell) < 0$? This problem can be solved in polynomial time, for example by a straightforward application of min-cost flow, and hence Theorem 4.3 implies the following.

Corollary 4.4 *A correlated equilibrium of a symmetric game can be found in time polynomial in its natural compact representation.*

While Corollary 4.4 is weaker than Theorems 3.1 and 3.6 in that it does not give an explicit description of the set of correlated equilibria, we derived it with considerably less work. As in Remark 3.7, Corollary 4.4 also holds when different players have different (symmetric) utility functions.

Graphical Games For a graphical game, its natural compact representation $\mathcal{P} = \{P_i^j\}$ has a class P_i^j for each player i and for each assignment j of strategies to the players that are neighbors of i. The separation problem for this representation is then the following: given rational numbers $y_i(j, \ell)$ for each player i, each set j of strategy choices of i's neighbors, and each choice ℓ for player i's strategy, is there a strategy profile s with $\sum_{(i,j,\ell)\,:\,s_i=\ell,s_{-i}\in P_i^j} y_i(j, \ell) < 0$? For graphical games defined on trees, this problem can be solved by dynamic programming.

Corollary 4.5 *A correlated equilibrium of a graphical game with a tree topology can be found in time polynomial in its natural compact representation.*

As we noted in the introduction, Corollary 4.5 was first proved by Kakade et al. [19], using tools from probabilistic inference. Corollary 4.5 is also easily generalized, for example to graphs of bounded treewidth.

For general topologies, however, the story is different. First, a reduction from Exact Cover By 3-Sets [16, SP2] shows the following.

Proposition 4.6 *The separation problem for the natural compact representation of a graphical game is NP-complete, even in bipartite graphs.*

In fact, something much stronger is true. A similar reduction, using the version of Exact Cover By 3-Sets where each element is contained in only a constant number of sets (see [16, SP2]), shows the following.

Proposition 4.7 *Assuming $P \neq NP$, there is no polynomial-time algorithm for computing a correlated equilibrium of a compactly represented graphical game that maximizes the expected sum of player payoffs.*

Proposition 4.7 dispels any lingering concern that we might have taken the wrong proof approach in our attempt to characterize the correlated equilibria of a graphical game: there is *no* linear system that characterizes the correlated equilibria of a general graphical game and can be optimized over in time polynomial in the game's compact representation (assuming

$P \neq NP$). Thus no small explicit description is possible (cf., Theorem 3.1), nor is there any description amenable to the ellipsoid algorithm (cf., Theorem 4.3).

Congestion Games Recall that a congestion game is specified by a ground set E, strategy sets $S_1, \ldots, S_k$, quantities $n_1, \ldots, n_k$ of players, and cost functions $\{c_e\}_{e \in E}$ defined on $\{1, 2, \ldots, \sum_i n_i\}$. Congestion games have the most economical description of all of the games studied in this paper, with nm numbers sufficing to describe all of the payoffs, where n and m are the number of players and elements, respectively. Perhaps because of this very small description, congestion games are in some sense also the least tractable class of games studied in this paper: analogously to Proposition 4.7, a reduction from EXACT COVER BY 3-SETS shows the following.

Proposition 4.8 *Assuming $P \neq NP$, there is no polynomial-time algorithm for computing a correlated equilibrium of a compactly represented congestion game that maximizes the expected sum of player payoffs.*

Proposition 4.8 holds even for congestion games with one player type ($k = 1$). Some positive results for efficiently optimizing over the correlated equilibria of a congestion game can be salvaged if somewhat larger representations are used; we defer a detailed discussion of this point to the full version of this paper.

5 Nash Equilibria of Symmetric Games

Finally, we give an algorithm for computing a symmetric Nash equilibrium in symmetric games.

Theorem 5.1 *The problem of computing a symmetric Nash equilibrium in a symmetric game with n players and k strategies can be solved to arbitrary precision in time polynomial in n^k, the number of bits required to describe the utility functions, and the number of bits of precision desired.*

Theorem 5.1 is a reduction from the so-called first-order theory of the reals. A different application of this idea to games was developed independently by Lipton and Markakis [25]. Due to space constraints, we omit further details. Since the compact representation of a symmetric game has size $\Omega(poly(n^k))$ when $k = O(\log n / \log \log n)$, we have the following corollary of Theorem 5.1.

Corollary 5.2 *The problem of computing a Nash equilibrium of a compactly represented n-player k-strategy symmetric game with $k = O(\log n / \log \log n)$ is in P.*

Theorem 5.1 and Corollary 5.2 can be extended to certain types of "partially symmetric" games (first considered by Nash in [30]), such as games with a constant number of player types and full symmetry among players of the same type.

Corollary 5.2 stands in contrast to the state of the art for general games, where no polynomial-time algorithm for computing a Nash equilibrium is known, even when all players have only two strategies.

We unfortunately have no progress to offer when n is small relative to k. We note, however, that finding an algorithm for computing a Nash equilibrium of a symmetric game in this case could be difficult. In particular, it has long been known that for games with a constant number of players, there is a polynomial-time reduction from general games to symmetric games [3, 15], and hence symmetry affords no computational advantage in this case.

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