SELFISH ROUTING

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A central and well-studied problem arising in the management of a large network is that of routing traffic to achieve the best possible network performance. In many networks, it is difficult or even impossible to impose optimal routing strategies on network traffic, leaving network users free to act according to their own interests. In general, the result of local optimization by many selfish network users with conflicting interests does not possess any type of global optimality; hence, this lack of regulation carries the cost of decreased network performance.

We study the degradation in network performance due to selfish, uncoordinated behavior by network users. Our contributions are twofold: we quantify the worst-possible loss in network performance arising from noncooperative behavior; and we design and analyze algorithms for building and managing networks so that selfish behavior leads to a socially desirable outcome.

To quantify the loss in network performance caused by selfish behavior, we investigate the following question: what is the worst-case ratio between the social cost of an uncoordinated outcome and the social cost of the best coordinated outcome? We provide an exact solution to this problem in a variety of traffic models, thereby identifying types of networks for which the cost of routing selfishly is mild.

The inefficiency inherent in an uncoordinated outcome and the inability to centrally implement a globally optimal solution motivate our second question: how can we ensure that the loss in network performance due to selfish behavior will be tolerable? We explore two algorithmic approaches to coping with the selfishness of network users. First, we consider the problem of designing networks that exhibit good performance when used selfishly. Second, we study how to route a small fraction of the traffic centrally to induce "good" (albeit selfish) behavior from the rest of the network users. We give both efficient algorithms with provable performance guarantees and hardness of approximation results for these two problems.

Biographical Sketch

Tim Roughgarden was born on July 20th, 1975. He received a B.S. in Applied Mathematics and Environmental Science from Stanford University in June 1997, an M.S. in Computer Science from Stanford University in June 1998, and expects to receive an M.S. in Mathematics and a Ph.D. in Computer Science from Cornell University in May 2002.

Acknowledgements

I have been very lucky to learn from a number of outstanding advisors at Cornell during my graduate studies. First and foremost, I am deeply indebted to my Ph.D. advisor, Éva Tardos; her insights (technical and otherwise) over the years have been invaluable to me. I also thank Jon Kleinberg, both for convincing me to come to Cornell for graduate school, and for his advice on talks, tennis rackets, and everything in between. I thank David Shmoys for many enjoyable discussions; like Éva and Jon, David has significantly influenced the way I think about research. I thank Lou Billera for serving on my committee and for pointers to relevant work in the game theory community.

I have also been fortunate to experience other exciting research environments over the past few years. The genesis of this thesis occurred during the 1999-2000 school year, when I was visiting U.C. Berkeley. I will always remember the Theory Lunch (that I was doubtlessly attending only for the free food) at which Leonard Schulman described Braess's Paradox and Christos Papadimitriou proposed comparing the social welfare of selfish and optimal solutions. Leonard and Christos have my eternal thanks for asking the questions that inspired my dissertation work. I am also grateful to the researchers at IBM Watson and IBM Almaden, and especially to David Williamson, for two great summers in the research industry.

Before coming to Cornell, I benefited from a number of mentors at Stanford. I am especially grateful to Serge Plotkin, who introduced me to the field of algorithms, got me excited about network flow, and provided support for my masters degree; to Steve Schneider, who advised my undergraduate thesis and showed me how much fun research can be; to Eric Roberts, for his contagious enthusiasm about computer science and inspiring work ethic; and to Gunnar Carlsson, who first introduced me to mathematical research. Going back even further in time, I am indebted to my parents for encouraging a love of learning at an early age.

Finally, of all the friends who have helped me out through the years, I'll single out only Eve Donnelly for special thanks; she's had to put up with me the most.

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Table of Contents

List of Tables

List of Figures

Part I

Overview and Preliminaries

Chapter 1

Introduction

1.1 Selfish Routing

What route should you take to work tomorrow? All else being equal, most of us would probably opt for the one that allows us to wake up at the least barbaric time—that is, most of us would prefer the shortest route available. As any morning commuter knows, the length of time required to travel along a given route depends crucially on the amount of traffic congestion—on the number of *other* commuters who choose interfering routes. In selecting a path to travel from home to work, do you take into account the additional congestion that you cause other commuters to experience? Not likely. Almost certainly you choose your route *selfishly*, aiming to get to work as quickly as possible without considering the adverse effects your choice creates for others. Naturally, you also expect your fellow commuters to behave in a similarly egocentric fashion. But what if all of you cooperated and coordinated routes? Is it possible to limit the interference between routes, thereby improving the average (or the maximum) commute time? If so, by how much?

In this thesis, we study the loss of social welfare due to selfish, uncoordinated behavior in networks. Our contributions are twofold: we quantify the worst-possible loss of social welfare arising from noncooperative behavior in a variety of traffic models; and we design and analyze algorithms for building and managing networks so that selfish behavior leads to a socially desirable outcome. Our results concern more than just the road networks described in the previous paragraph; we will see that they also have consequences for high-speed communication networks (with networks users seeking to minimize the end-to-end delay experienced by their traffic).

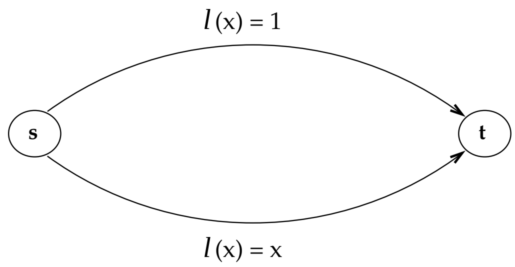


Figure 1.1: Pigou's example. A latency function $\ell(x)$ describes the delay experienced by drivers on a road as a function of the fraction of overall traffic using that road.

1.2 Two Motivating Examples

To motivate the questions investigated in this thesis (which we will describe in Section 1.3), let us informally explore two important examples. The first is essentially due to Pigou in his 1920 treatise [148, P.194] (see also Knight [99]), and the second is a famous "paradox" discovered by Braess in 1968 [28].

1.2.1 Pigou's Example

Posit a suburb and a nearby train station (denoted by s and t, respectively) connected by two non-interfering highways, and a fixed number of drivers who wish to commute from s to t at roughly the same time. Suppose the first highway is short but narrow with the delay experienced on it while driving from s to t increasing sharply with the number of drivers. Suppose the second is wide enough to accommodate all of the traffic without any crowding but takes a long, circuitous route. For concreteness, assume that all drivers on the latter highway require 1 hour to drive from s and t (irrespective of the number of other drivers on the road), while the delay (in hours) along the former route equals the fraction of the overall traffic choosing to use it. Pictorially, we are discussing the network of Figure 1.1, where the functions $\ell(\cdot)$ (which we will call *latency functions*) describe the latency or delay experienced by drivers on a road as a function of the fraction of the overall traffic using that road; thus, the top edge in Figure 1.1 represents the long wide highway, and the bottom edge the short narrow road.

Assuming that all drivers aim to minimize the time taken to drive from s and t, we have good reason to expect all traffic to follow the lower road and therefore, due to the ensuing congestion, to incur one hour of delay traveling from s to t. Indeed, any driver on the top road (experiencing 1 hour of latency) will soon become envious of the drivers on the lower route (who experience less than 1 hour of latency, provided some traffic chooses the other route) and will change his or her opinion about which route is superior.

Now suppose that, by whatever means, we can choose who drives where. Can we improve over the previous "selfish" outcome with the power of centralized control? To see that we can, consider the outcome of assigning half of the traffic to each of the two routes. The drivers forced onto the long, wide highway experience one hour of delay, and are thus no worse off than in the previous outcome; on the other hand, drivers allowed to use the short narrow road now enjoy lighter traffic conditions, and arrive at their destination after a mere 30 minutes. We have thus improved the state of affairs for half of the drivers while making no one worse off; moreover, the average delay experienced by traffic has dropped from 60 to 45 minutes, a significant improvement.

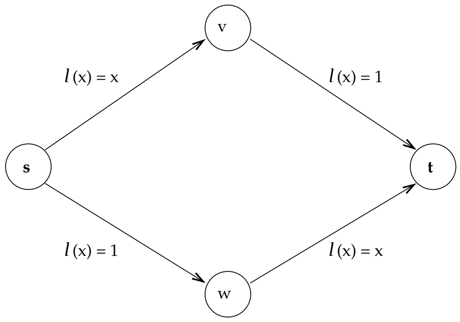
Pigou's example demonstrates a principle that is well-known and well-studied in economics and game theory: *selfish behavior by independent, noncooperative agents need not produce a socially desirable outcome.* In Part II of this thesis, we quantify this phenomenon in several traffic models by analyzing *how much worse* a selfishly-defined outcome can be relative to the best outcome achievable with complete coordination.

1.2.2 Braess's Paradox

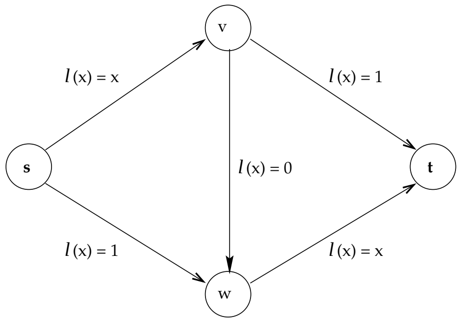
Pigou's example illustrates an important principle, that the outcome of selfish behavior need not optimize social welfare. However, it is perhaps unsurprising that the result of local optimization by many individuals with conflicting interests does not possess any type of global optimality. The next example, due to Braess [28] and subsequently reported by Murchland [128], is decidedly less intuitive.

We again begin with a suburb s, a train station t, and a fixed number of drivers who wish to commute from s to t. For the moment, we will assume two non-interfering routes from s and t, each comprising one long wide road and one short narrow road as shown in Figure 1.2(a). By symmetry, in a selfishly-defined outcome we expect each of the two routes to carry half of the overall traffic, so that all drivers incur 90 minutes of latency traveling from s to t.

Now, an hour and a half is quite a commute. Suppose that, in an effort to alleviate these unacceptable delays, we harness the finest available road technology



(a) Initial Network



(b) Augmented Network

Figure 1.2: Braess's Paradox. The addition of an intuitively helpful edge can adversely affect all of the users of a congested network.

to build a very short and very wide highway joining the midpoints of the two existing routes. The new network is shown in Figure 1.2(b), with the new road represented by edge (v, w) endowed with the constant latency function $\ell(x) = 0$ (independent of the road congestion). How will the drivers react?

We cannot expect the previous traffic pattern to persist in the new network; any driver can save roughly 30 minutes of travel time (assuming other drivers keep their choices fixed) by following route $s \to v \to w \to t$. Suppose that all drivers, in their haste to make use of the new road, simultaneously deviate from their previous routes to instead follow the path $s \to v \to w \to t$. Because of the heavy congestion on edges (s, v) and (w, t), all of these drivers now experience two hours of delay when driving from s to t; moreover, this congestion also implies that neither of the two alternative routes is superior and thus no driver has an incentive to change routes. Even worse, any other traffic pattern is unstable in the sense that some drivers will have an incentive to switch paths. It is therefore reasonable to expect all drivers to follow path $s \to v \to w \to t$ in the selfishly-defined outcome in the new network and thus experience 30 minutes more delay than in the original network. Braess's Paradox thus shows that the intuitively helpful (or at least innocuous) action of adding a new zero-latency link may negatively impact *all* of the traffic!

Braess's Paradox raises some interesting issues. First, it furnishes a second example of the suboptimality of selfishly-defined outcomes. Indeed, Braess's example demonstrates this principle in a stronger form than does Pigou's, in that all drivers would *strictly prefer* a coordinated outcome (namely, the original traffic pattern in the network of Figure 1.2(a)) to the one obtained by acting non-cooperatively.[1] More importantly, Braess's Paradox shows that the interactions between selfish behavior and the underlying network structure defy intuition and are not easy to predict. When we tackle the algorithmic questions of how to design and manage networks so that selfish behavior results in a socially desirable outcome (a task we undertake in Part III of this thesis), we must bear in mind the moral of Braess's paradox: "bigger" need not be "better".

1.3 Our Contributions

To describe our results precisely, we must be more formal about our model of selfish routing in a network. We consider a directed network in which each edge possesses a latency function describing the common latency incurred by all traffic on the edge as a function of the edge congestion (as in the two examples of Section 1.2). We are given a rate of traffic between each ordered pair of nodes in the network; in the two examples of Section 1.2 there was a positive traffic rate only for one ordered pair, but we will also be interested in networks where different users have different sources and destinations (that is, in *multicommodity* networks). We aspire toward an assignment of traffic to paths minimizing the sum of all travel times (the *total latency*)[2] of network users, although the examples of Section 1.2 demonstrate that selfish behavior need not achieve this goal.

We assume that an unregulated network user will always choose the minimum-latency path from its source to its destination (given the link congestion caused by the rest of the network users). As the route chosen by one network user affects the congestion (and hence the latency) experienced by others, the reader familiar with basic game theory will recognize the essential ingredients of a noncooperative game. Motivated by this analogy, when no network user has an incentive to reroute its traffic, we will follow the conventions of noncooperative game theory and say that the network is *at Nash equilibrium.* For instance, we saw that the network of Pigou's example (Figure 1.1) is at Nash equilibrium when all traffic is routed on

[1] This stronger form is also well known in the game theory literature; perhaps its most famous manifestation occurs in the so-called "Prisoner's Dilemma" [56, 150].

[2] Minimizing the *average* (rather than total) travel time may strike the reader as a more natural objective; however, these two objective functions differ only by a normalizing constant (namely, the amount of network traffic) and are therefore equivalent for our purposes. We work with total latency for technical convenience.

the bottom link, while the network of Braess's Paradox (Figure 1.2(b)) is at Nash equilibrium when all traffic follows the path $s \to v \to w \to t$. We can view a Nash equilibrium as a natural operating point of a network in which users are not centrally controlled and route their traffic selfishly—in the language of game theory, as a natural outcome of "rational behavior".

Finally, we will assume that each network user controls a negligible fraction of the overall traffic; assignments of traffic to paths in the network can then be modeled in a continuous manner by *network flow*, with the amount of flow between a pair of nodes in the network equal to the rate of traffic between the two nodes. A Nash equilibrium then corresponds to a flow in which all flow paths between a given source and destination have minimum latency (if a flow does not have this property, some traffic can improve its travel time by switching from a longer path to a shorter one).

Remark 1.3.1 While our examples have been phrased in the language of road networks, we emphasize that our model and results apply equally well to high-speed communication networks. The reader familiar with standard Internet routing protocols might object that, in most current networks, users cannot select paths for their traffic and are instead at the mercy of the network routers. As Friedman [74] points out, however, the paths used by routers are typically computed by a distributed shortest-path computation [98] and, provided end-to-end link delay is used as the metric on the network links, the only stable routings of traffic are Nash equilibria. Of course, neither communication nor road networks need exhibit stable behavior even when all traffic rates are held fixed; nevertheless, we believe that the study of Nash equilibria is a natural first step in understanding the behavior of actual networks.

1.3.1 Bounding the Price of Anarchy

In Chapters 3 and 4 of this thesis, we study the degradation in network performance caused by the selfish behavior of noncooperative network users in a variety of traffic models. Motivated by work of Koutsoupias and Papadimitriou [108] in a different context, we quantify this degradation with the following question: *what is the worst-case ratio between the total latency of a Nash equilibrium and that of the best coordinated outcome—of a flow minimizing the total latency?* This question carries particular importance for networks in which Nash equilibria are not too inefficient—proving strong upper bounds on this worst-case ratio obviates the need for centralized control (provided network users do indeed act in a purely selfish manner).

Computing the Price of Anarchy

We prove sharp upper bounds on this worst-case ratio (recently dubbed "the price of anarchy" by Papadimitriou [142]) for networks in which edge latency does not depend in a highly nonlinear fashion on the edge congestion. We can therefore conclude that the cost of foregoing centralized control in such networks is mild. For example, we prove the following.

- In networks with latency functions that are polynomials with nonnegative coefficients and degree at most p, the price of anarchy is $[1-p\cdot(p+1)^{-(p+1)/p}]^{-1}$, which is asymptotically $\Theta(\frac{p}{\ln p})$ as $p \to \infty$. The bound of $\frac{4}{3}$ (when $p = 1$) for networks with latency functions of the form $\ell(x) = ax + b$ for $a, b \geq 0$ shows that Pigou's example and Braess's Paradox (see Section 1.2) are worst-case examples for the inefficiency of Nash equilibria in such networks.[3]

- In networks with latency functions corresponding to the expected waiting time of an M/M/1 queue (functions of the form $\ell(x) = (u - x)^{-1}$ where u denotes an edge capacity or queue service rate), the price of anarchy is bounded if and only if the maximum allowable amount of traffic R_{max} is constrained to be less than the minimum allowable edge capacity u_{min}; in this case, the price of anarchy is $(1 + \sqrt{u_{min}/(u_{min} - R_{max})})/2$.

The first result demonstrates that the cost of routing selfishly depends crucially on the "steepness" of the network latency functions. The second has the following intuitive interpretation: since the worst-case ratio approaches 1 as $u_{min}/R_{max} \to +\infty$ and approaches $+\infty$ as $u_{min}/R_{max} \to 1$, the price of routing selfishly in a network with M/M/1 delay functions is always tolerable provided the network capacity is sufficiently large relative to the demand for bandwidth, and may be intolerable otherwise.

A Bicriteria Bound for Arbitrary Latency Functions

Our work above shows that Nash equilibria may incur much more latency than minimum-latency flows in networks with latency functions that exhibit steep growth

[3]Throughout this thesis, we will call such latency functions *linear* while admitting that *affine* would be a more accurate adjective.

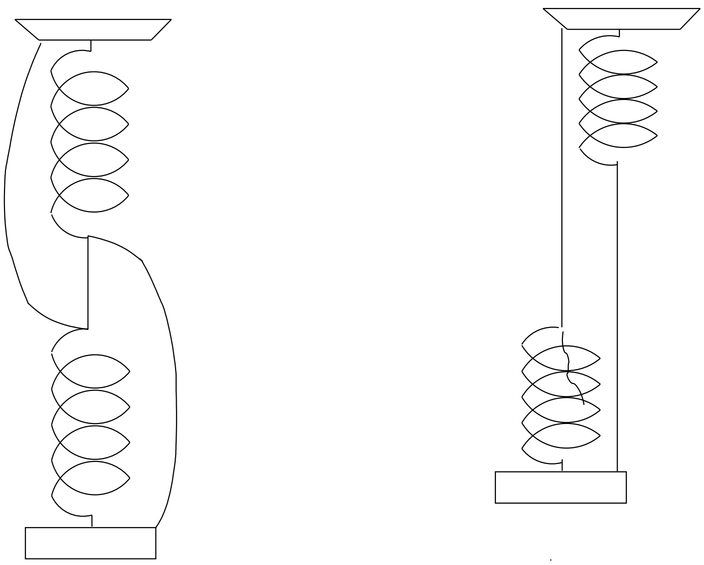
(such as networks with M/M/1 delay functions). Since such latency functions are common in important applications, such as in routing in the Internet and other communication networks [20, 98], we would nevertheless like to meaningfully bound the inefficiency of Nash equilibria in networks with arbitrarily steep latency functions. Toward this end, we consider *bicriteria* results. In particular, we compare the total latency of a Nash equilibrium with that of a minimum-latency flow that routes *additional traffic* between each pair of nodes. We prove that in a network with latency functions assumed only to be continuous and nondecreasing, the total latency incurred by traffic at Nash equilibrium is at most that of a minimum-latency flow forced to route twice as much traffic between each source-destination pair. This result has an alternative interpretation: in lieu of centralized control, the price of routing selfishly can be offset by a moderate increase in link speed (which for the M/M/1 delay functions $\ell(x) = (u - x)^{-1}$ mentioned above can be effected by doubling the capacity u of every edge).

The Price of Anarchy is Independent of the Network Topology

As a corollary of our methods for computing the price of anarchy, we prove that the "steepness" of a network's latency functions is in some sense the *only* cause of the inefficiency of Nash equilibria, and that the complexity of the network topology plays no role. Specifically, we show the following under weak hypotheses on the class of allowable latency functions: among all multicommodity flow networks, networks comprising only two nodes and a collection of parallel links furnish worst-case examples for the losses due to selfish routing. Thus, for any fixed class of latency functions, no nontrivial restriction on the class of allowable network topologies or on the number of commodities will improve the price of anarchy. In the special case of a class of latency functions that includes all of the constant functions, we prove that a network with only two parallel links suffices to achieve the worst-possible ratio. Informally, these results imply that the inefficiency inherent in a Nash equilibrium stems from the inability of selfish users to discern which of two competing routes is superior and *not* from the topological complexity arising from the diverse intersections of many paths belonging to different commodities.

Application to Counterintuitive Phenomena in Physical Systems

Braess's Paradox (as described in Subsection 1.2.2) is not particular to traffic in networks; perhaps the most compelling analogue occurs in a mechanical network



(a) Before

(b) After

Figure 1.3: Strings and springs. Severing a taut string results in the rise of a heavy weight.

of strings and springs, constructed by Cohen and Horowitz [36] and shown in Figure 1.3.[4] In this device, one end of a spring is attached to a fixed support, and the other end to a string. A second identical spring is hung from the free end of the string and carries a heavy weight. Finally, strings are connected (with some slack) from the support to the upper end of the second spring and from the lower end of the first spring to the weight. Assuming that the springs are ideally elastic, the stretched length of a spring is a linear function of the force applied to it. We may thus view the network of strings and springs as a traffic network, where force corresponds to flow and physical distance corresponds to latency. With a suitable choice of string and spring lengths and spring constants, the equilibrium position of this mechanical network is described by Figure 1.3(a). Contrary to intuition, severing the taut string causes the weight to *rise*, as shown in Figure 1.3(b). The explanation for this curiosity is the following. Initially, the two springs are connected "in series", and each bears the full weight and is stretched out to great length. After cutting the taut string, the two springs are only connected "in parallel"; each spring then carries only half of the weight, and accordingly is stretched to only half of

[4]We are indebted to Leslie Ann Goldberg for pointing out this application of our work.

its previous length. This counterintuitive effect corresponds to the improvement in the Nash equilibrium obtained by deleting the zero-latency edge of Figure 1.2(b) to obtain the network of Figure 1.2(a).

Our result above showing that the total latency of a Nash equilibrium in a network with linear latency functions is at most $\frac{4}{3}$ times that of a minimum-latency flow provides a quantitative limit on the extent to which this phenomenon can occur. In particular, we show that this result implies that for any system of strings and springs carrying a single weight, the distance between the support and the weight after severing an arbitrary collection of strings and springs is at least $\frac{3}{4}$ times the original support-weight distance.

Further examples of analogous counterintuitive phenomena have been exhibited in two-terminal electrical networks [36], and our results give analogous bounds on the largest possible increase in conductivity obtainable by removing conducting links.

Extensions to Other Models

Our techniques for computing the price of anarchy are not model-specific; we demonstrate this by extending several of the above results to more general and realistic models. In particular, we consider networks in which users can only evaluate path latency approximately, rather than exactly; networks with a finite number of network users, each controlling a strictly positive (as opposed to negligible) amount of traffic; and a more general class of games that need not take place in a network.

1.3.2 Braess's Paradox and Network Design

In Part III we turn our attention toward *coping with selfishness*—that is, toward methods for designing and managing networks so that selfish routing leads to a desirable outcome. In Chapter 5, we pursue this goal via *network design*; namely, armed with the knowledge that our networks will be host to selfish users, how can we design them to minimize the inefficiency inherent in a user-defined equilibrium?

A natural measure for the performance of a network with selfish routing is the total latency of a Nash equilibrium. Recall from Braess's Paradox (Subsection 1.2.2) the counterintuitive fact that removing edges from a network may *improve* its performance. This observation immediately suggests the following network design problem: given a network with latency functions on the edges and a traffic rate between each pair of vertices, which edges should be removed to obtain the best possible

Nash equilibrium? Equivalently, given a large network of candidate edges to build, which subnetwork will exhibit the best performance when used selfishly?

We give optimal inapproximability results and approximation algorithms for several network design problems of this type. For example, we prove that for networks with one source-destination pair and arbitrary (continuous and nondecreasing) edge latency functions, there is no $(\frac{n}{2} - \epsilon)$-approximation algorithm[5] for network design for any $\epsilon > 0$, where n is the number of vertices in the network (unless $P = NP$). We also prove this hardness result to be best possible by exhibiting an $\frac{n}{2}$-approximation algorithm for the problem. For networks in which the latency of each edge is a linear function of the congestion, we prove that there is no $(\frac{4}{3} - \epsilon)$-approximation algorithm for the problem (for any $\epsilon > 0$, unless $P = NP$), even in networks with a single source-destination pair. Since a $\frac{4}{3}$-approximation algorithm for this special case follows easily from our work bounding the price of anarchy, this hardness result is sharp.

Moreover, we prove that an optimal approximation algorithm for these network design problems is what we call the *trivial algorithm*: given a network of candidate edges, build the entire network. As a consequence of the optimality of the trivial algorithm, we prove that inefficiency due to harmful extraneous edges (as in Braess's Paradox) is impossible to detect efficiently, even in worst-possible instances.

In the course of proving our results, we introduce a new family of graphs generalizing the network of the original Braess's Paradox. This family may be of independent interest, as these networks give the first demonstration that the severity of Braess's Paradox can increase with the network size (for networks with nonlinear latency functions).

1.3.3 Stackelberg Routing

In Chapter 6 we continue to explore techniques for coping with selfishness, motivated by the following idea. In some networks, there will be a mix of "selfishly controlled" and "centrally controlled" traffic—that is, the network is used by both selfish individuals and by some central authority. We study the following question: given a network with centrally and selfishly controlled traffic, *how should centrally controlled traffic be routed to induce "good" (albeit selfish) behavior from the noncooperative*

[5] A *c-approximation algorithm* for a minimization problem runs in polynomial time and returns a solution no more than c times as costly as an optimal solution. The value c is the *approximation ratio* or *performance guarantee* of the algorithm.

users?

We formulate this goal as an optimization problem via *Stackelberg games*, games in which one player acts as a *leader* (here, the centralized authority interested in minimizing total latency) and the rest as *followers* (the selfish users). The problem is then to compute a strategy for the leader (a *Stackelberg strategy*) that induces the followers to react in a way that (at least approximately) minimizes the total latency in the network.

We prove that it is NP-hard to compute the optimal Stackelberg strategy in networks of parallel links and present simple strategies for such networks with provable performance guarantees. More precisely, we give a simple algorithm that computes a leader strategy in a network of parallel links inducing an equilibrium with total latency no more than a constant times that of the minimum-latency flow; a simple variant on Pigou's example (Subsection 1.2.1) shows that no result of this type is possible in the absence of centrally controlled traffic and a Stackelberg strategy. We also prove stronger performance guarantees for networks of parallel links with linear latency functions.

1.4 Comparison to Previous Work

In this section we place our contributions in context by describing previous work. We confine ourselves to a high-level review and to the most relevant references, postponing more specific and in-depth surveys to later chapters.

Bounding the Price of Anarchy

The traffic model studied in this thesis dates back to the 1950's [17, 186] and has been extensively studied ever since; we defer a survey of this literature until Chapter 3. However, the problem of quantifying the inefficiency inherent in a user-defined equilibrium has been considered only recently. To the best of our knowledge, the only previous work with this goal (and the inspiration for much of the work described in this thesis) is the paper of Koutsoupias and Papadimitriou [108]. However, the model of [108] is quite different from the one considered here (see Chapter 3 for details). More recently (and subsequent to some of our work), the model and results of [108] have been generalized in a series of papers by Mavronicolas and Spirakis [122], Czumaj and Vöcking [44], and Czumaj et al. [43].

Coping with Selfishness

For many decades, researchers have realized that selfish behavior can have undesirable consequences and have proposed numerous methods for coping with it. To mention just a few approaches ignored in this thesis, there have been significant recent advances in controlling selfish users in communication networks via pricing policies (see [6, 171, 174] and the references therein) and via centralized switch service disciplines and flow control protocols (for example, see Shenker [170] and the references therein for approaches that seek "fair" outcomes). An approach to coping with selfishness that possesses a rich history and that has led to an abundance of recent work is *mechanism design*—see the work of Nisan and Ronen [136, 137, 155] for recent research motivated by discrete optimization problems and for pointers to the classical literature. Most work in mechanism design is concerned with applications that are simpler than the problem of network routing (such as auctions), and requires a notion of currency to employ some kind of side payments to the selfish players.

A detailed survey of previous research on these topics is beyond the scope of this thesis. We will only attempt to describe work on the two methods of coping with selfishness that we study: network design and Stackelberg routing.

Braess's Paradox and Network Design

Ever since Braess's Paradox was reported [28, 128], researchers have attempted to solve variants of the network design problem described in Subsection 1.3.2; for example, the early work of Dafermos and Sparrow [50] alludes to such a problem. However, progress appears to have been elusive, both computationally and theoretically (see Chapter 5 for a survey of past efforts). Prior to our work, the network design problem discussed in Subsection 1.3.2 was not known to be NP-hard, nor was any heuristic for the problem known to have a finite approximation ratio. In addition, our construction of an infinite family of networks generalizing Braess's Paradox in both size and severity appears to be new.

Stackelberg Routing

Stackelberg games and Stackelberg equilibria have been extensively studied in the game theory literature and previously applied to problems in both networking and other fields (see Chapter 6 for an overview). Closest to our approach are the papers of Douligeris and Mazumdar [53] and Korilis et al. [104], which advocate system optimization via Stackelberg strategies. In [53], however, only experimental results are reported. Korilis et al. [104] seek necessary and sufficient conditions for the existence of a leader strategy inducing an optimal routing of all of the traffic; by contrast, we are interested in worst-case performance guarantees.

1.5 Tips for Reading this Thesis

1.5.1 Prerequisites

This dissertation assumes relatively few prerequisites. Foremost, we expect the reader to be comfortable with basic concepts of network flow theory, such as flows, cuts, and path decompositions. Our favorite reference for this material is Tarjan [179]. On occasion we assume a nodding acquaintance with the theory of NP-completeness, for which standard references include Garey and Johnson [77] and Papadimitriou [141], and with the basics of linear and nonlinear programming; accessible introductions to these two fields are given by Chvátal [34] and Peressini et al. [145], respectively. We assume no knowledge of game theory; however, some of our definitions and results may appear more natural to the reader familiar with basic game-theoretic concepts. Standard introductions to game theory include Fudenberg and Tirole [75], Osbourne and Rubinstein [139], and Owen [140]. For a gentler overview ideal for a long plane flight, we recommend Straffin [177].

1.5.2 Presentation Overview

Chapter 2 is devoted to technical preliminaries. In that chapter we formally define our traffic model, we define flows at Nash equilibrium and prove many of their basic properties, and we study the optimization problem of computing the minimum-latency traffic flow, thereby obtaining a useful characterization of such flows.

Subsequent chapters split naturally into two parts: in Part II we develop techniques for bounding the price of anarchy and in Part III we design and analyze methods for coping with selfishness. More specifically, in Chapter 3 we develop techniques for computing the price of anarchy, prove our bicriteria bound on the inefficiency of Nash equilibria in networks with arbitrary latency functions, and show that the price of anarchy is independent of the network topology (see Subsection 1.3.1 for a more detailed overview). In Chapter 4 we extend some of these results to more general traffic models and to more general types of games. Chapter 5 studies the problem of designing networks for selfish users, and presents all of the material outlined in Subsection 1.3.2. In the final technical chapter, Chapter 6, we define a model of Stackelberg routing and prove the results described in Subsection 1.3.3. Finally, in Chapter 7 we conclude with a discussion of recent work and some suggestions for further research.

1.5.3 Dependencies

Chapter 2 is a prerequisite for all that follows, though some sections are required only for a subset of our results; this is discussed in detail in the chapter's introduction. Chapters 3, 5, and 6 can be read independently of each other, although in Chapter 5 we assume some of the results (but not the proof techniques) of Chapter 3. Finally, Chapter 4 is meant to be read following Chapter 3.

1.6 Bibliographic Notes

Most of the work reported in this thesis has appeared previously in research papers [159, 160, 161, 162, 163, 164, 165]. Chapter 4 and portions of Chapter 3 and Appendix A are joint work with Éva Tardos and appeared in [164, 165]; the rest of Chapter 3 is drawn from [160, 163]. The results of Chapter 5 appeared in [159], the results of Chapter 6 in [161], and Theorem A.3.1 of Section A.3 in [162].

Chapter 2

Preliminaries

In this chapter we present the basic definitions and preliminary technical results needed in the rest of this work. In Section 2.1 we formally define the traffic model discussed in Chapter 1. In Section 2.2 we define flows at Nash equilibrium and prove some of their basic properties. Section 2.3 gives a characterization of minimum-latency flows that is crucial for a majority of our results. In Section 2.4 we illustrate the definitions and propositions of Sections 2.1–2.3 with several concrete examples. In Section 2.5 we build on the results of Section 2.3 to prove the existence and essential uniqueness of flows at Nash equilibrium, and in Section 2.6 we conclude by proving further useful properties about flows at Nash equilibrium.

Different portions of this dissertation depend on different subsets of this chapter; these dependencies are described in detail in Table 2.1.

2.1 The Model

We consider a directed network $G = (V, E)$ with vertex set V, edge set E, and k source-destination vertex pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$. We allow parallel edges between vertices but have no use for self-loops. We will sometimes refer to vertices as *nodes* and to edges as *links*. We denote the set of (simple) s_i-t_i paths by $\mathcal{P}_i$, and define $\mathcal{P} = \cup_i \mathcal{P}_i$. To avoid trivialities, we will always assume that $\mathcal{P}_i \neq \emptyset$ for each i. A *flow* is a function $f : \mathcal{P} \to \mathcal{R}^+$; for a fixed flow f and an edge $e \in E$ we define $f_e = \sum_{P:e \in P} f_P$ to be the total amount of flow on edge e. We sometimes refer to a source-destination pair $\{s_i, t_i\}$ and the s_i-t_i paths $\mathcal{P}_i$ as *commodity* i. When we wish to concentrate on the flow of a particular commodity i, we write f^i for the restriction of f to $\mathcal{P}_i$ and f^i_e for $\sum_{P \in \mathcal{P}_i : e \in P} f_P$.

Section	Prerequisite for...
2.1	rest of thesis
2.2	rest of thesis
2.3	Sections 2.4–2.5, 3.2–3.5, 4.1–4.2, 4.4, Chapter 6
2.4	none
2.5	rest of thesis/Section A.1
2.6	Section 5.4

Table 2.1: What in Chapter 2 is useful where. Section 2.4 is not logically necessary for what follows, but Subsections 2.4.1, 2.4.2, and 2.4.4 will show up frequently as examples in later chapters. The proof techniques of Section 2.5 are needed only in Section A.1, but we will use the fact that Nash flows exist and are essentially unique in the rest of the thesis.

We associate a finite and positive *traffic rate* r_i with each pair $\{s_i, t_i\}$, the amount of flow with source s_i and destination t_i; a flow f is said to be *feasible* if for all i, $\sum_{P \in \mathcal{P}_i} f_P = r_i$. Finally, each edge $e \in E$ is given a congestion-dependent *latency* that we denote by $\ell_e(\cdot)$. For each edge $e \in E$, we assume that the latency function ℓ_e is nonnegative, continuous, and nondecreasing. The *latency of a path P with respect to a flow f* is defined as the sum of the latencies of the edges in the path, denoted by $\ell_P(f) = \sum_{e \in P} \ell_e(f_e)$. We will call the triple (G, r, ℓ) an *instance*.

We define the *cost* $C(f)$ of a flow f as the total latency incurred by f, i.e., $C(f) = \sum_{P \in \mathcal{P}} \ell_P(f) f_P$. By summing over the edges in a path P and reversing the order of summation, we may also write

$$C(f) = \sum_{e \in E} \ell_e(f_e) f_e.$$

With respect to an instance (G, r, ℓ), a feasible flow minimizing $C(f)$ is said to be *optimal* or *minimum-latency*; such a flow always exists because the space of all feasible flows is a compact set and our cost function is continuous.

Remark 2.1.1 We will see in the coming sections that our assumptions that latency functions are nonnegative, continuous, and nondecreasing are essential for our theory of inefficiency of Nash equilibria in networks. We believe these assumptions to be reasonable for most network applications. Functions that violate the continuity assumption (for example, step functions) can be made continuous (even differentiable) with little loss of information. While some researchers have described scenarios where nonmonotone cost functions are natural[1], the applications we have in mind—routing traffic in computer and road networks—should always satisfy the required monotonicity condition.

2.2 Flows at Nash Equilibrium

We wish to study flows that represent an equilibrium among many noncooperative network users—that is, flows that behave "greedily" or "selfishly", without regard to the overall cost. We intuitively expect each unit of such a flow (no matter how small) to travel along the minimum-latency path available to it, where latency is measured with respect to the rest of the flow; otherwise, this flow would reroute itself on a path with smaller latency. Following Dafermos and Sparrow [50], we formalize this idea in the next definition.

Definition 2.2.1 A flow f feasible for instance (G, r, ℓ) is at *Nash equilibrium* (or is a *Nash flow*) if for all $i \in \{1, \ldots, k\}$, $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, and $\delta \in (0, f_{P_1}]$, we have $\ell_{P_1}(f) \le \ell_{P_2}(\tilde{f})$, where

$$\tilde{f}_P = \begin{cases} f_P - \delta & \text{if } P = P_1 \\ f_P + \delta & \text{if } P = P_2 \\ f_P & \text{if } P \notin \{P_1, P_2\}. \end{cases}$$

Letting δ tend to 0, continuity and monotonicity of the edge latency functions give the following useful characterization of a flow at Nash equilibrium, occasionally called a Wardrop equilibrium [84] or Wardrop's Principle [175, 176] in the literature, due to an influential paper of Wardrop [186].

Proposition 2.2.2 *A flow f feasible for instance (G, r, ℓ) is at Nash equilibrium if and only if for every $i \in \{1, \ldots, k\}$ and $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $\ell_{P_1}(f) \le \ell_{P_2}(f)$.*

Remark 2.2.3 While Definition 2.2.1 still makes sense without assuming continuity and monotonicity of the edge latency functions, Proposition 2.2.2 fails if either of these hypotheses is omitted (the forward direction fails in the absence of continuity and the reverse direction fails in the absence of monotonicity—see Section B.1).

[1] For example, Blonski [22] points out that people typically have nonmonotone preferences about the congestion in a restaurant or at a concert, preferring a moderate crowd to total isolation or to being packed in like a sardine.

Briefly, Proposition 2.2.2 states that, in a flow at Nash equilibrium, all flow travels on minimum-latency paths. In particular, if f is at Nash equilibrium then all s_i-t_i flow paths (s_i-t_i paths to which f assigns a positive amount of flow) have equal latency, say $L_i(f)$. We can thus express the cost $C(f)$ of a flow f at Nash equilibrium in a particularly nice form.

Proposition 2.2.4 *If f is a flow at Nash equilibrium for instance (G, r, ℓ), then*

$$C(f) = \sum_{i=1}^{k} L_i(f) r_i.$$

Remark 2.2.5 For the next two sections we will take for granted that Nash flows exist and are essentially unique; this will be proved in Section 2.5.

Remark 2.2.6 Our definition of a flow at Nash equilibrium corresponds to an equilibrium in which each network user chooses a single path of the network (a *pure strategy*), whereas in classical game theory a Nash equilibrium is defined via *mixed strategies* (with players of a game choosing probability distributions over pure strategies). However, since in our model each network user controls only a negligible fraction of the overall traffic, these two definitions are essentially equivalent—see [84] for a rigorous discussion.

2.3 A Characterization of Optimal Flows

We now investigate the properties of an optimal flow—that is, of a flow minimizing the total latency. Recalling that the cost of a flow f may be expressed $C(f) = \sum_{e \in E} \ell_e(f_e) f_e$, the problem of finding a minimum-latency feasible flow in a network is a special case of the nonlinear program

$$\begin{array}{lll} & \text{Min} & \displaystyle\sum_{e \in E} c_e(f_e) \\ & \text{subject to:} & \\ (NLP) & & \displaystyle\sum_{P \in \mathcal{P}_i} f_P = r_i \qquad \forall i \in \{1, \ldots, k\} \\ & & f_e = \displaystyle\sum_{P \in \mathcal{P} : e \in P} f_P \qquad \forall e \in E \\ & & f_P \geq 0 \qquad \forall P \in \mathcal{P} \end{array}$$

where in our problem, $c_e(f_e) = \ell_e(f_e) f_e$.

For simplicity we have given a formulation with an exponential number of variables, but it is not difficult to give an equivalent compact formulation (with decision variables only on edges and explicit node conservation constraints) that requires only polynomially many variables and constraints.

Next, we characterize the local optima of (NLP). Roughly, we expect a flow to be locally optimal if and only if moving flow from one path to another can only increase the flow's cost. Put differently, we expect a flow to be locally optimal when the marginal benefit of decreasing flow along any s_i-t_i flow path is at most the marginal cost of increasing flow along any other s_i-t_i path. Since the local and global minima of a convex function on a convex set coincide (see, e.g., [145, Thm 2.3.4]), this condition should be necessary and sufficient for a flow to be *globally* optimal whenever the objective function of (NLP) is convex.[2] This is the case when, for example, for each edge $e \in E$ we have $c_e(f_e) = \ell_e(f_e)f_e$ with a convex latency function ℓ_e.

We formalize this characterization of global optima of convex programs of the form (NLP) in the next lemma. For a differentiable cost function c_e, let c_e' denote the derivative $\frac{d}{dx}c_e(x)$ of c_e and define $c_P'(f)$ by $c'_P(f) = \sum_{e \in P} c_e'(f_e)$. We then have the following.[3]

Proposition 2.3.1 ([17, 50]) *A flow f is optimal for a convex program of the form (NLP) with differentiable cost functions if and only if for every $i \in \{1, \ldots, k\}$ and $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $c_{P_1}'(f) \le c_{P_2}'(f)$.*

The striking similarity between the characterizations of optimal solutions to a convex program of the form (NLP) and of flows at Nash equilibrium was noticed early on by Beckmann et al. [17], and provides an interpretation of an optimal flow as a flow at Nash equilibrium *with respect to a different set of edge latency functions.* To make this relationship precise, denote the marginal cost of increasing flow on edge e with differentiable latency function ℓ_e by $\ell_e^*(x) = \frac{d}{dy}(y \cdot \ell_e(y))(x) = \ell_e(x) + x \cdot \ell_e'(x)$. Propositions 2.2.2 and 2.3.1 then yield the following corollary.

Corollary 2.3.2 ([17, 50]) *Let (G, r, ℓ) be an instance with differentiable latency functions in which $x \cdot \ell_e(x)$ is a convex function for each edge e, with marginal cost*

[2]A function f defined on a convex subset S of $\mathcal{R}^n$ is *convex* if $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in S$ and $\lambda \in [0,1]$. Some authors call such functions *weakly convex.*

[3]For a formal derivation via the Karush-Kuhn-Tucker Theorem [145], see [17, 50].

functions ℓ^* defined as above. Then a flow f feasible for (G, r, ℓ) is optimal if and only if it is at Nash equilibrium for the instance (G, r, ℓ^*).

Remark 2.3.3 We will typically denote a minimum-latency flow for an instance by f^*. The marginal cost functions are denoted by ℓ^* since they are "optimal latency functions" in a sense made precise by Corollary 2.3.2: the optimal flow f^* arises as a flow at Nash equilibrium with respect to latency functions ℓ^*.

Remark 2.3.4 The function $\ell_e^*(x)$ describing the marginal cost of increasing flow on edge e has one term $\ell_e(x)$ capturing the per-unit latency incurred by the additional flow and a second term $x \cdot \ell_e'(x)$ accounting for the increased congestion experienced by the flow already using the edge. Essentially, the only difference between an optimal flow and a flow at Nash equilibrium is that the former accounts for this "conscientious" second term while the latter disregards it.

The conclusion of Proposition 2.3.1 is false without the convexity hypothesis. Instances to which Proposition 2.3.1 and Corollary 2.3.2 apply (those with latency functions satisfying the aforementioned convexity assumption) will play an important role in portions of this dissertation (in particular, in Sections 3.3–3.5 and in Chapter 6), important enough to warrant further terminology.

Definition 2.3.5 A latency function ℓ is *standard* if it is differentiable and if $x \cdot \ell(x)$ is convex on $[0, \infty)$.

Most but not all latency functions of interest are standard. All differentiable convex functions are standard, as are some well-behaved nonconvex functions such as $\log(1 + x)$. Differentiable approximations of step functions are the most notable examples of nonstandard latency functions.

We conclude this section with a final fact about networks with standard latency functions, useful in Chapter 6: since (NLP) is a convex program when all edge latency functions are standard, the optimal flow of such an instance can be found efficiently.

Fact 2.3.6 *If (G, r, ℓ) is an instance with standard latency functions, then the optimal flow for (G, r, ℓ) can be computed in polynomial time (up to an arbitrarily small additive constant).*

This algorithmic task can be accomplished with the ellipsoid method, even when network latency functions are not described by explicit formulae and are instead given only as oracles [81]. Under additional assumptions on the latency functions (e.g., that latency functions are sufficiently smooth with efficiently computable first and second derivatives), standard interior-point techniques (as described in, for example, Renegar [152]) can be used. In the special case of an instance with a single commodity, the problem reduces to that of computing a min-cost flow with respect to a convex separable objective function, a problem for which combinatorial polynomial-time algorithms are known—see [3, 19, 87] for a survey of the available techniques.

Remark 2.3.7 The additive error in Fact 2.3.6 is required, as an exact description of the optimal flow may require irrational numbers.

2.4 Examples

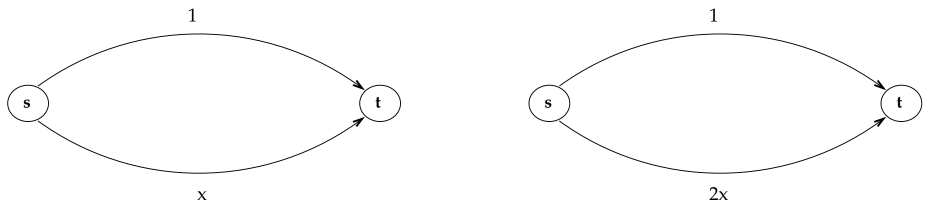
In this section we illustrate the definitions and characterizations of the previous sections in some concrete networks, and hope to develop the reader's intuition about Nash and optimal flows. We first return to the familiar examples of Subsection 1.2 (Pigou's example and Braess's Paradox) and then present three more examples that demonstrate further differences between Nash and optimal flows.

Remark 2.4.1 For simplicity, we have chosen examples in which all traffic shares the same source and destination. However, we are also interested in (and most of our results will apply to) networks in which different users have different sources and destinations.

2.4.1 Pigou's Example

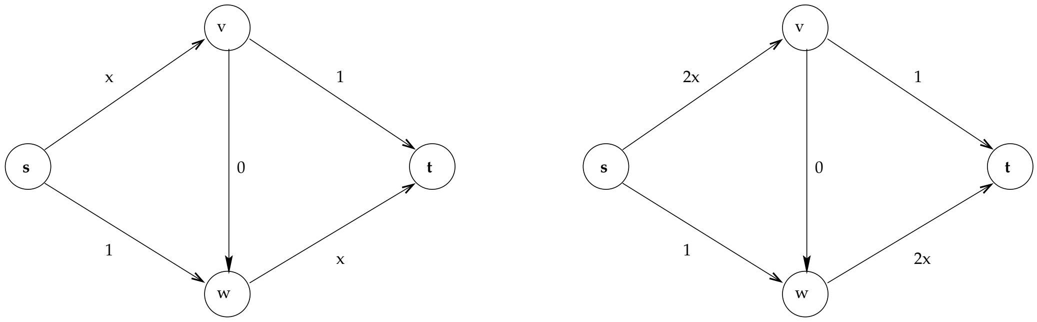
Recall that, in Pigou's example of Subsection 1.2.1, we have a network with two nodes s and t, two parallel edges with latency functions $\ell(x) = 1$ and $\ell(x) = x$, and a traffic rate of 1 (see Figure 2.1(a)). Routing all flow on the bottom link equalizes the latencies of the two available s-t paths at 1, and thus by Proposition 2.2.2 provides a flow f at Nash equilibrium. By Proposition 2.2.4 (or by inspection), the cost $C(f)$ of f is 1.

Next, notice that the marginal cost functions of the network are $\ell^*(x) = 1$ and $\ell^*(x) = 2x$ (see Figure 2.1(b)). Routing half of the traffic on each link thus equalizes



(a) Latency Functions (b) Marginal Cost Functions

Figure 2.1: Pigou's example revisited



(a) Latency Functions (b) Marginal Cost Functions

Figure 2.2: Braess's Paradox revisited

the marginal costs of the two s-t paths at 1, and so by Corollary 2.3.2 furnishes a minimum-latency flow f^*. The cost of f^* is $C(f^*) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}$.

2.4.2 Braess's Paradox

Next we consider the network of Braess's Paradox (Subsection 1.2.2) after the addition of the zero-latency edge; see Figure 2.2(a). Setting the traffic rate r to 1, we see that the flow f that routes all traffic on the path $s \to v \to w \to t$ equalizes the latency of the three s-t paths at 2, and thus (by Proposition 2.2.2) f is at Nash equilibrium with $C(f) = 2$. Switching to marginal cost functions (see Figure 2.2(b)), we find that the flow f^* that routes half the traffic on each of the two two-hop paths equalizes the marginal costs of the three s-t paths at 2, and is therefore (by Corollary 2.3.2) optimal. The cost $C(f^*)$ of f^* is $\frac{3}{2}$.

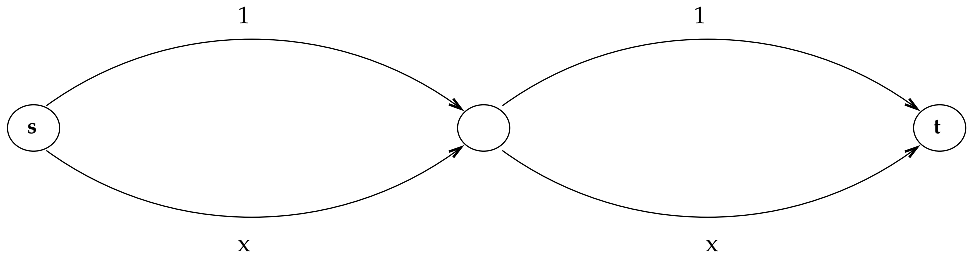


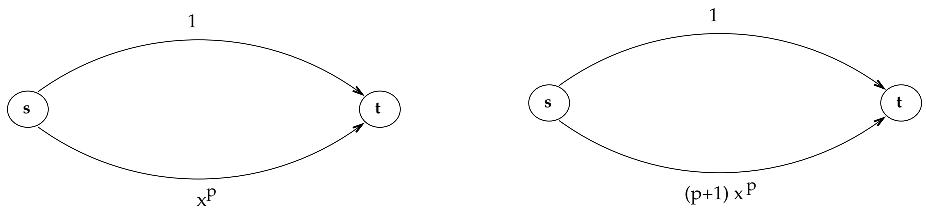
Figure 2.3: The Nash flow may be strictly Pareto-dominated by the optimal flow in the absence of Braess's Paradox.

2.4.3 Strict Pareto Suboptimality of Nash without Paradox

In Subsection 1.2.2 we remarked that Braess's Paradox demonstrates two different principles. First, all traffic may strictly prefer a coordinated outcome to the flow at Nash equilibrium[4]; second, if network users route selfishly, then augmenting a network with an additional link may strictly increase everyone's latency. While the second phenomenon implies the first in the augmented network (by coordinating, additional links can always be ignored), the converse is not true. Put differently, there are networks in which the flow at Nash equilibrium is strictly Pareto-dominated by an optimal flow and yet cannot be improved by the deletion of any number of network links.

To see this, consider the network shown in Figure 2.3, two copies of the network of Pigou's example glued together in series. Setting the traffic rate r to be 1, the flow f that routes all traffic along the two bottom links equalizes the latency of all four s-t paths at 2 and is thus at Nash equilibrium. On the other hand, in the optimal flow f^* that routes half of the traffic on the path comprising the top link of the first subnetwork and the bottom link of second subnetwork and the rest of the traffic on the path comprising the other two links, all traffic experiences only $\frac{3}{2}$ units of latency. All traffic is thus better off in the flow f^* than in the Nash flow f. Moreover, it is easy to check that the Nash flow does not improve when any subset of the links of the network is removed.

[4]In the language of economics, we would say that the Nash flow is *strictly Pareto-dominated* by the optimal flow.



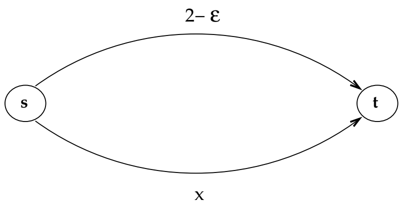
(a) Latency Functions (b) Marginal Cost Functions

Figure 2.4: A nonlinear variant of Pigou's example

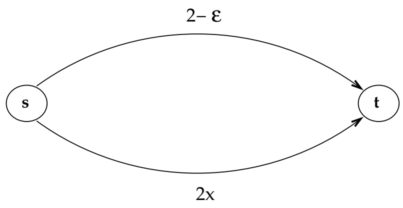
2.4.4 A Nonlinear Variant of Pigou's Example

In all three of our examples thus far, the Nash flow fails to minimize the total latency and is a factor of precisely $\frac{4}{3}$ more costly than the optimal flow. This is not entirely a coincidence, as in the next chapter we will see that no worse ratio is possible in any multicommodity flow network provided the latency of every edge increases linearly with the edge congestion (as is the case in the previous three examples). We now show that this strong hypothesis on the network latency functions is necessary for such a strong result, and that flows at Nash equilibrium can be arbitrarily more costly than optimal flows in networks with nonlinear edge latency functions.

Consider the minor modification of Pigou's example shown in Figure 2.4(a), where we have replaced the latency function $\ell(x) = x$ by the highly nonlinear one $\ell(x) = x^p$ (for concreteness, think of p as 100 or 1000). With the usual traffic rate of 1, the Nash flow f is the same as in Pigou's example; all flow is routed on the bottom link and the total latency is 1 (for any choice of p). On the other hand, the discrepancy between the latency functions (in Figure 2.4(a)) and the marginal cost functions (in Figure 2.4(b)) is much larger; now, the flow f^* that routes $(p+1)^{-1/p}$ units on the lower link and the remainder on the upper link equalizes the marginal cost of the two links at 1 and is thus optimal. The cost $C(f^*)$ of f^* is $1 - p \cdot (p+1)^{-(p+1)/p}$, which tends to 0 as $p \to \infty$. Thus, if arbitrarily steep latency functions are allowed (even restricting to polynomials), a flow at Nash equilibrium can be arbitrarily more costly than an optimal flow. This negative result motivates our work on bicriteria bounds for Nash flows in networks with arbitrary latency functions (Section 3.6) and on ensuring that the cost of a selfish solution in such a network is close to optimal by carefully routing a small fraction of the traffic centrally (Chapter 6).



(a) Latency Functions



(b) Marginal Cost Functions

Figure 2.5: The optimal flow may sacrifice some traffic to a path with large latency to minimize the total latency.

2.4.5 The Unfairness of Optimal Flows

In all of our examples thus far, the optimal flow has been superior to the Nash flow in a very strong sense. Rather than merely achieving a smaller *total* latency than a Nash flow, in the previous examples *all* traffic is at least as well-off in the optimal flow as in the flow at Nash equilibrium; that is, the optimal flow has *Pareto-dominated* the flow at Nash equilibrium. Our next example shows that this will not always be the case; in general, an optimal flow will sacrifice some traffic to paths with large latency in order to minimize the total latency experienced by all of the traffic.

Consider the network of Figure 2.5(a), a small variation on Pigou's example in which we replace the latency function $\ell(x) = 1$ with the latency function $\ell(x) = 2-\epsilon$ for a very small positive constant $\epsilon > 0$. As usual, we set the traffic rate r to 1. In the flow at Nash equilibrium, all traffic is routed on the bottom link and experiences one unit of latency. In the optimal flow, however, only $1-\epsilon/2$ units of flow are routed on the bottom link—routing the other $\epsilon/2$ units of traffic on the upper link equalizes the marginal costs of the two edges at $2-\epsilon$. Intuitively, a small fraction of the traffic is sacrificed to the slow edge in order to (slightly) reduce the congestion experienced by the overwhelming majority of network users.

The "unfairness" of optimal flows is an unfortunate property. There is good news, however; for networks in which all traffic shares the same source and destination, we can quantify this unfairness and prove that it cannot be too large. Since this issue is somewhat removed from the main themes of this thesis, we defer a further discussion of it to Section A.3 of Appendix A.

2.5 Existence and Uniqueness of Nash Flows

In this section, we exploit the similarity between the characterizations of Nash and of minimum-latency flows (Propositions 2.2.2 and 2.3.1) to prove the existence and essential uniqueness of flows at Nash equilibrium. This result is originally due to Beckmann et al. [17] and was later reproved by Dafermos and Sparrow [50]; we include a proof for completeness.

Proposition 2.5.1 ([17, 50]) *An instance (G, r, ℓ) with continuous, nondecreasing latency functions admits a flow at Nash equilibrium. Moreover, if $f, \tilde{f}$ are flows at Nash equilibrium, then $\ell_e(f_e) = \ell_e(\tilde{f}_e)$ for each edge e.*

Proof. Set $h_e(x) = \int_0^x \ell_e(t)dt$. By continuity of the latency function ℓ_e, the function h_e is differentiable with nondecreasing derivative ℓ_e and is therefore convex. Now consider the convex program

$$\begin{array}{llll} & \text{Min} & \sum_{e \in E} h_e(f_e) & \\ & \text{subject to:} & & \\ (NLP2) & & \sum_{P \in \mathcal{P}_i} f_P = r_i & \forall i \in \{1, \ldots, k\} \\ & & f_e = \sum_{P \in \mathcal{P} : e \in P} f_P & \forall e \in E \\ & & f_P \geq 0 & \forall P \in \mathcal{P} \end{array}$$

and observe that the optimality conditions of Proposition 2.3.1 are identical to the characterization of flows at Nash equilibrium in Proposition 2.2.2. The optimal solutions for $(NLP2)$ are thus precisely the flows at Nash equilibrium for (G, r, ℓ). Existence of a flow at Nash equilibrium for (G, r, ℓ) then follows from the facts that $(NLP2)$ has a continuous objective function and a compact feasible region. Next, suppose $f, \tilde{f}$ are flows at Nash equilibrium for (G, r, ℓ) (and hence global optima for $(NLP2)$). By convexity of the objective function of $(NLP2)$, whenever $f \neq \tilde{f}$ the objective function must be linear between these two values; otherwise any convex combination of $f, \tilde{f}$ would be a feasible solution for $(NLP2)$ with smaller objective function value. Since the objective function is convex separable, h_e must be linear between f_e and $\tilde{f}_e$ for each edge e. By continuity of each latency function ℓ_e, each ℓ_e must be constant between f_e and $\tilde{f}_e$. This implies that $\ell_e(f_e) = \ell_e(\tilde{f}_e)$ for all $e \in E$, and the proof is complete. ■

Remark 2.5.2

(a) The proof of Proposition 2.5.1 shows that if each latency function ℓ_e is *strictly* increasing, then the flow f_e on edge e in a Nash flow is uniquely determined for each edge e. Even in this setting there may be several Nash flows, however, as distinct flows (i.e., distinct functions on the paths $\mathcal{P}$) may induce identical f_e-values on the edges.

(b) In the absence of strictly increasing latency functions, different Nash flows may place different amounts of flow on the same edge—consider the trivial example of two nodes and two parallel links endowed with the constant latency function $\ell(x) = 1$ (every flow in this network is at Nash equilibrium). Thus in some sense the uniqueness statement of Proposition 2.5.1 is the strongest possible.

(c) Both the existence and uniqueness conclusions of Proposition 2.5.1 fail if the hypothesis of continuity is omitted. This fact illustrates a technical distinction between our traffic routing model (with infinitely many players) and the classical theory of noncooperative games developed by Nash [130] where, assuming only finitely many players and mixed strategies but arbitrary cost functions, a Nash equilibrium always exists. In addition, if latency functions are allowed to be decreasing or nonmonotone, the uniqueness conclusion fails (see Section B.1 for counterexamples). For this reason and others, the assumption of continuous, nondecreasing network latency functions is crucial for our work.

(d) The proof of Proposition 2.5.1 shows that flows at Nash equilibrium are precisely the optimal solutions to a related convex program defined over the same feasible region. Thus, under mild smoothness conditions on the network latency functions, a flow at Nash equilibrium can be computed (up to an arbitrarily small additive constant) in polynomial time; see Fact 2.3.6 and the comments thereafter for more details.

(e) The reader familiar with noncooperative game theory will recognize the unusual ease with which we proved the existence and essential uniqueness of flows at Nash equilibrium; in general non-zero sum (matrix) games, establishing the existence of Nash equilibria (in mixed strategies) requires recourse to a nonconstructive fixed-point theorem [130]. That flows at Nash equilibrium arise as the optimum solutions to a well-behaved optimization problem is both useful and remarkable, and the recent study of *congestion games* and *potential* *games* by the game theory community can be viewed as an ongoing quest for broad classes of games that share this property. We will have more to say about these two classes of games in Section 4.4.

(f) The proof of Proposition 2.5.1 leads to a nontrivial "quick and dirty" upper bound on the price of anarchy. This bound is easy to apply but does not in general give the best possible results (unlike the techniques that we will develop in Chapter 3); for this reason, we defer further discussion of this bound to Section A.1 in Appendix A.

In much of this thesis, we will be content with the following corollary of Proposition 2.5.1, which states that all flows at Nash equilibrium have the same cost.

Corollary 2.5.3 *If $f, \tilde{f}$ are flows at Nash equilibrium for the instance (G, r, ℓ), then $C(f) = C(\tilde{f})$.*

Proof. The corollary follows directly from Propositions 2.2.4 and 2.5.1. ■

2.6 Acyclicity of Nash Flows

The goal of this final section is to prove that every instance (G, r, ℓ) admits a Nash flow without flow cycles (thereby strengthening the existence guarantee of Proposition 2.5.1). Along the way, we will prove some useful properties about the structure of minimum-latency paths with respect to a Nash flow.

We begin with an extension of Proposition 2.2.2. While Proposition 2.2.2 characterizes Nash flows as those with all s_i-t_i flow paths having minimum latency (for each commodity i), the following lemma gives an analogous characterization with s_i and t_i replaced by an arbitrary pair of vertices.

Proposition 2.6.1 *Let f be a flow feasible for the instance (G, r, ℓ). For a vertex v in G and a commodity i, let $d^i(v)$ denote the length, with respect to edge lengths $\ell_e(f_e)$, of a shortest s_i-v path in G. Then f is at Nash equilibrium if and only if for every pair v, w of vertices in G, every commodity i, and every v-w path P:*

(a) $d^i(w) - d^i(v) \leq \sum_{e \in P} \ell_e(f_e)$

(b) if $f_e^i > 0$ for every edge $e \in P$, then $d^i(w) - d^i(v) = \sum_{e \in P} \ell_e(f_e)$.

Proof. First suppose f is feasible for (G, r, ℓ) and satisfies the two conditions of the proposition. For a commodity i, we can take $v = s_i$ and $w = t_i$ and apply properties (a) and (b) to find that every s_i-t_i flow path of f has minimum latency among all s_i-t_i paths (namely, $d^i(t)$). Since this holds for all commodities, Proposition 2.2.2 implies that f is at Nash equilibrium.

Conversely, suppose f is at Nash equilibrium for (G, r, ℓ). It suffices to prove that properties (a) and (b) hold when P is a single edge (for a general path, sum up the inequalities or equalities corresponding to the constituent edges). Then, (a) follows by definition of $d^i(v)$ and $d^i(w)$. To prove (b), consider a commodity i and an edge e with $f_e^i > 0$ and suppose for contradiction that $d^i(w) < d^i(v) + \ell_e(f_e)$. Let P_e denote an s_i-t_i path containing e with $f_{P_e} > 0$. We may obtain another s_i-t_i path P' via the union of a shortest s_i-w path and the w-t_i path contained in P_e. Since the latency of the s_i-w path contained in P_e is at least $d^i(v) + \ell_e(f_e) > d^i(w)$, we have $\ell_{P_e}(f) > \ell_{P'}(f)$; by Proposition 2.2.2, this contradicts our assumption that f is at Nash equilibrium. ■

It is important to note that the path P in the statement of Proposition 2.6.1 does *not* need to be a subpath of any flow path of f; in particular, in property (b) the flow on different edges of P can be carried by distinct flow paths of f.

By a *flow cycle for commodity* i of a flow f we mean a collection C of edges in the underlying graph that form a directed cycle and for which $f_e^i > 0$ for all $e \in C$; we emphasize that flow on different edges may be carried by different s_i-t_i flow paths. Call a flow f *acyclic* if it has no flow cycles (for any commodity). We now prove that every instance admits an acyclic Nash flow, a fact that we believe to be "folklore".

Proposition 2.6.2 *An instance (G, r, ℓ) admits an acyclic flow at Nash equilibrium.*

Proof. An instance (G, r, ℓ) admits a (not necessarily acyclic) Nash flow f by Proposition 2.5.1. We will first show that flow cycles must comprise only zero-latency edges, and will then show how to remove such cycles.

For a commodity i, define the s_i-v distance $d^i(v)$ of a vertex v with respect to the flow f as in Proposition 2.6.1. By Proposition 2.6.1 and nonnegativity of edge latencies, if edge $e = (v, w)$ carries flow then $d^i(w) \geq d^i(v)$. Thus, in a directed cycle C of flow edges (i.e., of edges e satisfying $f_e^i > 0$), all vertices of C have equal d^i-values and hence (again by Proposition 2.6.1) all edges of C must have zero latency with respect to f.

We next wish to remove zero-latency flow cycles from f; this is not entirely trivial as the flow on different edges of a flow cycle may be carried by different flow paths (recall f is defined as a function on paths, rather than on edges). We extract a new feasible flow $\tilde{f}$ from f by running the following procedure for $i = 1, 2, \ldots, k$:

(1) view f^i as a function on edges with $f^i_e = \sum_{P \in \mathcal{P}_i : e \in P} f_P$

(2) repeatedly discard flow cycles from f^i to obtain an s_i-t_i flow $\bar{f}^i$ (still defined only on edges) without flow cycles

(3) let $\tilde{f}^i$ be an arbitrary path decomposition of $\bar{f}^i$.

The reader unfamiliar with path decompositions and the discarding of flow cycles should consult any text on network flow, such as Tarjan [179].

The flow $\tilde{f}$ is acyclic by construction and is feasible for (G, r, ℓ) since only flow cycles were removed from the feasible flow f; it remains only to show that $\tilde{f}$ is at Nash equilibrium. For each edge e, we have either $f_e = \tilde{f}_e$ or $\tilde{f}_e < f_e$ with $\ell_e(f_e) = 0$ (and hence $\ell_e(\tilde{f}_e) = 0$). It follows that $\ell_e(\tilde{f}_e) = \ell_e(f_e)$ for every edge e, which in turn implies that the flows f and $\tilde{f}$ induce identical d^i-values on the vertices of G for every commodity i. Appealing to the characterization of Nash flows given in Proposition 2.6.1, that f is a Nash flow implies that $\tilde{f}$ is, as well. ■

Part II

Bounding the Price of Anarchy

Chapter 3

How Bad is Selfish Routing?

3.1 Introduction

In this chapter we quantify the inefficiency of Nash equilibria in the traffic model described in the previous two chapters. Recall from our previous examples that traffic flows at Nash equilibrium (flows in which no network user has an incentive to reroute its traffic) do not in general minimize the total latency, our measure of social welfare. In this chapter we study the cost of routing selfishly via the following question: given an arbitrary multicommodity flow network with congestion-dependent edge latencies, what is the worst-possible ratio between the total latency of a flow at Nash equilibrium and that of the best coordinated outcome—of a flow minimizing the total latency?

3.1.1 Summary of Results

As discussed in Subsection 1.3.1, this worst-case ratio ("the price of anarchy" [142]) depends crucially on the "steepness" of the network latency functions. We will present our techniques for computing the price of anarchy in an incremental fashion, with each section considering successively more general classes of edge latency functions.

The simplest nontrivial networks are those with linear latency functions (where every edge latency function is of the form $\ell(x) = ax + b$ for $a, b \geq 0$); for this reason and others, such networks have been studied extensively in the past [71, 72, 143, 144, 175, 176]. Our first result is that in any multicommodity flow network with linear latency functions, the total latency of a flow at Nash equilibrium is at most

Description	Typical Representative	Price of Anarchy
Linear	$ax + b$	$\frac{4}{3} \approx 1.333$
Quadratic	$ax^2 + bx + c$	$\frac{3\sqrt{3}}{3\sqrt{3}-2} \approx 1.626$
Cubic	$ax^3 + bx^2 + cx + d$	$\frac{4\sqrt[3]{4}}{4\sqrt[3]{4}-3} \approx 1.896$
Polynomials of degree $\leq p$	$\sum_{i=0}^{p} a_i x^i$	$\frac{(p+1)\sqrt[p]{p+1}}{(p+1)\sqrt[p]{p+1}-p} = \Theta(\frac{p}{\ln p})$
M/M/1 Delay Functions	$(u - x)^{-1}$	$\frac{1}{2}\left(1 + \sqrt{\frac{u_{min}}{u_{min}-R_{max}}}\right)$
M/G/1 Delay Functions	$\frac{1}{u} + \frac{x(1+\sigma^2 u^2)}{2u(u-x)}$	See Section 3.5

Table 3.1: The price of anarchy for common classes of edge latency functions. Polynomial coefficients are assumed nonnegative. The parameters u and σ are the expectation and standard deviation of the associated queue service rate distribution. R_{max} denotes the maximum allowable amount of network traffic, and u_{min} denotes the minimum allowable edge service rate (or capacity).

$\frac{4}{3}$ times that of a minimum-latency flow (with a matching lower bound provided by any of the first three examples of Section 2.4). We also demonstrate how this result provides a quantitative limit on the extent to which counterintuitive phenomena can occur in certain physical systems, such as the strings and springs example of Figure 1.3.

The assumption of linear latency functions is quite restrictive; we next demonstrate how to enhance the techniques developed for networks with linear latency functions to compute the price of anarchy with respect to an (almost) arbitrary class of latency functions. We apply this generalization to additional classes of latency functions that are well-studied in the networking and queueing theory literature; in particular, all of the results shown in Table 3.1 follow directly from these techniques.

These methods also show that the underlying network topology plays no role in the determination of the price of anarchy. Specifically, we show that under weak hypotheses on the class of allowable latency functions[1], the worst-case ratio between the total latency of a flow at Nash equilibrium and that of a minimum-latency flow in any multicommodity flow network is achieved by a single-commodity instance on

[1]For example, it suffices for the class to satisfy a mild convexity assumption, to be closed under multiplication by positive scalars, and to possess some latency function that is positive when evaluated with zero congestion. Almost all classes of latency functions previously considered in the literature meet the required hypotheses.

a set of parallel links. In the special case of a class of latency functions that includes all of the constant functions, we prove that a network with only two parallel links suffices to achieve the worst-possible ratio. Informally, these results imply that the inefficiency inherent in a flow at Nash equilibrium stems from the inability of selfish users to discern which of two competing routes is superior and *not* from the topological complexity arising from the diverse intersections of many paths belonging to different commodities.

Finally, we employ a bicriteria approach to bound the inefficiency of Nash flows in networks with arbitrary latency functions (where our previous work shows that Nash flows may be arbitrarily more costly than optimal flows). We show that in a multicommodity flow network with latency functions assumed only to be continuous and nondecreasing, the total latency incurred by traffic at Nash equilibrium is at most that of a minimum-latency flow forced to route twice as much traffic between each source-destination pair. We show that this result also has the following alternative interpretation: in lieu of centralized control, the price of routing selfishly can be offset by a moderate increase in link speed (which for queueing delay functions can be effected by a moderate increase in the network capacity).

3.1.2 Related Work

Traffic Equilibria

Unregulated traffic has been modeled as network flow with all flow paths between a given source-destination pair having minimum latency since the 1950's [17, 186] (though as mentioned in Chapter 1, Pigou [148] and Knight [99] informally discussed a similar model thirty years earlier). Wardrop [186] formulated the notions of a flow at Nash equilibrium and of a minimum-latency flow, and suggested that these were the two types of traffic flows deserving special study. Beckmann et al. [17], observing that a flow at Nash equilibrium is an optimal solution to a related convex program (see Section 2.3), gave existence and uniqueness results for traffic equilibria. Dafermos and Sparrow [50] reproved many of the results of [17] and were also interested in computing flows at Nash equilibrium efficiently; in particular, they proposed two iterative algorithms for computing Nash flows and proved convergence results for certain networks.

Since these early works, the traffic model studied in this dissertation has been generalized in many different directions. In the transportation science literature, this model has been generalized to allow for the latency of an edge to depend on the entire traffic pattern (rather than merely on the flow using that edge), to include different types (or "modes") of traffic, to allow elastic (rather than fixed) traffic rates, and so on; research on these more general models have focused on establishing existence and uniqueness of traffic equilibria [1, 26, 32, 45, 46, 47, 66, 67, 68, 84, 85, 102, 129, 173, 192], on designing algorithms to compute an equilibrium [1, 11, 18, 45, 46, 65, 66, 67, 68, 69, 78, 115, 118, 129, 134, 135, 187, 192], and on sensitivity analysis [45, 49, 83, 190]. For an introduction to this literature, we recommend the survey of Florian and Hearn [68] and the book of Sheffi [169]. More recently, Nesterov [131] (see also Nesterov and De Palma [132]) has proposed an interesting alternative to (rather than a generalization of) the traffic model considered in this thesis.

In the networking literature, many recent papers alter the model considered here by relaxing the assumption that every network user controls a negligible fraction of the overall traffic (as will we in Sections 4.2 and 4.3); rather, a finite number of users each control a positive amount of flow. Most of these papers allow players to split their flow among several routes but disallow randomization, and then give necessary and sufficient conditions (on the network topology, the amount of flow that users control, and on the edge latency functions) for the existence and uniqueness of (pure-strategy) Nash equilibria [4, 8, 25, 59, 138] and for convergence to a Nash equilibrium under natural models of user behavior [5, 138]. Some of these results have been extended to networks in which users cannot split flow and must instead route all of their traffic on a single path [117].

Finally, the game theory community has generalized the traffic model studied here to broad classes of games that need not take place in a network. This literature (which we will review in Section 4.4, when we study a related class of games) aims to identify games that enjoy many of the desirable properties possessed by traffic equilibria, such as existence of Nash equilibria in pure strategies, rather than to model any specific type of application.

The Price of Anarchy

In contrast to the previous work on traffic equilibria, we are interested in quantifying the difference in social welfare between equilibrium and optimal traffic flows. The idea of bounding the inefficiency of Nash equilibria was first proposed by Koutsoupias and Papadimitriou [108] for the following simple load-balancing model. A finite number of users share a collection of parallel links, and each user chooses a probability distribution on the set of links (specifying the probability that the user will route all of its flow on a given link). Each user wishes to minimize the expected congestion it will experience, while the global objective is to minimize the expected load on the most congested edge; the worst-case Nash equilibrium is then compared to a globally optimal choice of distributions. Koutsoupias and Papadimitriou [108] obtained a tight analysis of this worst-case ratio (which they call the *coordination ratio*) in two-node, two-link networks and partial results for two-node networks with three or more parallel links; tight results were subsequently found for parallel networks with any number of links by Mavronicolas and Spirakis [122] (for a special case) and by Czumaj and Vöcking [44] (for the general case). More recently, the original model of [108] has been generalized by Czumaj et al. [43] (for example, to include more general objective functions), who also prove a variety of facts about the coordination ratio in this more general model.

3.1.3 Organization

In Sections 3.2–3.5 we present our methods for computing the price of anarchy; these sections should be read in order. In Section 3.2 we study networks with linear latency functions and prove that the price of anarchy for such networks is $\frac{4}{3}$. We also demonstrate the connection between networks with linear latency functions and the networks of strings and springs advertised in Subsection 1.3.1. In Sections 3.3 and 3.4 we generalize these techniques to show that the price of anarchy is independent of the network topology, and in Section 3.5 we show how this fact permits computation of the price of anarchy with respect to an arbitrary class of latency functions. Section 3.5 also illustrates our techniques by computing the price of anarchy for important classes of latency functions not considered earlier.

Finally, in Section 3.6 we prove that a Nash flow in a network with arbitrary latency functions costs no more than an optimal flow forced to route twice as much traffic. This section does not depend on earlier results of this chapter and can be read immediately following Section 2.2.

3.2 The Price of Anarchy with Linear Latency Functions

In this section, we consider the scenario where the latency of each edge e is linear in the edge congestion—that is, where for each edge $e \in E$, $\ell_e(x) = a_e x + b_e$ for some $a_e, b_e \geq 0$. This is the setting in which Braess's paradox was originally discovered [28, 128], and several subsequent papers focused entirely on this model [71, 72, 143, 144, 175, 176]. In addition, linear latency functions are important for other applications: we will see later in this section that the mechanical networks of strings and springs described in Subsection 1.3.1 can be modeled as traffic networks with linear latency functions, and Friedman [74] shows how linear latency functions naturally arise in a simple model of selfish users transferring files over a network employing a congestion control protocol (such as TCP).

We have already seen in Subsections 2.4.1–2.4.3 three examples with linear latency functions for which the ratio of the cost of a flow at Nash equilibrium and the cost of an optimal flow is $\frac{4}{3}$. Our main result for this section (Theorem 3.2.6) is a matching upper bound for networks with linear latency functions. The proof techniques of this section will also form the basis for our subsequent work bounding the price of anarchy for networks with nonlinear latency functions.

To make these statements precise, we require some additional notation. For an instance (G, r, ℓ) admitting an optimal flow f^* and a flow at Nash equilibrium f, we denote the ratio $\frac{C(f)}{C(f^*)}$ by $\rho = \rho(G, r, \ell)$; this ratio is well defined by Corollary 2.5.3.

We begin by noting that the propositions of Section 2.3 have particularly simple and useful forms in the special case of networks with linear latency functions. First, the total latency $C(f)$ of a flow f is given by $C(f) = \sum_e a_e f_e^2 + b_e f_e$; since $a_e \geq 0$ for all e, the nonlinear program (NLP) of Section 2.3 is a convex (quadratic) program and thus Proposition 2.3.1 characterizes its optimal solutions. Also, in the notation of Section 2.3, if the latency function ℓ_e of edge e is $\ell_e(x) = a_e x + b_e$, then the marginal cost function ℓ_e^* of e is simply $\ell_e^*(x) = 2a_e x + b_e$. For convenience, we summarize this discussion together with specialized versions of Propositions 2.2.2 and 2.3.1 in the following lemma.

Lemma 3.2.1 *Let (G, r, ℓ) be an instance with edge latency functions $\ell_e(x) = a_e x + b_e$ for each $e \in E$. Then,*

(a) a feasible flow f is at Nash equilibrium for (G, r, ℓ) if and only if for each

commodity i and $P, P' \in \mathcal{P}_i$ with $f_P > 0$,

$$\sum_{e \in P} a_e f_e + b_e \leq \sum_{e \in P'} a_e f_e + b_e$$

(b) a feasible flow f^* is optimal for (G, r, ℓ) if and only if for each commodity i and $P, P' \in \mathcal{P}_i$ with $f^*_P > 0$,

$$\sum_{e \in P} 2a_e f^*_e + b_e \leq \sum_{e \in P'} 2a_e f^*_e + b_e.$$

As an aside, we note that Lemma 3.2.1 immediately gives a simple proof of the following nontrivial result regarding networks in which the latency of each edge is proportional to its congestion; this result is implicit in the work of Dafermos and Sparrow [50], and other properties of this special case have been investigated in the context of electrical networks [24, 39].

Corollary 3.2.2 *Let G be a network in which each edge latency function ℓ_e is of the form $\ell_e(x) = a_e x$. Then for any rate vector r, a flow feasible for (G, r, ℓ) is optimal if and only if it is at Nash equilibrium.*

Proof. A feasible flow for such an instance satisfies the conditions of Lemma 3.2.1(a) if and only if it satisfies the conditions of Lemma 3.2.1(b). ∎

A second corollary of Lemma 3.2.1 will play a crucial role in our proof of the main theorem of this section.

Lemma 3.2.3 *Suppose (G, r, ℓ) has linear latency functions and f is a flow at Nash equilibrium. Then,*

(a) the flow $f/2$ is optimal for $(G, r/2, \ell)$

(b) the marginal cost of increasing the flow on a path P with respect to $f/2$ equals the latency of P with respect to f.

Proof. For part (a), simply note that if f satisfies the conditions of Lemma 3.2.1(a) for (G, r, ℓ), then $f/2$ satisfies the conditions of Lemma 3.2.1(b) for $(G, r/2, \ell)$. For the second part, recall that if edge e has latency function $\ell_e(x) = a_e x + b_e$ then e has marginal cost function $\ell^*_e(x) = 2a_e x + b_e$. Thus, $\ell^*_e(f_e/2) = \ell_e(f_e)$ for each edge e and hence $\ell^*_P(f/2) = \ell_P(f)$ for each path P. ∎

An outline of the proof of the main theorem is as follows. It will be useful to think about creating an optimal flow for the instance (G, r, ℓ) via a two-step process: in the first step, a flow optimal for the instance $(G, r/2, \ell)$ is sent through G (which from Lemma 3.2.3(a) we know to be simply half of a Nash flow for (G, r, ℓ)), and in the second step this flow is augmented to one optimal for (G, r, ℓ). It is important to note that this augmentation may increase *or decrease* the amount of flow on any given edge—e.g., in Braess's Paradox (Figure 2.2) the Nash flow f (and hence the flow $f/2$) makes use of the zero-latency edge (v, w) while the optimal flow eschews it. We will show that the first flow has cost at least $\frac{1}{4}C(f)$ and that the augmentation has cost at least $\frac{1}{2}C(f)$, where f is some flow at Nash equilibrium.

We will see in the proof of Theorem 3.2.6 that the first lower bound follows easily from Lemma 3.2.3(a), but the second (for the cost of the augmentation, given that the first flow has already been routed) requires more work, and in particular the following lemma. Intuitively, the lemma simply claims that the per-unit cost of increasing the amount of flow through a network is at least the marginal cost of increasing flow on any path with respect to the current optimal flow.

Lemma 3.2.4 *Suppose (G, r, ℓ) is an instance with linear latency functions for which f^* is an optimal flow. Let $L_i^*(f^*)$ be the minimum marginal cost of increasing flow on an s_i-t_i path with respect to f^*. Then for any $\delta > 0$, a feasible flow for the instance $(G, (1+\delta)r, \ell)$ has cost at least*

$$C(f^*) + \delta \sum_{i=1}^{k} L_i^*(f^*) r_i.$$

Proof. A heuristic proof of this lemma is as follows. Since the marginal cost of increasing flow on any s_i-t_i path with respect to f^* is at least $L_i^*(f^*)$, routing δr_i additional units of flow from s_i to t_i should cost at least $\delta L_i^*(f^*) r_i$. Summing over all commodities i should then yield the lemma. This argument provides good intuition for why the lemma is true, but is not sufficient because not all feasible flows for $(G, (1+\delta)r, \ell)$ are obtainable from f^* by simply routing additional flow through the network.

To prove the lemma, fix $\delta > 0$ and suppose f is feasible for $(G, (1+\delta)r, \ell)$. In general f_e may be larger or smaller than f_e^*. For any edge $e \in E$, convexity of the function $x \cdot \ell_e(x) = a_e x^2 + b_e x$ implies that

$$\ell_e(f_e) f_e \geq \ell_e(f_e^*) f_e^* + (f_e - f_e^*) \ell_e^*(f_e^*).$$

In essence, this inequality states that estimating the cost of changing the flow value on edge e from f_e^* to f_e by $(f_e - f_e^*)\ell_e^*(f^*)$ (i.e., by the marginal cost of flow increase at f_e^* times the size of the perturbation) only underestimates the actual cost of an increase (when $f_e > f_e^*$) and overestimates the actual benefit of a decrease (when $f_e < f_e^*$). We may thus derive

$$\begin{aligned} C(f) &= \sum_{e\in E} \ell_e(f_e)f_e \\ &\geq \sum_{e\in E} \ell_e(f_e^*)f_e^* + \sum_{e\in E}(f_e - f_e^*)\ell_e^*(f_e^*) \\ &= C(f^*) + \sum_{i=1}^{k}\sum_{P\in\mathcal{P}_i} \ell_P^*(f^*)(f_P - f_P^*). \end{aligned}$$

Since we have $\ell_P^*(f^*) \geq L_i^*(f^*)$ for each i and each $P \in \mathcal{P}_i$ and equality holding whenever $f_P^* > 0$ (see Lemma 3.2.1(b)), we obtain

$$\begin{aligned} C(f) &\geq C(f^*) + \sum_{i=1}^{k} L_i^*(f^*) \sum_{P\in\mathcal{P}_i}(f_P - f_P^*) \\ &= C(f^*) + \delta\sum_{i=1}^{k} L_i^*(f^*)r_i, \end{aligned}$$

completing the proof. ■

Remark 3.2.5 Lemma 3.2.4 and its proof remain valid in much more general settings; all that is required is convexity of the function $x \cdot \ell_e(x)$ for each edge e (i.e., that each latency function is *standard*—recall Definition 2.3.5).

We are now prepared to prove the main theorem.

Theorem 3.2.6 *If (G, r, ℓ) has linear latency functions, then $\rho(G, r, \ell) \leq \frac{4}{3}$.*

Proof. Let f be a flow at Nash equilibrium for (G, r, ℓ). Let $L_i(f)$ be the latency of an s_i-t_i flow path, so that $C(f) = \sum_i L_i(f)r_i$ (see Proposition 2.2.4). By Lemma 3.2.3(a), $f/2$ is an optimal flow for the instance $(G, r/2, \ell)$. Moreover, by Lemma 3.2.3(b), $L_i^*(f/2) = L_i(f)$ for each i—in words, marginal costs with respect to $f/2$ and latencies with respect to f coincide. This establishes the necessary connection between the cost of augmenting $f/2$ to a flow feasible for (G, r, ℓ) and the cost of a flow at Nash equilibrium.

Taking $\delta = 1$ in Lemma 3.2.4, we find that the cost of any flow f^* feasible for (G, r, ℓ) satisfies

$$C(f^*) \;\geq\; C(f/2) + \sum_{i=1}^{k} L_i^*(f/2)\frac{r_i}{2}$$

$$
\begin{aligned}
&= C(f/2) + \frac{1}{2}\sum_{i=1}^{k} L_i(f) r_i \\
&= C(f/2) + \frac{1}{2}C(f).
\end{aligned}
$$

Finally, it's easy to lower bound the cost of $f/2$:

$$
\begin{aligned}
C(f/2) &= \sum_e \frac{1}{4}a_e f_e^2 + \frac{1}{2}b_e f_e \\
&\geq \frac{1}{4}\sum_e a_e f_e^2 + b_e f_e \\
&= \frac{1}{4}C(f)
\end{aligned}
$$

and thus $C(f^*) \geq \frac{3}{4}C(f)$. ■

We note that the analysis of this section can easily be extended to prove that in an instance (G, r, ℓ) where for some p, $\ell_e(x) = a_e x^p + b_e$ (with $a_e, b_e \geq 0$) for each edge e, $\rho(G, r, \ell) \leq (1 - p \cdot (p+1)^{-(p+1)/p})^{-1} = \Theta(\frac{p}{\ln p})$. The nonlinear variant of Pigou's example (Subsection 2.4.4) shows that this result is tight. In Section 3.5 we will see that this upper bound holds more generally for instances with polynomial latency functions with nonnegative coefficients and any number of terms with degree at most p.

Consequences for Strings and Springs

We now return to the mechanical networks of strings and springs discovered by Cohen and Horowitz [36] and discussed in Subsection 1.3.1 and Figure 1.3. Viewing the support as a source and the suspended weight as a destination, with each string and spring as an edge, the equilibrium position of the mechanical device can be modeled as a flow at Nash equilibrium in a traffic network G, with force corresponding to flow and support-weight distance corresponding to the common latency of every source-destination flow path. Strings (as perfectly inelastic objects) are modeled as links with constant latency functions while (perfectly elastic) springs correspond to links with latency functions that include a term of the form ax. Severing a string or spring corresponds to deleting an edge from a traffic network; thus any realizable equilibrium of the mechanical network (after possibly destroying some of its constituent parts) corresponds to a Nash equilibrium in a subgraph of the corresponding traffic network G.

Although Theorem 3.2.6 is concerned with the total latency of flows (a concept with no natural analogue in our mechanical networks), we can use the result in the following way. By Theorem 3.2.6, every traffic flow in G (and in particular every flow at Nash equilibrium in a subgraph of G) has total latency at least $\frac{3}{4}$ times that of a Nash flow f in G. By Proposition 2.2.4, it follows that if the common latency of every flow path of f is L and f^* is a flow at Nash equilibrium in a subgraph of G, then the common latency of every flow path of f^* is at least $\frac{3}{4}L$. Reinterpreting this result for networks of strings and springs, we obtain the following corollary of Theorem 3.2.6.

Corollary 3.2.7 *In any network of strings and springs carrying a single weight with support-weight distance D, the support-weight distance after severing an arbitrary collection of strings and springs is at least $\frac{3}{4}D$.*

Cohen and Horowitz [36] also showed, by an analogous construction, that removing a diode from a two-terminal electrical network of resistors and diodes can decrease the voltage drop from source to ground—thus removing a conducting link can increase the network conductivity. By the same arguments as above, Theorem 3.2.6 implies that the voltage drop from source to ground in such an electrical network after removing any number of resistors and diodes is at least $\frac{3}{4}$ times the voltage drop in the original network.

3.3 The Price of Anarchy with Standard Latency Functions

The goal of this section is to provide an upper bound on the worst-case ratio between the cost of a Nash flow and of an optimal flow for instances with nonlinear latency functions. As we have seen in the nonlinear variant of Pigou's example (Subsection 2.4.4), the price of anarchy depends crucially on how "steep" the allowable latency functions can be, and one may therefore ask whether any meaningful upper bound is possible for networks with arbitrary latency functions. The answer is affirmative, provided that the upper bound is a *function of the class of allowable latency functions.*

To state the main result of this section precisely, recall that $\rho(G, r, \ell)$ denotes the ratio between the cost of a Nash and of an optimal flow for instance (G, r, ℓ). We will associate a real number $\alpha(\mathcal{L}) \geq 1$ to each class $\mathcal{L}$ of allowable edge latency functions that quantifies the "steepness" of the latency functions in $\mathcal{L}$, and will then prove that for any instance (G, r, ℓ) with latency functions in the class $\mathcal{L}$, $\rho(G, r, \ell) \leq \alpha(\mathcal{L})$. In Section 3.4 we will provide a matching lower bound, by exhibiting (for any class $\mathcal{L}$) instances with latency functions in $\mathcal{L}$ and ρ-value arbitrarily close to $\alpha(\mathcal{L})$.

3.3.1 The Anarchy Value

Our first task is to find a definition that captures how "steep" a given class of allowable latency functions is. A first attempt attempt might involve the first several derivatives of the latency functions; for example, we might hope to prove that if the first few derivatives of all allowable latency functions are everywhere bounded by some universal constant, then the price of anarchy is constant. However, any instance (G, r, ℓ) with latency functions of class C^k (latency functions that are k times continuously differentiable) can be "scaled down" to an instance $(G, r, \frac{1}{M}\ell)$ in which the first k derivatives of all edge latency functions are as small as desired (by taking M sufficiently large). Moreover, $\rho(G, r, \frac{1}{M}\ell) = \rho(G, r, \ell)$ since a feasible flow is optimal (respectively, Nash) for $(G, r, \frac{1}{M}\ell)$ if and only if is optimal (respectively, Nash) for (G, r, ℓ). Thus, networks with latency functions with (any number of) bounded derivatives are not "better-behaved" than networks with polynomial latency functions; and from the nonlinear variant of Pigou's example (Subsection 2.4.4), we already know that networks with polynomial latency functions and no upper bound on the allowable degree are not well-behaved at all.

Before giving our definition capturing how "steep" a given class of allowable latency functions is (which, admittedly, is not immediately intuitive), we will consider a motivating example. It will be convenient to apply Corollary 2.3.2 to compute the optimal flow in this example; for this reason and others that will become clear later in this section, we restrict ourselves to networks with standard latency functions (see Definition 2.3.5). Since the focus of this section is on classes of allowable latency functions, we make another definition.

Definition 3.3.1 A class $\mathcal{L}$ of latency functions is *standard* if $\mathcal{L}$ contains a nonzero function and if each $\ell \in \mathcal{L}$ is standard.

We now introduce the motivating example. Suppose we are given a standard class $\mathcal{L}$ of allowable latency functions, and wish to construct a network in which the Nash flow incurs much more latency than the optimal flow. A natural idea is to mimic the bad example of Subsection 2.4.4 as best we can, given that $\mathcal{L}$ is the class of latency functions that we are allowed to work with. For simplicity, assume that the constant function $\ell_1(x) = 1$ lies in $\mathcal{L}$. Then, we can consider the usual two-node, two-link network, assign the first link the latency function ℓ_1 and the second link the "steepest" latency function that we can find. More formally, suppose $\ell_2 \in \mathcal{L}$ is assigned to the second link where ℓ_2 satisfies $\ell_2(0) < 1$ and $\ell_2(x) > 1$ for x sufficiently large. Choosing $r > 0$ to satisfy $\ell_2(r) = 1$, we find that a Nash flow with traffic rate r routes all of its flow on the second edge for a total latency of r. Using Corollary 2.3.2 and letting $\lambda \in (0, 1)$ satisfy $\ell_2^*(\lambda r) = 1$, we find that the optimal flow routes λr units of flow on the second link and $(1 - \lambda)r$ units of flow on the first link, for a total latency of $\lambda r \ell_2(\lambda r) + (1 - \lambda)r$. Letting $\mu \in [0, 1]$ denote $\ell_2(\lambda r)$, the ratio between the total latency of the Nash flow and of the optimal flow is $[\lambda\mu + (1 - \lambda)]^{-1}$. Taking into account that this argument can be used with ℓ_1 replaced by any constant function, and that $\ell_2 \in \mathcal{L}$ was chosen arbitrarily, we arrive at the following definition.

Definition 3.3.2 Let ℓ be a nonzero standard latency function. The *anarchy value* $\alpha(\ell)$ of ℓ is

$$\alpha(\ell) = \sup_{r>0\,:\,\ell(r)>0} [\lambda\mu + (1 - \lambda)]^{-1}$$

where $\lambda \in (0, 1)$ satisfies $\ell^*(\lambda r) = \ell(r)$ and $\mu \in [0, 1]$ is defined by $\mu = \ell(\lambda r)/\ell(r)$.

That the scalar $\lambda \in (0, 1)$ exists follows from the Intermediate Value Theorem and the fact that $\ell^*(0) = \ell(0) \leq \ell(r) \leq \ell^*(r)$. In most cases of interest λ will be uniquely determined by ℓ and r; otherwise, our assumption that ℓ is standard ensures that the anarchy value is well defined (i.e., that $[\lambda\mu + (1 - \lambda)]^{-1}$ is independent of the choice of λ satisfying $\ell^*(\lambda r) = \ell(r)$).

The anarchy value of a latency function ℓ should be interpreted as the worst possible ratio between the cost of a Nash flow and of an optimal flow in a two-node, two-link network where one edge possesses latency function ℓ and the other possesses a constant latency function; the worst-case is taken over choices of the constant and of the traffic rate.

Since we are interested only in the "steepest" latency functions of a class, the next definition should be unsurprising.

Definition 3.3.3 The *anarchy value* $\alpha(\mathcal{L})$ of a standard class $\mathcal{L}$ of latency functions is

$$\alpha(\mathcal{L}) = \sup_{0 \neq \ell \in \mathcal{L}} \alpha(\ell).$$

Remark 3.3.4

(a) The anarchy value of a class lies in $[1, \infty]$ and need not be finite.

(b) The anarchy value seems a fearsome expression to compute analytically, but we will see in Section 3.5 that it can typically be worked out in cases of practical interest.

(c) There are also simpler definitions of "steepness" that provide nontrivial but suboptimal upper bounds on the price of anarchy; see Sections A.1 and A.2 of Appendix A.

We have already argued informally that if $\mathcal{L}$ is a standard class of latency functions containing the constant functions, then there are instances $\mathcal{I}$ on a network with two nodes and two links and latency functions in $\mathcal{L}$ with ratio ρ arbitrarily close to $\alpha(\mathcal{L})$. On the other hand, there is no reason *a priori* to expect the anarchy value to have any connection to instances defined on more general networks (even to those defined on parallel networks with more than two links). The central result of this section is that, under very weak conditions on the class of allowable latency functions, $\alpha(\mathcal{L})$ upper bounds $\rho(G, r, \ell)$ for *any* instance (G, r, ℓ) with latency functions in $\mathcal{L}$ (with an arbitrary network topology and an arbitrary number of commodities).

3.3.2 Proof Approach

We next discuss our proof approach. At the highest level, the proof of this section is inspired by that of the last section, which shows that the price of anarchy for networks with linear latency functions is precisely $\frac{4}{3}$. Let us recapitulate the three main steps of that proof. First, we used the characterizations of Nash and optimal flows (via Corollary 2.3.2) to show that if f is a flow at Nash equilibrium for an instance (G, r, ℓ) with linear latency functions, then the scaled-down flow $f/2$ is optimal for the instance $(G, r/2, \ell)$. Second, we lower bounded the cost of $f/2$ in terms of the cost of f; this was not difficult since the scaled-down flow $f/2$ was a "significant fraction" of f. Finally, we lower bounded the cost of augmenting the flow $f/2$ to a flow optimal for (G, r, ℓ) in terms of the cost of f. This was the most difficult part of the proof; roughly, we leveraged the connection between Nash and optimal flows given in Corollary 2.3.2 to show that the marginal cost of routing new flow with respect to $f/2$ is high, and thus augmenting the flow $f/2$ to a flow feasible for the full set of traffic rates r is costly.

A direct attempt at adapting the three-step approach of the previous section to more general latency functions fails immediately. In particular, for nonlinear latency functions (even for quadratic latency functions), there is no constant c for which a scaled-down version f/c of a Nash flow f is optimal for the reduced traffic rates r/c. Thus, it is not at all clear how to exploit our characterizations of Nash and optimal flows to relate their respective costs. To circumvent this problem, we view the proof approach of the previous section in the following more general way: chop up an optimal flow into two "pieces" (in the linear latency case, $f/2$ and an augmentation from $f/2$ to a flow feasible for rates r) such that each piece can be lower-bounded in terms of the cost of a Nash flow. Guided by a desire to define the second piece of the optimal flow as an augmentation of the first and to lower bound its cost by means of marginal cost functions (as in the linear latency case), we will define the first piece in a way that ensures that any augmentation with respect to it has large marginal cost. Unfortunately, this requires scaling down a Nash flow f by *different factors* on different edges, thereby producing an object which is *not* a flow (it is a more general object that need not obey conservation constraints, which we call a *pseudoflow*). While this does not significantly complicate the lower bound for the cost of the scaled-down pseudoflow (it is a "significant fraction" of the Nash flow, as before), a more careful analysis is now required to lower bound the cost of an augmentation from the scaled-down pseudoflow to a flow feasible for the original instance (as we are augmenting with respect to an object more complicated than simply a flow at reduced traffic rates).

3.3.3 Proof of Upper Bound

We now turn toward making these ideas precise. We first define what we mean by a "scaled-down pseudoflow". The idea is to scale down the amount of Nash flow on a single edge until the value of the marginal cost function equals the original latency incurred by the Nash flow on that edge (this original latency is then our definition of "large marginal cost"). Formally, if f is a flow at Nash equilibrium, our scaled-down pseudoflow will be defined by $\{\lambda_e f_e\}_{e \in E}$ where λ_e satisfies $\ell_e^*(\lambda_e f_e) = \ell_e(f_e)$ (as in Definition 3.3.2). As discussed following Definition 3.3.2, these scaling factors always exist but need not be unique; our analysis must work with an arbitrary choice of scaling factors.

The next lemma formalizes the notion of "breaking up the optimal flow into two pieces". Again, the idea is to express the cost of the optimal flow as one term that is a scaled-down version of a Nash flow, and a second term that corresponds to an augmentation with respect to large marginal costs.

Lemma 3.3.5 *Let f^* and f be optimal and Nash flows, respectively, for instance (G, r, ℓ) with standard latency functions. For an edge e, let $\lambda_e \in (0,1)$ be a solution to $\ell_e^*(\lambda_e f_e) = \ell_e(f_e)$. Then,*

$$C(f^*) \geq \sum_e \left[\ell_e(\lambda_e f_e)\lambda_e f_e + (f_e^* - \lambda_e f_e)\ell_e(f_e)\right].$$

Proof. Since each edge latency function ℓ_e is standard, each marginal cost function ℓ_e^* is nondecreasing. For an edge e, we may thus write

$$\begin{aligned}
\ell_e(f_e^*)f_e^* &= \ell_e(\lambda_e f_e)\lambda_e f_e + \int_{\lambda_e f_e}^{f_e^*} \ell_e^*(x)dx \\
&\geq \ell_e(\lambda_e f_e)\lambda_e f_e + (f_e^* - \lambda_e f_e)\ell_e^*(\lambda_e f_e) \\
&= \ell_e(\lambda_e f_e)\lambda_e f_e + (f_e^* - \lambda_e f_e)\ell_e(f_e)
\end{aligned}$$

with the final equality holding by the definition of λ_e. Summing over all edges proves the lemma. ■

Note that neither the statement nor the proof of Lemma 3.3.5 assumes that the expression $f_e^* - \lambda_e f_e$ is nonnegative for all edges e; as in the previous section, the augmentation from the pseudoflow defined by $\{\lambda_e f_e\}_{e \in E}$ to a flow f^* optimal for the original instance may increase or decrease the amount of flow on an edge.

To lower bound the right-hand side of Lemma 3.3.5, we require two more easy lemmas. The next lemma simply rephrases Definitions 3.3.2 and 3.3.3.

Lemma 3.3.6 *Let $\mathcal{L}$ be a standard class of latency functions with anarchy value $\alpha(\mathcal{L})$. For $\ell \in \mathcal{L}$ and $f > 0$, let $\lambda \in (0,1)$ satisfy $\ell^*(\lambda f) = \ell(f)$ and define μ by $\mu = \ell(\lambda f)/\ell(f)$ (if $\ell(f) = 0$, put $\mu = 1$). Then $\lambda\mu + (1-\lambda) \geq \frac{1}{\alpha(\mathcal{L})}$.*

Our final lemma states that if f is a Nash flow for (G, r, ℓ), then f is a min-cost flow (in the classical sense of network flow theory [179]) with respect to the cost vector $\ell_e(f_e)$.

Lemma 3.3.7 *Let f be at Nash equilibrium and f^* feasible for instance (G, r, ℓ). Then,*

$$\sum_e \ell_e(f_e)f_e \leq \sum_e \ell_e(f_e)f_e^*.$$

Proof. Let $L_i(f)$ denote the common latency of every s_i-t_i flow path of f, so that

$$\sum_e \ell_e(f_e) f_e = C(f) = \sum_{i=1}^{k} L_i(f) r_i$$

by Proposition 2.2.2. Since f is at Nash equilibrium, $\ell_P(f) \geq L_i(f)$ for every s_i-t_i path P; we may thus write

$$\sum_e \ell_e(f_e) f_e^* = \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} \ell_P(f) f_P^* \geq \sum_{i=1}^{k} L_i(f) r_i,$$

proving the lemma. ■

With all of the preliminaries now in place, we state and prove the main result of this section: the anarchy value of a standard class $\mathcal{L}$ of latency functions upper bounds the ratio ρ for any instance with latency functions in $\mathcal{L}$.[2]

Theorem 3.3.8 *Let $\mathcal{L}$ be a standard class of latency functions with anarchy value $\alpha(\mathcal{L})$. Let (G, r, ℓ) denote an instance with latency functions drawn from $\mathcal{L}$. Then $\rho(G, r, \ell) \leq \alpha(\mathcal{L})$.*

Proof. By Lemma 3.3.5 we have

$$\begin{aligned} C(f^*) &\geq \sum_e [\ell_e(\lambda_e f_e)\lambda_e f_e + (f_e^* - \lambda_e f_e)\ell_e(f_e)] \\ &= \sum_e [\mu_e \lambda_e f_e + (1-\lambda_e) f_e + (f_e^* - f_e)]\ell_e(f_e) \\ &= \sum_e [\mu_e \lambda_e f_e + (1-\lambda_e) f_e]\ell_e(f_e) + \sum_e [f_e^* - f_e]\ell_e(f_e) \end{aligned}$$

where $\lambda_e \in (0,1)$ is chosen (arbitrarily) to satisfy $\ell_e^*(\lambda_e f_e) = \ell_e(f_e)$ and where $\mu_e \in [0,1]$ is defined by $\mu_e = \ell_e(\lambda_e f_e)/\ell_e(f_e)$ (if $\ell_e(f_e) = 0$, put $\mu_e = 1$). The second sum is nonnegative by Lemma 3.3.7, so we may derive

$$C(f^*) \geq \sum_e [\mu_e \lambda_e f_e + (1-\lambda_e) f_e]\ell_e(f_e);$$

applying Lemma 3.3.6 we obtain

$$\begin{aligned} C(f^*) &\geq \sum_e [\mu_e \lambda_e + (1-\lambda_e)]\ell_e(f_e) f_e \\ &\geq \frac{1}{\alpha(\mathcal{L})} \sum_e \ell_e(f_e) f_e \\ &= \frac{C(f)}{\alpha(\mathcal{L})} \end{aligned}$$

and the theorem is proved. ■

[2] We are indebted to Amir Ronen for substantially simplifying our original proof of this theorem.

3.4 The Price of Anarchy is Independent of the Network Topology

With Theorem 3.3.8 in hand, it is now a relatively easy matter to prove that the price of anarchy is independent of the network topology. In Subsection 3.4.1 we prove that, with respect to a standard class of allowable edge latency functions that contains the constant functions, the worst possible value of $\rho(G, r, \ell)$ for a multicommodity instance (G, r, ℓ) is realized (up to an arbitrarily small additive factor) by a single-commodity instance on a two-node, two-link network. In Subsection 3.4.2, we prove that under significantly weaker conditions on the class of allowable latency functions, the worst-case value of $\rho(G, r, \ell)$ is achieved (again, up to an arbitrarily small factor) by a single-commodity instance on a network of parallel links.

3.4.1 Lower Bounds in Two-Link Networks

We begin by formalizing an argument of the previous section; the following lemma is essentially a restatement of Definitions 3.3.2 and 3.3.3.

Lemma 3.4.1 *Let G_2 denote the graph with one source vertex, one destination vertex, and two edges directed from source to destination. Let $\mathcal{L}$ denote a standard class of latency functions containing the constant functions, with anarchy value $\alpha(\mathcal{L})$. If $\mathcal{I}_2$ denotes the set of all instances with underlying network G_2 and latency functions in $\mathcal{L}$, then*

$$\sup_{(G_2,r,\ell)\in\mathcal{I}_2} \rho(G_2, r, \ell) \geq \alpha(\mathcal{L}).$$

Proof. We will assume that $\alpha(\mathcal{L})$ is finite, and will leave the straightforward modifications necessary for the $\alpha(\mathcal{L}) = +\infty$ case to the interested reader.

For any $\epsilon > 0$, choose a nonzero latency function $\ell_1 \in \mathcal{L}$, a positive number $r > 0$ with $\ell_1(r) > 0$, and a scalar $\lambda \in (0, 1)$ satisfying $\ell_1^*(\lambda r) = \ell_1(r)$ so that $[\lambda\mu + (1-\lambda)]^{-1} \geq \alpha(\mathcal{L}) - \epsilon$, where $\mu = \ell_1(\lambda r)/\ell_1(r)$. Let $\ell_2 \in \mathcal{L}$ denote the constant function that is everywhere equal to $\ell_1(r)$. Define an instance on G_2 with latency functions ℓ_1 and ℓ_2 and traffic rate r. The total latency incurred by the Nash flow is $\ell_1(r)r$, while that of the optimal flow is $\ell_1(r)r[\lambda\mu + (1 - \lambda)]$; hence the ρ-value of this instance is at least $\alpha(\mathcal{L}) - \epsilon$. Since $\epsilon > 0$ was arbitrary, the lemma follows. ■

Combining Theorem 3.3.8 and Lemma 3.4.1, we find that the price of anarchy with respect to a standard class of latency functions containing the constant functions is independent of the class of allowable network topologies (thereby generalizing Theorem 3.2.6 and the matching lower bound of Pigou's example).

Theorem 3.4.2 *Let G_2 denote the graph with one source vertex, one destination vertex, and two edges directed from source to destination. Let $\mathcal{L}$ be a standard class of latency functions containing the constant functions. If $\mathcal{I}$ denotes the set of all instances with latency functions in $\mathcal{L}$ and $\mathcal{I}_2 \subseteq \mathcal{I}$ the instances with underlying network G_2, then*

$$\sup_{(G_2,r,\ell)\in\mathcal{I}_2} \rho(G_2, r, \ell) = \alpha(\mathcal{L}) = \sup_{(G,r,\ell)\in\mathcal{I}} \rho(G, r, \ell).$$

3.4.2 Lower Bounds in Networks of Parallel Links

We now relax the assumption that the class of allowable latency functions contains all of the constant functions, and assume instead the following much weaker condition: for any positive real number a, there is a latency function ℓ satisfying $\ell(0) = a$. We call such a class of latency functions *diverse*. For any class of latency functions that is closed under multiplication by positive scalars[3], diversity merely asserts that some latency function is positive when evaluated at 0. Under these weaker hypotheses, we have the following.

Lemma 3.4.3 *Let G_m denote the graph with one source vertex, one destination vertex, and m edges directed from source to destination. Let $\mathcal{L}$ be a standard and diverse class of latency functions with anarchy value $\alpha(\mathcal{L})$. If $\mathcal{I}_m$ denotes the set of all instances with underlying network G_m and latency functions in $\mathcal{L}$, then*

$$\sup_{(G,r,\ell)\in\cup_m\mathcal{I}_m} \rho(G, r, \ell) \geq \alpha(\mathcal{L}).$$

Proof. We again assume for simplicity that $\alpha(\mathcal{L})$ is finite. For any $\epsilon > 0$, choose a nonzero latency function $\ell_1 \in \mathcal{L}$, a positive number $r > 0$ with $\ell_1(r) > 0$, and a scalar $\lambda \in (0, 1)$ satisfying $\ell_1^*(\lambda r) = \ell_1(r)$ so that $[\lambda\mu + (1 - \lambda)]^{-1} \geq \alpha(\mathcal{L}) - \epsilon/2$, where $\mu = \ell_1(\lambda r)/\ell_1(r)$. Since $\mathcal{L}$ is diverse, there is a function $\ell_2 \in \mathcal{L}$ satisfying $\ell_2(0) = \ell_1(r)$. The main idea of the proof is to use many links, all with latency function ℓ_2, to approximately "simulate" a single link with the constant latency function everywhere equal to $\ell_1(r)$.

[3]Since a scalar multiplication of the latency functions can be effected simply by changing the units in which we measure latency, we expect most classes of interest to satisfy this property.

Let m be so large that $\ell_2(\frac{(1-\lambda)r}{m-1}) \leq \ell_1(r) + \delta$, where δ is a sufficiently small positive number (depending on ϵ) to be chosen later; existence of the integer m follows from continuity of ℓ_2 at 0. Define an instance on G_m with traffic rate r, latency function ℓ_1 on one link, and latency function ℓ_2 on the other $m-1$ links. The total latency incurred by the Nash flow is $\ell_1(r)r$, as all flow is routed on the link with latency function ℓ_1. The flow that routes λr units of flow on the link with latency function ℓ_1 at a cost of $\lambda r \mu \ell_1(r)$ and $(1-\lambda)r/(m-1)$ units of flow on each of the other $m-1$ links has total latency at most $\ell_1(r)r[\lambda\mu + (1-\lambda) + \frac{1-\lambda}{\ell_1(r)}\delta]$, by choice of m. Choosing δ sufficiently small, we obtain an instance with ρ-value at least $\alpha(\mathcal{L}) - \epsilon$. Since $\epsilon > 0$ was arbitrary, the lemma follows. ■

Theorem 3.3.8 and Lemma 3.4.3 together imply the main result of this section.

Theorem 3.4.4 *Let G_m denote the graph with one source vertex, one destination vertex, and m edges directed from source to destination. Let $\mathcal{L}$ be a standard and diverse class of latency functions. If $\mathcal{I}$ denotes the set of all instances with latency functions in $\mathcal{L}$ and $\mathcal{I}_m \subseteq \mathcal{I}$ the instances with underlying network G_m, then*

$$\sup_{(G,r,\ell)\in\cup_m \mathcal{I}_m} \rho(G,r,\ell) = \alpha(\mathcal{L}) = \sup_{(G,r,\ell)\in\mathcal{I}} \rho(G,r,\ell).$$

Remark 3.4.5 The conclusion of the theorem is false with $\cup_m \mathcal{I}_m$ replaced by $\mathcal{I}_2$ (for a counterexample, take $\mathcal{L} = \{\ell(x) = x\} \cup \{\ell(x) = a(1+x) : a > 0\}$). The conclusion of the theorem is also false when the class of allowable latency functions need not be diverse (for a counterexample, let $\mathcal{L} = \{\ell(x) = 1 + x\}$).

Remark 3.4.6 That the price of anarchy is independent of the network topology is a remarkable fact; to better appreciate this, we will foreshadow some forthcoming results. In Chapter 4 we will generalize the traffic model studied in this chapter in several different directions, and we will often see that general network topologies are more poorly-behaved than networks of parallel links. In Chapter 5 we will study Braess's Paradox and generalizations of it, and will discover that the severity of the paradox grows with the network size and cannot occur in networks of parallel links. Finally, in Chapter 6 we will generalize the traffic model studied here to accommodate a different equilibrium concept, Stackelberg equilibria, and will prove that the inefficiency of such equilibria can be strictly larger in general network topologies than in networks of parallel links.

3.5 Computing the Price of Anarchy

In this section, we leverage the results of Sections 3.3 and 3.4 to show that computing the price of anarchy with respect to an (almost) arbitrary standard class of latency functions reduces to computing the anarchy value of the class, even when the diversity condition of Theorem 3.4.4 fails (as in the important case of M/M/1 delay functions with some minimum allowable queue service rate). This provides a general reduction from a combinatorial problem (finding a worst-case instance among all possible multicommodity flow instances) to a simpler analytical one (finding the "steepest" latency function in a given class). Subsection 3.5.1 describes this method, and Subsection 3.5.2 computes the price of anarchy for several important function classes.

3.5.1 More Techniques for Computing the Price of Anarchy

In the previous section, we saw that the price of anarchy with respect to a standard and diverse class of latency functions is precisely the anarchy value of the class (Theorem 3.4.4). In this subsection we will show that this fact remains true under even weaker hypotheses. From a computational perspective, this result has the following interpretation: to compute the price of anarchy with respect to an (almost) arbitrary standard class of latency functions $\mathcal{L}$, it suffices to compute the worst-possible ratio between the cost of a Nash and of an optimal flow in a two-node, two-link network where one link possesses a constant latency function and the other link possesses a latency function of the form $\nu\ell$ for $\ell \in \mathcal{L}$ and a positive scalar $\nu > 0$ (even though $\mathcal{L}$ need *not* contain $\nu\ell$ or any constant functions).

The first step of this reduction is the following lemma, which implies that we can always assume that our class of latency functions is closed under multiplication by positive scalars.

Lemma 3.5.1 *Let $\mathcal{L}$ be a standard class of latency functions, and define $\overline{\mathcal{L}}$ as the closure of $\mathcal{L}$ under multiplication by positive scalars (so $\overline{\mathcal{L}} = \{\nu\ell : \ell \in \mathcal{L}, \nu > 0\}$). Let $\mathcal{I}$ denote the set of instances with latency functions in $\mathcal{L}$, and $\overline{\mathcal{I}}$ the set of instances with latency functions in $\overline{\mathcal{L}}$. Then,*

$$\sup_{(G,r,\ell)\in\mathcal{I}} \rho(G,r,\ell) = \sup_{(G,r,\ell)\in\overline{\mathcal{I}}} \rho(G,r,\ell).$$

Proof. The left-hand side trivially lower bounds the right-hand side since $\overline{\mathcal{L}} \supseteq \mathcal{L}$ and hence $\overline{\mathcal{I}} \supseteq \mathcal{I}$. For the reverse inequality, we will show that for any instance $(\overline{G}, \overline{r}, \overline{\ell}) \in \overline{\mathcal{I}}$ and any $\epsilon > 0$, there is an instance $(G, r, \ell) \in \mathcal{I}$ satisfying $\rho(G, r, \ell) \geq \rho(\overline{G}, \overline{r}, \overline{\ell}) - \epsilon$. Fix an instance $(\overline{G}, \overline{r}, \overline{\ell}) \in \overline{\mathcal{I}}$ and $\epsilon > 0$, and for an edge e of $\overline{G}$ write $\overline{\ell}_e = \nu_e \ell_e$ for $\nu_e > 0$ and $\ell_e \in \mathcal{L}$. The ratio ρ is a continuous function of each scalar ν_e (holding the network $\overline{G}$ and the traffic rate vector $\overline{r}$ fixed), and we may thus replace each ν_e by a sufficiently close positive rational number η_e to obtain a new instance with ρ-value at least $\rho(\overline{G}, \overline{r}, \overline{\ell}) - \epsilon$. Clearing denominators, we may assume that each scalar η_e is a positive integer (multiplying all latency functions of an instance by a common positive number does not affect its ρ-value). Finally, replacing each edge e with a directed path of η_e edges, each endowed with latency function ℓ_e, we obtain a network with latency functions in $\mathcal{L}$ and with ρ-value at least $\rho(\overline{G}, \overline{r}, \overline{\ell}) - \epsilon$. ■

The following observation will also be useful.

Lemma 3.5.2 *Let $\mathcal{L}$ be a standard class of latency functions, and define $\overline{\mathcal{L}}$ as the closure of $\mathcal{L}$ under multiplication by positive scalars. Then $\mathcal{L}$ and $\overline{\mathcal{L}}$ have equal anarchy value.*

Proof. Simply note that the functions ℓ and $\nu\ell$ (for $0 \neq \ell \in \mathcal{L}$ and $\nu > 0$) have equal anarchy value. ■

Lemmas 3.5.1 and 3.5.2 yield the following theorem, which reduces computing the price of anarchy (the combinatorial problem of finding a worst-possible multicommodity flow instance) to computing the anarchy value (the simpler analytical problem of determining the worst behavior exhibited by any function in a given class).

Theorem 3.5.3 *Let $\mathcal{L}$ be a standard class of latency functions containing a function ℓ satisfying $\ell(0) > 0$, with anarchy value $\alpha(\mathcal{L})$. If $\mathcal{I}$ denotes the set of instances with latency functions in $\mathcal{L}$, then*

$$\sup_{(G,r,\ell)\in\mathcal{I}} \rho(G, r, \ell) = \alpha(\mathcal{L}).$$

Proof. Since $\mathcal{L}$ is standard and contains a function that is positive at 0, the class $\overline{\mathcal{L}} = \{\nu\ell : \ell \in \mathcal{L}, \nu > 0\}$ is standard and diverse. By Theorem 3.4.4, we have

$$\sup_{(G,r,\ell)\in\overline{\mathcal{I}}} \rho(G, r, \ell) = \alpha(\overline{\mathcal{L}})$$

where $\overline{\mathcal{I}}$ denotes the set of instances with latency functions in $\overline{\mathcal{L}}$. Applying Lemmas 3.5.1 and 3.5.2, we obtain the desired equality. ■

Remark 3.5.4 The conclusion of Theorem 3.5.3 fails if the hypothesis that some function is positive with zero congestion is omitted (recall Corollary 3.2.2 and consider the counterexample class $\mathcal{L} = \{ax : a > 0\}$). We do not know if the assumption that the function class is standard can be omitted. We leave open the problem of computing the price of anarchy for classes of latency functions that fail to satisfy these two hypotheses, though it is not clear if such function classes have any practical import.

3.5.2 Applications

We are finally prepared to put our techniques to use in computing the price of anarchy for some concrete function classes. We give only three illustrative examples; it will be obvious that many other function classes can be treated in a similar way.

Polynomial Latency Functions

For a positive integer p, let $\mathcal{L}_p$ denote the set of latency functions that are polynomials with nonnegative coefficients and degree at most p. As a first showcase for our machinery, we next compute the price of anarchy with respect to latency functions $\mathcal{L}_p$.

Proposition 3.5.5 *If $\mathcal{I}_p$ is the set of instances with latency functions in $\mathcal{L}_p$, then*

$$\sup_{(G,r,\ell)\in\mathcal{I}_p} \rho(G,r,\ell) = [1 - p \cdot (p+1)^{-(p+1)/p}]^{-1} = \Theta\left(\frac{p}{\ln p}\right).$$

Proof. Since $\mathcal{L}_p$ is standard and contains the constant functions, Theorem 3.4.2 implies that the price of anarchy is simply the anarchy value of $\mathcal{L}_p$. We claim that it suffices to compute the anarchy value of the smaller function class consisting of functions of $\mathcal{L}_p$ comprising only one term, namely $\widetilde{\mathcal{L}}_p \equiv \{ax^i : a \geq 0, i \in \{0, 1, 2, \ldots, p\}\}$. This claim is valid because an instance (G, r, ℓ) with latency functions in $\mathcal{L}_p$ can be transformed into an equivalent instance with latency functions in $\widetilde{\mathcal{L}}_p$ by replacing an edge e of G with latency function $\ell_e(x) = \sum_{i=0}^{p} a_i x^i$ by a directed path of $p+1$ edges, with the ith edge of the path possessing latency function

$\widetilde{\ell}_{e,i}(x) = a_i x^i$.[4]

We next compute the anarchy value $\alpha(\ell)$ of an arbitrary nonzero function $\ell(x) = ax^i$ of $\widetilde{\mathcal{L}}_p$ (recall Definition 3.3.2). If $i = 0$ then $\alpha(\ell) = 1$; otherwise, ℓ^* is strictly increasing and the scalar λ is uniquely determined by the choice of r. In this case, for $r > 0$ we have $\lambda = (i+1)^{-1/i}$, hence $\mu = \lambda^i = (i+1)^{-1}$, hence $[\lambda\mu + (1-\lambda)]^{-1} = [(i+1)^{-(i+1)/i} + (1-(i+1)^{-1/i})]^{-1} = [1 - i \cdot (i+1)^{-(i+1)/i}]^{-1}$. Since this expression is independent of $r > 0$, we obtain $\alpha(\ell) = [1 - i \cdot (i+1)^{-(i+1)/i}]^{-1}$. This expression is independent of a and is increasing in i on $[0, p]$ (as shown by a simple derivative test), so the functions of $\widetilde{\mathcal{L}}_p$ with largest anarchy value are those of the form ax^p for $a > 0$; hence, $\alpha(\mathcal{L}_p) = \alpha(\widetilde{\mathcal{L}}_p) = [1 - p \cdot (p+1)^{-(p+1)/p}]^{-1}$. ■

Remark 3.5.6 A sharp lower bound on the left-hand side of Proposition 3.5.5 is provided by the nonlinear variant of Pigou's example (Subsection 2.4.4); the content of the proposition is that no worse example is possible, even in arbitrary multicommodity flow networks.

Delay Functions of M/M/1 Queues

Latency functions of the form $\ell(x) = (u - x)^{-1}$ arise as the (expected) delay function of an M/M/1 queue[5] with service rate (or capacity) u [80], and for this reason have been extensively studied in the networking literature [20, 105, 106, 113, 138]. These latency functions do not directly fit into our framework, since they are defined only on the set $[0, u)$, rather than on all of $[0, \infty)$. Nevertheless, only minor generalizations of our results are needed to compute the price of anarchy in this setting.

We will fix two parameters, the largest allowable sum of all traffic rates R_{max} and the smallest allowable edge capacity u_{min}. We will assume that $R_{max} < u_{min}$; while it may seem unreasonable to assume that any edge of the network has the capacity to carry all of the demand, our computations below will show that, in the absence of further assumptions, the price of anarchy is $+\infty$ if the sum of traffic rates can be arbitrarily close to (or greater than) the smallest edge capacity. Under this assumption, the restricted domains of the latency functions pose no difficulty;

[4]This maneuver illustrates a general principle: if $\mathcal{L}$ is the cone generated by a (possibly infinite) standard class of latency functions S (i.e., $\mathcal{L}$ is all finite affine combinations of functions in S), then $\alpha(\mathcal{L}) = \alpha(S)$.

[5]By M/M/1, we mean a single queue with Poisson arrivals and exponentially distributed service times [80].

every feasible flow routes at most R_{max} units of flow on every edge and hence has a well-defined cost.

Let $\mathcal{L}$ denote the set of latency functions $\{\ell(x) = (u-x)^{-1} : u \geq u_{min}\}$ and, for the purposes of this example only, redefine the anarchy value $\alpha(\ell)$ of a latency function ℓ to be $\alpha(\ell) = \sup_{r:0<r\leq R_{max}}[\lambda\mu + (1-\lambda)]^{-1}$, where λ is the unique scalar satisfying $\ell^*(\lambda r) = \ell(r)$ and $\mu = \ell(\lambda r)/\ell(r)$. The key difference between this definition and the original definition of anarchy value (Definition 3.3.2) is that the range of traffic rates we consider is restricted to lie in $(0, R_{max}]$ rather than $(0, \infty)$; this ensures that the equations defining λ and μ make sense.

Next, it is straightforward to check that Theorem 3.3.8 and hence Theorem 3.5.3 remain valid with our new definition of anarchy value, provided we only care about the worst-possible value of ρ achieved by instances whose sum of all traffic rates is at most R_{max}. Since the class $\mathcal{L}$ satisfies both hypotheses of Theorem 3.5.3, computing the price of anarchy with respect to $\mathcal{L}$ for instances with sum of all traffic rates at most R_{max} reduces to computing the anarchy value of $\mathcal{L}$.

Proposition 3.5.7 *If $\mathcal{I}$ is the set of instances with latency functions in $\mathcal{L}$ and sum of all traffic rates at most R_{max}, then*

$$\sup_{(G,r,\ell)\in\mathcal{I}} \rho(G,r,\ell) = \frac{1}{2}\left(1 + \sqrt{\frac{u_{min}}{u_{min} - R_{max}}}\right).$$

Proof. The previous discussion implies that we need only check that $\alpha(\mathcal{L}) = [1 + \sqrt{u_{min}/(u_{min} - R_{max})}]/2$. We begin by computing the anarchy value of an arbitrary function in $\mathcal{L}$, say $\ell(x) = (u-x)^{-1}$ for $u \geq u_{min}$. The marginal cost function ℓ^* is given by $\frac{d}{dx}(x(u-x)^{-1})$ and hence

$$\ell^*(x) = \frac{u}{(u-x)^2}.$$

Now fix $r \in (0, R_{max}]$; λ is defined to solve the equation $\ell^*(\lambda r) = \ell(r)$ and hence satisfies

$$\frac{u}{(u-\lambda r)^2} = \frac{1}{u-r}.$$

Solving, we obtain $\lambda = (u - \sqrt{u(u-r)})/r$. Next,

$$\mu = \frac{\ell(\lambda r)}{\ell(r)} = \frac{u-r}{u-\lambda r} = \frac{u-r}{u-(u-\sqrt{u}\sqrt{u-r})} = \frac{\sqrt{u-r}}{\sqrt{u}}.$$

It remains to compute $[\lambda\mu + (1-\lambda)]^{-1}$:

$$\begin{aligned}
[\lambda\mu + (1-\lambda)]^{-1} &= \left[\frac{u-\sqrt{u}\sqrt{u-r}}{r}\frac{\sqrt{u-r}}{\sqrt{u}} + 1 - \frac{u-\sqrt{u}\sqrt{u-r}}{r}\right]^{-1} \\
&= \left[\frac{2u\sqrt{u-r} - 2u\sqrt{u} + 2r\sqrt{u}}{r\sqrt{u}}\right]^{-1} \\
&= \frac{1}{2}\frac{r}{\sqrt{u}\sqrt{u-r} - (u-r)} \cdot \frac{\sqrt{u}\sqrt{u-r} + (u-r)}{\sqrt{u}\sqrt{u-r} + (u-r)} \\
&= \frac{1}{2}\frac{r[\sqrt{u}\sqrt{u-r} + (u-r)]}{u(u-r) - (u-r)^2} \\
&= \frac{1}{2}\frac{\sqrt{u}\sqrt{u-r} + (u-r)}{u-r} \\
&= \frac{1}{2}\left(1 + \sqrt{\frac{u}{u-r}}\right).
\end{aligned}$$

Since this expression is increasing in r, it follows that

$$\alpha(\ell) = \frac{1}{2}\left(1 + \sqrt{\frac{u}{u-R_{max}}}\right).$$

Since the anarchy value is decreasing in the edge capacity, we have

$$\alpha(\mathcal{L}) = \frac{1}{2}\left(1 + \sqrt{\frac{u_{min}}{u_{min}-R_{max}}}\right),$$

as claimed. ■

Remark 3.5.8

(1) We note that the above class $\mathcal{L}$ is *not* diverse (since $\ell(0) \leq \frac{1}{u_{min}}$ for all $\ell \in \mathcal{L}$); indeed, the worst-possible value of the ratio ρ need not be achieved on a set of parallel links for this class of latency functions (cf. Theorem 3.4.4).[6] Thus, the extensions provided in Subsection 3.5.1 are crucial in this application.

(2) The anarchy value of $\mathcal{L}$ and hence the worst possible value of ρ go to $+\infty$ as $R_{max} \to u_{min}$; fulfilling a previous promise, this shows that the hypothesis that R_{max} is bounded away from u_{min} is necessary for the price of anarchy for networks with M/M/1 delay functions to be finite.

[6]However, the proof of Lemma 3.5.1 shows that subdivisions of parallel networks suffice to achieve the worst-case bound.

Delay Functions of M/G/1 Queues

As a final example, we extend the preceding analysis to queues that need not have exponentially distributed service times—that is, to M/G/1 delay functions (we retain our assumptions of a single queue and Poisson arrivals). Our solution will not be as clean as in the M/M/1 case, but will demonstrate that our techniques for computing the price of anarchy remain useful even for relatively complex classes of allowable latency functions.

Recall that if a queue service distribution (specifying the number of customers served in a time step) has expectation μ and standard deviation σ (both of which we assume to be finite), then the expected waiting time under Poisson arrivals with rate λ is

$$\frac{1}{\mu} + \frac{\lambda(1+\sigma^2\mu^2)}{2\mu(\mu-\lambda)};$$

see [80] or [94] for a derivation. To rephrase this formula in our usual notation, we view the parameter μ as the edge capacity u and the Poisson rate λ as the amount of traffic assigned to an edge; we are then interested in latency functions ℓ of the following form:

$$\ell(x) = \frac{1}{u} + \frac{x(1+\sigma^2u^2)}{2u(u-x)}.$$

As in the M/M/1 case, to achieve an interesting result we will need to assume a minimum allowable capacity u_{min} and a maximum allowable sum of all traffic rates $R_{max} < u_{min}$.

The anarchy value of such a function can be computed by the same method as for M/M/1 case (though the calculations are more tedious). It turns out that the anarchy value of a latency function ℓ with the above formula is

$$\alpha(\ell) = \left(1+\sqrt{\frac{u}{u-R_{max}}}\right)\frac{2u+R_{max}(\sigma^2u^2-1)}{4u+(u+R_{max}-\sqrt{u(u-R_{max})})(\sigma^2u^2-1)}.$$

Applying Theorem 3.5.3, we obtain the following proposition.

Proposition 3.5.9 *Let $\mathcal{L}$ be a non-empty collection of M/G/1 delay functions with expected service rate at least u_{min}. Then, the price of anarchy for instances with latency functions in $\mathcal{L}$ and sum of all traffic rates at most $R_{max} < u_{min}$ is precisely*

$$\sup_{\ell\in\mathcal{L}} \left(1+\sqrt{\frac{u_\ell}{u_\ell-R_{max}}}\right)\frac{2u_\ell+R_{max}(\sigma_\ell^2u_\ell^2-1)}{4u_\ell+(u_\ell+R_{max}-\sqrt{u_\ell(u_\ell-R_{max})})(\sigma_\ell^2u_\ell^2-1)}$$

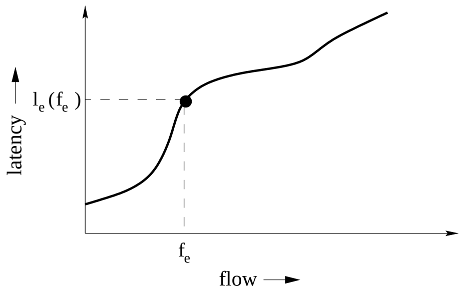
where u_ℓ and σ_ℓ denote the expectation and standard deviation of the service rate distribution associated with ℓ.

Without more assumptions on the class $\mathcal{L}$, we cannot simplify the expression of Proposition 3.5.9 further; this reflects the relative complexity of M/G/1 delay functions (which are specified by two independent parameters u_ℓ and σ_ℓ, unlike the simpler M/M/1 case). On the other hand, reducing the computation of the price of anarchy to computing the expression of Proposition 3.5.9 is both nontrivial and useful. When the class $\mathcal{L}$ possesses structure beyond merely being some collection of M/G/1 delay functions, the expression of Proposition 3.5.9 may become simple and transparent (as in the special case of M/M/1 delay functions, where $\sigma_\ell u_\ell = 1$ for all ℓ). Even for classes for which no analytical simplification is possible, Proposition 3.5.9 should permit the approximate (if not exact) computation of the price of anarchy with respect to $\mathcal{L}$ by straightforward numerical methods; in the simplest case where $\mathcal{L}$ is finite and not astronomically large (and we suspect almost all classes of M/G/1 delay functions can be closely approximated by such an $\mathcal{L}$), the price of anarchy can be computed simply by enumeration. We note that without the assurance that simple network topologies always provide worst-case examples, an enumerative approach to computing the price of anarchy would be unthinkable.

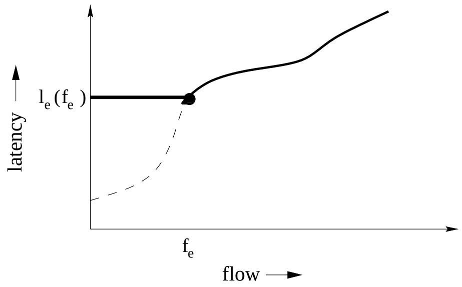
3.6 A Bicriteria Bound for Networks with Arbitrary Latency Functions

The nonlinear variant of Pigou's example (Subsection 2.4.4) shows that, assuming only continuity and monotonicity of the edge latency functions, the price of anarchy cannot be bounded above (even as a function of the network size). On the other hand, this example does not rule out interesting *bicriteria* results. Toward this end, we compare the cost of a flow at Nash equilibrium to that of an optimal flow feasible for *increased rates*.[7] In the example of Subsection 2.4.4, an optimal flow feasible for rate $r > 1$ assigns the additional flow to the upper link, now incurring a cost that tends to $r - 1$ as $p \to \infty$. In particular, for any p an optimal flow feasible for twice the rate ($r = 2$) has total latency at least that of the flow at Nash equilibrium (feasible for the original rates). We next prove that this statement holds in *any* network with continuous, nondecreasing edge latency functions.

[7]This approach is in the spirit of the analyses of online scheduling algorithms via *resource augmentation* given by Kalyanasundaram and Pruhs [91] and Phillips et al. [147].



(a) Graph of latency function ℓ_e and its value at flow value f_e



(b) Graph of latency function $\bar{\ell}_e$

Figure 3.1: Construction in the proof of Theorem 3.6.1 of modified latency function $\bar{\ell}_e$ given original latency function ℓ_e and Nash flow value f_e. Solid lines denote graphs of functions.

Theorem 3.6.1 *If f is a flow at Nash equilibrium for (G, r, ℓ) and f^* is feasible for $(G, 2r, \ell)$, then $C(f) \le C(f^*)$.*

Proof. Suppose f, f^* satisfy the hypotheses of the theorem. For $i = 1, \ldots, k$, let $L_i(f)$ be the latency of an s_i-t_i flow path of f, so that $C(f) = \sum_i L_i(f) r_i$ (see Proposition 2.2.4). We seek a set of latency functions $\bar{\ell}$ that on one hand approximates the original ones (in the sense that the cost of a flow with respect to latency functions $\bar{\ell}$ is close to its original cost) and, on the other hand, allows us to easily lower bound the cost (with respect to $\bar{\ell}$) of *any* feasible flow. With this goal in mind, we define new latency functions $\bar{\ell}$ as follows:

$$\bar{\ell}_e(x) = \begin{cases} \ell_e(f_e) & \text{if } x \le f_e \\ \ell_e(x) & \text{if } x \ge f_e. \end{cases}$$

Figure 3.1 illustrates this construction.

First we compare the cost of the flow f^* under the new latency functions $\bar{\ell}$ to its original cost $C(f^*)$. For any edge e, $\bar{\ell}_e(x) - \ell_e(x)$ is zero for $x \ge f_e$ and bounded above by $\ell_e(f_e)$ for $x < f_e$, so $x(\bar{\ell}_e(x) - \ell_e(x)) \le \ell_e(f_e) f_e$ for all $x \ge 0$. Notice that the left-hand side (the discrepancy between $x\bar{\ell}_e(x)$ and $x\ell_e(x)$) is maximized when x is slightly smaller than f_e and when $\ell_e(x) = 0$; in this case, the value of the left-hand side is essentially the area of the rectangle enclosed by dashed lines in Figure 3.1(a). The difference between the new cost (with respect to $\bar{\ell}$) and the old cost (with respect to ℓ) can now be bounded as follows:

$$\begin{aligned}\sum_e \bar{\ell}_e(f_e^*)f_e^* - C(f^*) &= \sum_{e\in E} f_e^*(\bar{\ell}_e(f_e^*) - \ell_e(f_e^*)) \\ &\leq \sum_{e\in E} \ell_e(f_e)f_e \\ &= C(f).\end{aligned}$$

In other words, evaluating f^* with latency functions $\bar{\ell}$ (rather than ℓ) increases its cost by at most an additive $C(f)$ factor.

On the other hand, if f_0 denotes the zero flow in G, then by construction $\bar{\ell}_P(f_0) \geq L_i(f)$ for any path $P \in \mathcal{P}_i$. Since $\bar{\ell}_e$ is nondecreasing for each edge e, it follows that $\bar{\ell}_P(f^*) \geq L_i(f)$ for each path $P \in \mathcal{P}_i$. Thus, the cost of f^* with respect to $\bar{\ell}$ can be bounded below in the following manner:

$$\begin{aligned}\sum_P \bar{\ell}_P(f^*)f_P^* &\geq \sum_i \sum_{P\in\mathcal{P}_i} L_i(f)f_P^* \\ &= \sum_i 2L_i(f)r_i \\ &= 2C(f).\end{aligned}$$

Combining these two results we obtain the theorem:

$$\begin{aligned}C(f^*) &\geq \sum_P \bar{\ell}_P(f^*)f_P^* - C(f) \\ &\geq 2C(f) - C(f) \\ &= C(f).\end{aligned}$$

■

The same proof also shows the following more general result.

Theorem 3.6.2 *If f is a flow at Nash equilibrium for (G, r, ℓ) and f^* is feasible for $(G, (1+\xi)r, \ell)$, then $C(f) \leq \frac{1}{\xi}C(f^*)$.*

Remark 3.6.3 Referring back to the network of Subsection 2.4.4 (the network with two nodes and two edges with latency functions $\ell(x) = 1$ and $\ell(x) = x^p$), we see that Theorem 3.6.2 is essentially tight for all values of ξ. More precisely, by taking p sufficiently large we can obtain an instance admitting an optimal flow feasible for a traffic rate arbitrarily close to $(1+\xi)$ with cost strictly less than ξ (recall the cost of the flow at Nash equilibrium for the original rate $r = 1$ is 1) and an optimal flow feasible for rate $1+\xi$ with cost arbitrarily close to ξ.

Theorem 3.6.1 has a natural interpretation for networks with the M/M/1 delay functions mentioned in the previous section. To see this, we first note that comparing a Nash flow to an optimal flow forced to route more traffic is the same as comparing a Nash flow with "faster" latency functions to an optimal flow in the original network. Formally, we have the following corollary of Theorem 3.6.1.

Corollary 3.6.4 *Let (G, r, ℓ) be an instance and define the modified latency function $\tilde{\ell}_e$ by $\tilde{\ell}_e(x) = \frac{1}{2}\ell_e(\frac{x}{2})$ for each edge e. If $\tilde{f}$ is a flow at Nash equilibrium for $(G, r, \tilde{\ell})$ and f^* is feasible for (G, r, ℓ), then the cost of $\tilde{f}$ (with respect to latency functions $\tilde{\ell}$) is at most the cost of f^* (with respect to latency functions ℓ).*

Proof. Let f be a Nash flow for $(G, r/2, \ell)$ and f^* a flow feasible for (G, r, ℓ); by Theorem 3.6.1, $\sum_e \ell_e(f_e)f_e \leq \sum_e \ell_e(f_e^*)f_e^*$. Now consider the flow $\tilde{f} = 2f$, viewed as a feasible flow for $(G, r, \tilde{\ell})$. Since $\tilde{\ell}_e(\tilde{f}_e) = \frac{1}{2}\ell_e(f_e)$ for each edge e and f is a Nash flow for (G, r, ℓ), $\tilde{f}$ is a Nash flow for $(G, r, \tilde{\ell})$; moreover,

$$\sum_e \tilde{\ell}_e(\tilde{f}_e)\tilde{f}_e = \sum_e (\frac{1}{2}\ell_e(f_e))(2f_e) = \sum_e \ell_e(f_e)f_e.$$

We have shown that $\sum_e \tilde{\ell}_e(\tilde{f}_e)\tilde{f}_e \leq \sum_e \ell_e(f_e^*)f_e^*$ with $\tilde{f}$ at Nash equilibrium for $(G, r, \tilde{\ell})$; the corollary now follows from the essential uniqueness of Nash flows (Corollary 2.5.3). ■

Now consider the special case of an instance (G, r, ℓ) in which all latency functions are the delay functions of M/M/1 queues, and thus for each edge e we have $\ell_e(x) = (u_e - x)^{-1}$ on $[0, u_e)$ for some edge capacity u_e. Assume for simplicity that every flow f feasible for (G, r, ℓ) satisfies $f_e < u_e$ for each edge e (e.g., by insisting that the minimum edge capacity is larger than the sum of all traffic rates), so that we can ignore the restricted domains of the latency functions. Here, if $\ell_e(x) = 1/(u_e - x)$ then $\tilde{\ell}_e(x) = 1/2(u_e - x/2) = 1/(2u_e - x)$. Thus in a network with M/M/1 delay functions, Corollary 3.6.4 offers the following advice: to match the performance of a centrally controlled network with selfish routing, simply *double the capacity of every edge.*

Remark 3.6.5 We made the strong assumption that every feasible flow f satisfies $f_e < u_e$ for each edge e for simplicity. We can alternatively define the latency of an edge with an M/M/1 delay function and capacity u to be $+\infty$ on $[u, \infty)$; arithmetic with $+\infty$ is defined in the usual way. Some care must be taken with this approach, however, as the essential uniqueness of Nash flows (Proposition 2.5.1 and Corollary 2.5.3) fails when edge latencies can take on $+\infty$ as a value; in particular, some but not all Nash flows may have infinite cost. The guarantee of Theorem 3.6.1 also fails for networks in which Nash flows may have infinite cost. What remains true is this: doubling the capacity of a network with M/M/1 delay functions is guaranteed to offset the cost of selfish routing, assuming only that all Nash flows in the augmented network possess finite cost. This significantly weakens our earlier assumption that all feasible flows have finite cost in the original network.

Chapter 4

Extensions to Other Models

In Chapter 3 we studied the price of anarchy in the network model put forth in Chapter 2. In this chapter we show how this work can be extended both to more realistic models of network routing and to a broader class of games. We do not endeavor to generalize all of the results of the previous chapter to the greatest possible extent; we merely wish to point out that our techniques are not entirely model-specific, and to indicate some directions in which they are readily extended.

The extensions presented in the first three sections of this chapter are motivated by some of the deficiencies of the traffic model defined in Chapter 2. First, network users can often only evaluate path latency approximately, rather than exactly. Section 4.1 extends the notion of a flow at Nash equilibrium and Theorems 3.2.6 and 3.6.1 to this setting. Second, our basic model represents an idealized scenario with infinitely many users each controlling a negligible fraction of the overall traffic, while in reality we encounter a finite number of network users, each controlling a strictly positive amount of traffic. In Section 4.2 we prove an analogue of Theorem 3.6.1 for the case of finitely many network users, provided each user can route its flow fractionally over any number of paths. In Section 4.3 we show that such an assumption is essentially necessary, in that no bicriteria bound analogous to Theorem 3.6.1 holds when there are only finitely many network users, each of whom must route its flow on a single path; however, a version of Theorem 3.6.1 does hold if network users do not control too much flow and the edge latency functions are not too steep. In the last section (Section 4.4), we show how all of the results of Chapter 3 can be extended to a broad class of games (that need not involve a network) previously studied in the game theory literature.

4.1 Flows at Approximate Nash Equilibrium

It is often unreasonable to expect network users to be able to evaluate the latency of different paths with arbitrary precision. We next investigate the sensitivity of our results to this assumption. We suppose that a network user can only distinguish between paths that differ significantly in their latency (say by more than a $(1+\epsilon)$ factor for some $\epsilon > 0$). Our definition of a flow at *ϵ-approximate Nash equilibrium* is then an obvious modification of Definition 2.2.1:

Definition 4.1.1 A flow f feasible for instance (G, r, ℓ) is at *ϵ-approximate Nash equilibrium* if for all $i \in \{1, \ldots, k\}$, $P_1, P_2 \in \mathcal{P}_i$, and $\delta \in (0, f_{P_1}]$, we have $\ell_{P_1}(f) \leq (1+\epsilon)\ell_{P_2}(\tilde{f})$, where

$$\tilde{f}_P = \begin{cases} f_P - \delta & \text{if } P = P_1 \\ f_P + \delta & \text{if } P = P_2 \\ f_P & \text{if } P \notin \{P_1, P_2\}. \end{cases}$$

The analogue of Proposition 2.2.2 is then:

Lemma 4.1.2 *A flow f is at ϵ-approximate Nash equilibrium if and only if for every $i \in \{1, \ldots, k\}$ and $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $\ell_{P_1}(f) \leq (1+\epsilon)\ell_{P_2}(f)$.*

The next theorem provides an analogue of Theorem 3.6.1 for flows at ϵ-approximate Nash equilibrium.

Theorem 4.1.3 *If f is at ϵ-approximate Nash equilibrium with $\epsilon < 1$ for (G, r, ℓ) and f^* is feasible for $(G, 2r, \ell)$, then $C(f) \leq \frac{1+\epsilon}{1-\epsilon} C(f^*)$.*

Proof. Suppose f, f^* satisfy the hypotheses of the theorem. For $i = 1, \ldots, k$, let $L_i(f)$ be the minimum latency of any s_i-t_i path (with respect to f); since f is at ϵ-approximate Nash equilibrium, every s_i-t_i flow path has latency at most $(1+\epsilon)L_i(f)$ and hence $C(f) \leq (1+\epsilon)\sum_i L_i(f) r_i$.

As in the proof of Theorem 3.6.1, we define a new set of latency functions $\bar{\ell}$ by

$$\bar{\ell}_e(x) = \begin{cases} \ell_e(f_e) & \text{if } x \leq f_e \\ \ell_e(x) & \text{if } x \geq f_e. \end{cases}$$

As before, the cost of a flow with respect to $\bar{\ell}$ exceeds its cost with respect to ℓ by at most an additive factor of $C(f)$.

Letting f_0 denote the zero flow in G, we have $\bar{\ell}_P(f_0) \geq L_i(f)$ for any path $P \in \mathcal{P}_i$. Since $\bar{\ell}_e$ is nondecreasing for each edge e, it follows that $\bar{\ell}_P(f^*) \geq L_i(f)$ for each path $P \in \mathcal{P}_i$. This allows us to bound the cost of f^* with respect to $\bar{\ell}$ from below:

$$\begin{aligned} \sum_P \bar{\ell}_P(f^*) f_P^* &\geq \sum_i \sum_{P \in \mathcal{P}_i} L_i(f) f_P^* \\ &= \sum_i 2L_i(f) r_i \\ &\geq \frac{2}{1+\epsilon} C(f). \end{aligned}$$

To conclude, we derive

$$\begin{aligned} C(f^*) &\geq \sum_P \bar{\ell}_P(f^*) f_P^* - C(f) \\ &\geq \frac{2}{1+\epsilon} C(f) - C(f) \\ &= \frac{1-\epsilon}{1+\epsilon} C(f). \end{aligned}$$

■

Remark 4.1.4 A simple example in a network similar to that of Braess's Paradox (Figure 1.2(b)) shows that the factor of $\frac{1+\epsilon}{1-\epsilon}$ cannot be improved (see Section B.2). However, it is not difficult to improve this factor to $1+\epsilon$ in networks of parallel links (see Section B.2); this provides a counterpoint to our work in Sections 3.3–3.4 showing that worst-case examples for the price of anarchy for flows at exact Nash equilibrium always occur in networks of parallel links.

Further extensions of the theorems of Chapter 3 to the current setting are possible. As an example, we sketch an approximate version of Theorem 3.2.6 for networks with linear latency functions. As in Section 3.2, the idea is to start with a flow f at ϵ-approximate Nash equilibrium and consider the scaled-down flow $f/2$. The claim $C(f/2) \geq \frac{1}{4}C(f)$ holds as in Section 3.2, but now $f/2$ is only *approximately* optimal for $(G, r/2, \ell)$; because of this, proving that an augmentation from $f/2$ to a flow feasible for (G, r, ℓ) is costly will require a bit of care.

Next, let $L_i(f)$ denote the minimum latency of any s_i-t_i path with respect to f. The following are true:

(1) $C(f) \leq (1+\epsilon) \sum_{i=1}^k L_i(f) r_i$.

(2) If P is an s_i-t_i path, then the marginal cost of P with respect to $f/2$ is at least $L_i(f)$.

(3) If P is an s_i-t_i flow path of f, then the marginal cost of P with respect to $f/2$ is at most $(1+\epsilon)L_i(f)$.

Now consider augmenting $f/2$ to a flow f^* optimal for (G, r, ℓ). As in Lemmas 3.2.4 and 3.3.5, convexity of the objective function $C(\cdot)$ with linear (or more generally, standard) latency functions allows us to lower bound the cost of this augmentation on each edge by the change in flow value times the marginal cost with respect to $f/2$. At worst, this augmentation will remove $r_i/2$ units of flow between each commodity i at a marginal benefit of $(1+\epsilon)L_i(f)$ per flow unit and will add r_i units of flow at a marginal cost of $L_i(f)$ per flow unit. This argument gives

$$\begin{aligned} C(f^*) &\geq \frac{1}{4}C(f) - (1+\epsilon)\sum_{i=1}^{k} L_i(f)\frac{r_i}{2} + \sum_{i=1}^{k} L_i(f) r_i \\ &= \frac{1}{4}C(f) + \left(\frac{1-\epsilon}{2}\right)\sum_{i=1}^{k} L_i(f) r_i \\ &\geq C(f)\left(\frac{1}{4} + \frac{1-\epsilon}{2(1+\epsilon)}\right) \\ &= \frac{3-\epsilon}{4+4\epsilon}C(f). \end{aligned}$$

A straightforward modification of the bad example of Section B.2 for Theorem 4.1.3 shows that the factor $\frac{3-\epsilon}{4+4\epsilon}$ is best possible for $\epsilon \leq 1$.

4.2 Finitely Many Users: Splittable Flow

Our basic model makes the convenient assumption that there are an infinite number of noncooperative network users, each controlling a negligible fraction of the overall traffic. In this section we extend the basic model to the case of finitely many network users, each of whom controls a strictly positive amount of traffic. In this section we allow a network user to split flow among any number of paths; this model has been studied extensively in the networking literature [4, 5, 8, 25, 59, 138]. In the next section we will investigate the setting in which each network user must route all of its flow on a single path.

We are given a network G with continuous nondecreasing latency functions ℓ as before, and in addition k *users*. We assume that user i intends to send r_i units of flow from source s_i to destination t_i. Distinct users may have identical source-destination pairs. We continue to denote an instance by (G, r, ℓ), and we call the instance *finite splittable*. A *flow* f now consists of k functions, with one function $f^{(i)} : \mathcal{P}_i \to \mathcal{R}^+$ for each user i. For a flow f, we denote by $C_i(f)$ the total latency experienced by user i; thus, $C_i(f) = \sum_{P \in \mathcal{P}_i} \ell_P(f) f_P^{(i)}$. As usual, a flow is *at Nash equilibrium* if no user can decrease the latency it experiences by rerouting its flow. In this setting, a flow f is at Nash equilibrium if and only if for each i, $f^{(i)}$ minimizes $C_i(f)$ given $f^{(j)}$ for $j \neq i$. We will focus on networks with standard latency functions (see Definition 2.3.5); under this assumption, results of Rosen [156] imply that a flow at Nash equilibrium must exist (see [138] for a proof sketch).

Our main result for this model is an analogue of Theorem 3.6.1.

Theorem 4.2.1 *If f is at Nash equilibrium for the finite splittable instance (G, r, ℓ) with standard latency functions and f^* is feasible for the finite splittable instance $(G, 2r, \ell)$, then $C(f) \leq C(f^*)$.*

Proof. Fix f, f^* and define latency functions $\bar{\ell}$ as in the proofs of Theorems 3.6.1 and 4.1.3. As in those proofs, evaluating f^* with latency functions $\bar{\ell}$ (rather than ℓ) increases its cost by at most an additive $C(f)$ factor.

We claim that f is optimal for the instance $(G, r, \bar{\ell})$. We proceed by contradiction, showing that if f is not optimal for $(G, r, \bar{\ell})$ then f fails to be at Nash equilibrium for (G, r, ℓ). Suppose f is not optimal; since the instance $(G, r, \bar{\ell})$ defines a convex optimization problem of the form (NLP) (see Section 2.3), by Proposition 2.3.1 there are two paths P_1, P_2, a user i such that $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1}^{(i)} > 0$, and a sufficiently small $\delta \in (0, f_{P_1}^{(i)}]$ such that moving δ units of flow from P_1 to P_2 yields a new flow with cost (with respect to $\bar{\ell}$) strictly less than that of f. Our goal is to show that the same local move will be beneficial for user i in the instance (G, r, ℓ). We may assume that P_1, P_2 are disjoint (otherwise, the following argument applies to the symmetric difference of P_1 and P_2). The benefit (with respect to $\bar{\ell}$) of removing δ units of flow from path P_1 is then $\delta \cdot \bar{\ell}_{P_1}(f) = \delta \cdot \ell_{P_1}(f)$ (since $\bar{\ell}_e(x) = \ell_e(f_e)$ when $x \leq f_e$) while the cost (with respect to $\bar{\ell}$) of adding δ units of flow to P_2 is $\sum_{e \in P_2} [\ell_e(f_e + \delta)(f_e + \delta) - \ell_e(f_e) f_e]$; we are assuming that the former exceeds the latter. On the other hand, user i is capable of making an identical local change to $f^{(i)}$ in the instance (G, r, ℓ), and doing so provides a benefit to user i of at least $\delta \cdot \ell_{P_1}(f)$ with respect to ℓ (since latency functions are nondecreasing) and a cost (with respect to ℓ) of

$$\sum_{e \in P_2} [\ell_e(f_e + \delta)(f_e^{(i)} + \delta) - \ell_e(f_e) f_e^{(i)}]$$

which is at most

$$\sum_{e \in P_2} [\ell_e(f_e + \delta)(f_e + \delta) - \ell_e(f_e)f_e]$$

since ℓ_e is nondecreasing and $f_e^{(i)} \leq f_e$ for each edge e. Thus, moving δ units of flow from path P_1 to path P_2 yields a better outcome for user i in the instance (G, r, ℓ), so f fails to be at Nash equilibrium for (G, r, ℓ).

We have determined that any flow feasible for $(G, r, \bar{\ell})$ must have cost at least $C(f)$. Since every latency function is nondecreasing, it follows that any flow feasible for $(G, 2r, \bar{\ell})$ must have cost at least $2C(f)$: such a flow may be expressed as the sum of two flows feasible for $(G, r, \bar{\ell})$, and the cost of their sum is at least the sum of their individual costs. Since the cost of f^* with respect to $\bar{\ell}$ exceeds its cost with respect to ℓ by at most $C(f)$, the theorem follows. ■

Theorem 3.6.1 can be regarded as the limiting case of the above theorem, as the number of users tends to infinity and the amount of flow controlled by each user tends to 0.

4.3 Finitely Many Users: Unsplittable Flow

In this section we continue our investigation of selfish routing with finitely many users, each controlling a non-negligible amount of flow. It is easy to imagine scenarios in which users cannot route flow on several different paths, but must instead select a single path for routing. Our previous results have made crucial use of the "infinitely divisible" nature of flow, and we next show that this assumption is essentially necessary.

Consider an instance (G, r, ℓ) as in the previous section (with k users and the ith user controlling r_i units of flow), but with the additional constraint that each user selects a *single* path on which to route all of its flow. We call such an instance *finite unsplittable*; such instances have also been studied by Libman and Orda [116, 117] (though not from the perspective of quantifying the inefficiency of Nash equilibria). Adapting the definition of the previous section to this new setting, a flow f (now consisting only of k paths) is at Nash equilibrium if and only if for each i, user i routes its flow on a path minimizing $\ell_P(f)$ (with P ranging over all paths in $\mathcal{P}_i$), given the paths chosen by the other $k-1$ users. One technical difficulty of this model is that Nash equilibria need not exist (unless network users can randomize) [117];

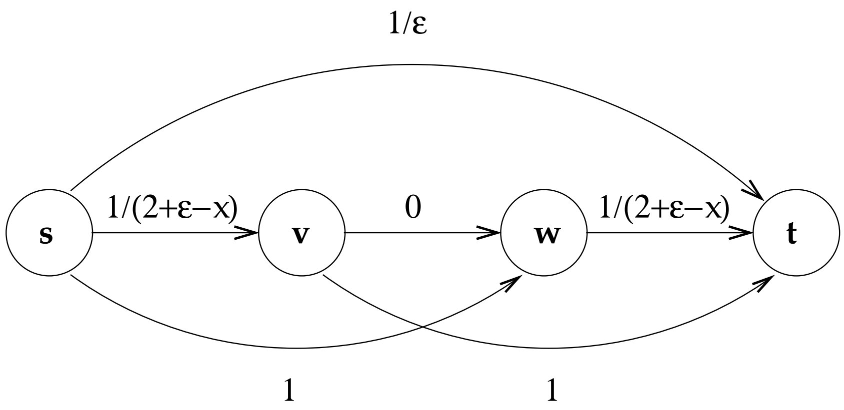


Figure 4.1: A Bad Example for Unsplittable Flow

we will see below that even when they do, they may be inefficient in a very strong sense.

We first consider a simple example showing that a flow at Nash equilibrium may have cost arbitrarily larger than that of an optimal flow. Consider the network shown in Figure 4.1, and suppose there are two users, each of whom has source s, destination t, and one unit of flow to send; $\epsilon > 0$ is arbitrary. In the optimal solution, one user chooses path $s \to v \to t$ and the other $s \to w \to t$; the cost of this solution is less than 4 (for any $\epsilon > 0$). On the other hand, a solution with one user choosing path $s \to v \to w \to t$ and the other routing on the $s \to t$ link is a flow at Nash equilibrium with cost greater than $\frac{1}{\epsilon}$; by choosing ϵ arbitrarily small this cost is arbitrarily large, and hence arbitrarily more costly than optimal.

In light of the example at the beginning of Section 3.6, such a result is hardly surprising; however, we can extend this example to show that bicriteria bounds analogous to Theorems 3.6.1 and 4.2.1 are false when we require users to route flow unsplittably. For a positive integer q, consider the network G_q consisting of $2q + 2$ vertices arranged in a path $s, v_1, v_2, \ldots, v_{2q}, t$ with edges along the path alternately having latency functions $\ell(x) = \frac{1}{q+1+\epsilon-x}$ and $\ell(x) = 0$, a direct s-t link with constant latency function $\ell(x) = \frac{1}{\epsilon}$, and edges from s to v_{2i} and from v_{2i-1} to t with constant latency function $\ell(x) = q$; this construction produces the network of Figure 4.1 when $q = 1$. Suppose there are $q + 1$ users, each with 1 unit of flow to send from s to t. Analogous to the previous paragraph, one Nash equilibrium consists of q users routing flow on the long path $s \to v_1 \to v_2 \to \cdots \to v_{2q} \to t$ and the final user routing its flow on the direct s-t link. This Nash equilibrium has total latency at least $\frac{1}{\epsilon}$. On the other hand, for any $\epsilon > 0$ it is possible for each of the $q+1$ users to route q units of flow unsplittably through G_q with total cost at most $(2q+1)(q+1)^2$: the first user routes on the path $s \to v_1 \to t$, the last on $s \to v_{2q} \to t$, and otherwise the ith user routes on the path $s \to v_{2i-2} \to v_{2i-1} \to t$. Letting ϵ tend to 0 for each fixed value of q, we see that an optimal flow can send *arbitrarily more flow* at *arbitrarily less cost* than a flow at Nash equilibrium.

In the above bad example, the network has latency functions with unbounded derivatives; in this situation, routing a strictly positive amount of additional flow on an edge may increase the latency of that edge by an arbitrarily large amount. We note that this example is not "pathological", in the sense that latency functions of the form $\ell(x) = 1/(u-x)$ naturally arise in networking applications [20] (see also Section 3.5). However, in networks where the largest possible change in edge latency resulting from a single user rerouting its flow is not too large, we can apply Theorem 4.1.3 to derive the following.

Theorem 4.3.1 *Suppose f is at Nash equilibrium in the finite unsplittable instance (G, r, ℓ), and for some $\xi < 2$, we have $\ell_e(x+r_i) \le \xi \cdot \ell_e(x)$ for all users $i \in \{1, \ldots, k\}$, edges $e \in E$, and $x \in [0, \sum_{j \neq i} r_j]$. Then for any flow f^* feasible for $(G, 2r, \ell)$, $C(f) \le \frac{\xi}{2-\xi} \cdot C(f^*)$.*

Proof. We may interpret f and f^* as (fractional) flows feasible for instances (G, r', ℓ) and $(G, 2r', \ell)$ of the usual fractional type (in the sense of Section 2.1), where r'_i is the total amount of flow controlled by users with source s_i and destination t_i in the original instance. The hypotheses ensure that f is at $(\xi - 1)$-approximate Nash equilibrium for (G, r', ℓ), so the result follows from Theorem 4.1.3. ■

For example, in an instance with linear latency functions (say $\ell_e(x) = a_e x + b_e$) with $b_e > 0$ for all edges e, we may apply Theorem 4.3.1 with $\xi = 1 + \max_i r_i \cdot \max_e a_e/b_e$.

4.4 Nonatomic Congestion Games

Both the traffic model of Chapter 2 and the theorems of Chapter 3 can be recast in a more general and abstract setting; in this section, we pursue this generalization and explain its connections to recent work by the game theory community. Because of the similarity to the definitions and results of Chapters 2 and 3, our development will be brief.

4.4.1 Definitions

A *nonatomic congestion game*[1] *(NCG)* is defined by: a finite set E of *resources*, each possessing a nonnegative, nondecreasing, and continuous *cost function* ℓ_e; a finite number k of *player types*; and for each player type i, a positive real number r_i describing the amount of players of type i and a (finite) set $\mathcal{S}_i \subseteq 2^E \setminus \{\emptyset\}$ of *strategies*. By an *assignment* (of players to strategies) for a NCG, we mean k functions $f^i : \mathcal{S}_i \to \mathcal{R}^+$ satisfying $\sum_{S \in \mathcal{S}_i} f_S = r_i$ for $i = 1, 2, \ldots, k$; we will also denote the assignment $f^1, \ldots, f^k$ by f. We define f_e to be the amount of resource e consumed by the assignment f, namely

$$f_e = \sum_{i=1}^{k} \sum_{S \in \mathcal{S}_i : e \in S} f_S^i.$$

Remark 4.4.1 It is easy to see that the traffic routing model of Chapter 2 is a nonatomic congestion game, with network edges as resources and commodities defining player types with strategy sets equal to collections of source-destination paths. On the other hand, nonatomic congestion games are more general than the traffic routing model: the strategy sets $\mathcal{S}_i$ are not assumed to possess any special structure (such as that enjoyed by paths with a common source and destination) and are not assumed to be disjoint for different player types.

The *cost* $\ell_S(f)$ of a strategy S with respect to an assignment f is $\sum_{e \in S} \ell_e(f_e)$; the *social cost* $C(f)$ of an assignment is

$$C(f) \equiv \sum_{i=1}^{k} \sum_{S \in \mathcal{S}_i} \ell_S(f) f_S^i = \sum_{e \in E} \ell_e(f_e) f_e.$$

We will call an assignment minimizing $C(\cdot)$ *min-cost* or *optimal*. Our final definition extends Definition 2.2.1 of flows at Nash equilibrium to NCGs.

Definition 4.4.2 An assignment for a NCG is at *Nash equilibrium* (or is a *Nash assignment*) if for all $i \in \{1, \ldots, k\}$, $S_1, S_2 \in \mathcal{S}_i$ with $f_{S_1}^i > 0$, and $\delta \in (0, f_{S_1}^i]$, we have $\ell_{S_1}(f) \leq \ell_{S_2}(\tilde{f})$, where

$$\tilde{f}_S^j = \begin{cases} f_S^j - \delta & \text{if } j = i \text{ and } S = S_1 \\ f_S^j + \delta & \text{if } j = i \text{ and } S = S_2 \\ f_S^j & \text{otherwise.} \end{cases}$$

[1]See Subsection 4.4.2 below for an explanation of this terminology.

4.4.2 Related Work

Many types of games have been studied under the moniker of "congestion games" in the game theory literature; the salient feature of all such games is that the payoff to a player depends only on the player's strategy and on the number of other players choosing the same or some "interfering" strategy. Rosenthal [157, 158] was the first to describe how the traffic model described in Chapter 2 can be naturally generalized to a more abstract setting, and introduced the name "congestion game"; our model is in many respects inspired by his. However, Rosenthal studied games with a finite number of discrete (or *atomic*) players restricted to playing pure strategies, a requirement that in our notation corresponds to insisting that each function f^i assigns a value of r_i to one strategy and a value of 0 to all other strategies. Rosenthal [157] used a discretized version of the proof of Proposition 2.5.1 to show the existence of a Nash equilibrium (in pure strategies) provided all r_i's are equal, and exhibited a game violating this hypothesis with no Nash equilibrium.

More recently, Rosenthal's work has been extended in several different directions. Monderer and Shapley [127] introduced a class of atomic games they call *potential games*, which by definition are games for which a Nash equilibrium in pure strategies arises as the optimum solution to a related optimization problem (the objective function of which they call a *potential function*). Potential games strictly generalize Rosenthal's congestion games and have since been studied for their own sake [62, 110, 111, 181, 185]. Holzman and Law-Yone [88] studied necessary and sufficient conditions for a congestion game (in the sense of Rosenthal [157]) to possess a Nash equilibrium in pure strategies with certain "nice" properties (such as Pareto-optimality). Several authors considered congestion games in which all strategies are single resources (rather than subsets of resources) but in which different player types experience different amounts of congestion [103, 123, 149] and gave sufficient conditions for the existence of Nash equilibria (as well as "nice" Nash equilibria) in pure strategies; see also Voorneveld et al. [185] for a survey of this work. In addition, many researchers have studied various *nonatomic* versions of congestion games, wherein the number of players is assumed to be so large that an individual has negligible effect on the outcome of the game; see [121, 151, 166] for foundational work on nonatomic games and [22, 73, 124, 126] for work on nonatomic congestion games in particular. The nonatomic setting is the one to which the techniques of Chapter 3 naturally generalize, and is the object of our study for the rest of this section.

Despite the growing body of research on congestion games and their variants, our work is the first to quantify the inefficiency of Nash equilibria in such games.

4.4.3 Bounding the Inefficiency of Nash Equilibria in NCGs

We now observe that all of the results of Chapter 3 carry over (with the same proofs) to the more general setting of NCGs. To see this, we first note that the optimization problem of computing an optimal assignment can be modeled as a convex program identical to the program (NLP) of Section 2.3; because of this, all of the key propositions of Sections 2.2, 2.3, and 2.5 extend to the setting of NCGs without difficulty. For example, we have the following analogues of Proposition 2.2.2 and Corollary 2.3.2: an assignment f for a NCG is at Nash equilibrium if and only if for every $i \in \{1, \ldots, k\}$ and $S_1, S_2 \in \mathcal{S}_i$ with $f_{S_1} > 0$, $\ell_{S_1}(f) \leq \ell_{S_2}(f)$; in a NCG with standard cost functions (in the sense of Definition 2.3.5), an assignment f is optimal if and only if it is at Nash equilibrium with respect to cost functions ℓ^* (defined as in Section 2.3).

In addition, careful scrutiny of the proofs of Chapter 3 reveals that the combinatorial structure of the underlying network was never used. Because of this, these proofs extend, *mutatis mutandis*, to the more general setting of NCGs. We summarize (somewhat informally) the main consequences below.

Extension of Theorem 3.3.8: Let $\mathcal{L}$ be a class of standard cost functions, with anarchy value $\alpha(\mathcal{L})$ defined as in Definitions 3.3.2 and 3.3.3. In a NCG with cost functions in $\mathcal{L}$, the social cost of an assignment at Nash equilibrium is most $\alpha(\mathcal{L})$ times that of an optimal assignment.

Extension of Theorem 3.4.2: Let $\mathcal{L}$ be a class of standard cost functions containing the constant functions. Then the worst-case ratio between the social cost of Nash and optimal assignments in NCGs with cost functions in $\mathcal{L}$ is achieved (up to an arbitrarily small additive factor) by NCGs with two resources and one player type with only singleton strategies.

Extension of Theorem 3.4.4: Let $\mathcal{L}$ be a class of standard cost functions that is diverse in the sense that $\{\ell(0) : \ell \in \mathcal{L}\} = (0, \infty)$. Then the worst-case ratio between the social cost of Nash and optimal assignments in NCGs with cost functions in $\mathcal{L}$ is achieved (up to an arbitrarily small additive factor) by NCGs with one player type with only singleton strategies.

Extension of Theorem 3.6.1: The social cost of an assignment at Nash equilibrium for a NCG with arbitrary (continuous, nondecreasing) cost functions is at most the social cost of an optimal assignment for the same NCG with twice as many players of each type.

Remark 4.4.3 Further extensions are possible. For example, we can associate a positive real number $a^i_{S,e}$ to each resource e contained in strategy S of $\mathcal{S}_i$ that represents in some sense the "rate of consumption" of resource e by players of type i selecting strategy S. In this new setting, the total consumption f_e of a resource e by an assignment f is defined to be $f_e = \sum_{i=1}^k \sum_{S \in \mathcal{S}_i} a^i_{S,e} f^i_S$. Since we expect a player requiring large (respectively, small) amounts of a particular resource to be correspondingly sensitive (respectively, insensitive) to the cost of that resource, we define the cost $\ell_S(f)$ of strategy $S \in \mathcal{S}_i$ with respect to assignment f by $\ell_S(f) = \sum_{e \in S} a^i_{S,e} \ell_e(f_e)$. For example, if resources are types of food and strategies are dinner possibilities, rates of consumption correspond to amounts of different ingredients needed and the cost $\ell_S(f)$ is the total cost of raw ingredients, with the per-unit price ℓ_e of ingredient e a function of the overall demand f_e. It is then straightforward to adapt the proofs of Chapters 2 and 3 to this setting; in particular, the four results above carry over to this generalization of NCGs. We omit further details, and refer the interested reader to [164].

Remark 4.4.4 The reader may well wonder at this point whether *any* conceivable statement that holds for the network model of Chapter 2 fails to also hold for NCGs. One argument that does not immediately generalize from networks to NCGs is provided by the forthcoming proof of Theorem 5.4.1, which bounds the worst-case severity of losses due to harmful extraneous edges—as in Braess's Paradox—in networks with general latency functions and selfish routing. This proof makes use of the graph-theoretic notions of acyclicity and cuts, notions without analogues in NCGs. Indeed, even the (best-possible) guarantee of Theorem 5.4.1 is a function of the number of network vertices; since NCGs possess the analogue of an edge set (namely, resources) but not of a vertex set, it is unclear what sort of generalization of Theorem 5.4.1 to NCGs can be hoped for.

Part III

Coping with Selfishness

Chapter 5

Designing Networks for Selfish Users

Motivated by our previous examples showing that flows at Nash equilibrium may be quite inefficient, in this chapter and the next we study methods for *coping with selfishness*—that is, for ensuring that selfish behavior results in a desirable outcome. In this chapter, we explore the following idea for ameliorating the degradation in network performance due to selfish routing: armed with the knowledge that our networks will be host to selfish users, how can we design them to minimize the inefficiency inherent in a user-defined equilibrium?

5.1 Introduction

A natural measure for the performance of a network host to selfish users sharing a common source s and destination t is the common latency experienced by each user in a flow at Nash equilibrium (see Proposition 2.2.2). Recall that Braess's Paradox (see Subsections 1.2.2 and 2.4.2) demonstrates how removing edges from a network may *improve* its performance. This phenomenon suggests the following network design problem: given a network with latency functions on the edges and a traffic rate, which edges should be removed to obtain the best possible flow at Nash equilibrium? Equivalently, given a large network of candidate edges to build, which subnetwork will exhibit the best performance when used selfishly?

5.1.1 Summary of Results

We give optimal inapproximability results and approximation algorithms for several network design problems of the following type: given a network with edge latency functions, a single source-destination pair, and a rate of traffic, find the subnetwork minimizing the travel time of all (selfish) network users in a flow at Nash equilibrium. Specifically, we prove the following for any $\epsilon > 0$ (assuming $P \neq NP$):

- General Latency Network Design: for networks with continuous, nonnegative, nondecreasing edge latency functions, there is no $(n/2 - \epsilon)$-approximation algorithm for network design, where n is the number of vertices in the network. We also prove this hardness result to be best possible by exhibiting an $n/2$-approximation algorithm for the problem.

- Linear Latency Network Design: for networks in which the latency of each edge is a linear function of the congestion, there is no $(\frac{4}{3} - \epsilon)$-approximation algorithm for network design. The existence of a $\frac{4}{3}$-approximation algorithm follows easily from our work bounding the price of anarchy in such networks, proving this hardness result sharp.

Moreover, we prove that an optimal approximation algorithm for these problems is what we call the *trivial algorithm*: given a network of candidate edges, build the entire network. As a consequence of the optimality of the trivial algorithm, we prove that inefficiency due to harmful extraneous edges is impossible to detect efficiently, even in worst-possible instances.

Finally, we consider additional classes of latency functions (such as polynomials of bounded degree) and show that our strong hardness results are not particular to the classes of general and linear latency functions.

5.1.2 Related Work

Paradoxes

Braess's Paradox [28] (as described in Subsections 1.2.2 and 2.4.2) has intrigued researchers ever since its discovery, appearing frequently in textbooks [40, 76, 107, 129, 133, 169] and the popular science literature [10, 14, 15, 35, 101, 146]. In addition, Braess's Paradox has led to many further research developments: it has catalyzed the search for other "paradoxes" in traffic networks [9, 33, 51, 63, 82, 92, 93, 97, 172, 175, 191] and for analogues of Braess's paradox in queueing networks [31, 37, 38, 188] as well as in seemingly unrelated contexts (such as the strings and springs example of Subsection 1.3.1) [16, 30, 36, 96]; renewed interest in the older "Downs-Thomson paradox" [29, 55, 180]; and even stirred debate over its implications for classical philosophical problems [89, 120].

Network Design

Motivated by the discovery of Braess's Paradox and evidence of similarly counter-intuitive and counterproductive traffic behavior following the construction of new roads in congested cities [64, 100, 101, 128], researchers have tried to develop network design strategies that avoid Braess's Paradox. However, scant progress has been made on the network design problem that we study, either computationally or theoretically. Indeed, early computational work on the problem either focused on very small networks [114] or admitted to ignoring congestion effects entirely, due to the difficulties involved [27, 52, 86, 154, 168, 189]; in a 1984 survey, Magnanti and Wong describe the problem as "essentially unsolved" from a practical perspective [119, P.15]. On the theoretical side, all existing work on the network design problem that we study and on the problem of detecting the presence of harmful extraneous edges either exclusively considers the four-node networks of Figure 1.2 [71, 143, 144] or other very special classes of networks [72, 125], or focuses entirely on the special case where only *one* edge is to be added or deleted from the network (as opposed to seeking the best *subgraph* of a network, which may contain many fewer edges than the entire network) [48, 125, 176, 178]. Prior to our work the network design problem studied in this chapter was not known to be NP-hard, nor was any heuristic for the problem known to have a finite approximation ratio.

Indirectly related to our work is a series of papers [7, 60, 105, 106, 116] that study traffic models that differ from ours in that latency functions are assumed to be capacitated (often with the M/M/1 delay functions described in Section 3.5) and network users are assumed to control a strictly positive (rather than negligible) fraction of the overall traffic. These works consider the problem of allocating a fixed amount of additional capacity to network edges to obtain the largest possible improvement in the Nash equilibrium. Since these papers either confine themselves to networks of parallel links or provide only sufficient (and far from necessary) conditions for a given capacity allocation to improve network performance, they are not directly relevant for our network design problem.

Finally, we point out that the network design problem considered here is fundamentally different from most of the network design problems studied in the theoretical computer science literature (such as those described by Goemans and Williamson [79]), which typically ask for the cheapest network satisfying certain desiderata such as high connectivity or small diameter. Problems of this sort are only nontrivial in the presence of costs on vertices and/or edges; otherwise, the best solution is to simply build the largest possible network. On the other hand, Braess's Paradox shows this approach to be suboptimal for our network design problem; even in the absence of costs, it is not at all clear which network should be preferred.

5.1.3 Organization

We begin in Section 5.2 by quickly formalizing how we encode network design instances in a machine-readable way. In each of the next three sections, we prove matching upper and lower bounds on the approximability of network design for a different class of allowable edge latency functions. Linear latency functions are considered in Section 5.3, general (continuous and nondecreasing) latency functions in Section 5.4, and polynomial latency functions in Section 5.5.

5.2 Encodings of Latency Functions

We have thus far studied analytic problems (bounding the inefficiency of Nash flows) rather than algorithmic ones; for this reason, we have not yet needed to describe how an instance (G, r, ℓ) should be encoded as input to a (mathematical model of a) computer. Before we present complexity results for the problem of network design, we must be precise about our encodings of network latency functions (encoding the network G and a rational rate vector r can be done via any standard method—see Aho et al. [2], for example). In this chapter, we assume that every edge latency function is either a polynomial with rational coefficients (in which case the input complexity is the number of bits needed to represent the values and positions of the coefficients) or a piecewise linear function described by a finite number of rational slopes and breakpoints (in which case the input complexity is the number of bits needed to describe the slopes and breakpoints). Less restrictive assumptions are of course possible, but they render hardness of approximation results less compelling.

5.3 Linear Latency Functions: An Approximability Threshold of $\frac{4}{3}$

We begin with the setting in which the latency of every edge of the network is a linear function of the congestion (that is, each latency function ℓ_e may be written $\ell_e(x) = a_e x + b_e$ for $a_e, b_e \geq 0$), as our proof of the inapproximability of network design is particularly simple in this special case.

We next formalize our network design problem. Throughout this chapter, we assume all instances to be single-commodity. By Propositions 2.2.2 and 2.5.1, the following definition makes sense: for an instance (G, r, ℓ) with source s and destination t admitting a Nash flow f, we define $L(G, r, \ell)$ to be the common latency (with respect to f) of every s-t flow path of f. If G has no s-t path, we define $L(G, r, \ell) = +\infty$. When no confusion results, we will abbreviate the expression $L(G, r, \ell)$ by $L(G)$. We may then formally state our network design problem as follows:

Given an instance (G, r, ℓ), find the subgraph H of G minimizing $L(H, r, \ell)$.

Recall that the *trivial algorithm*, when presented with instance (G, r, ℓ), outputs the network G (i.e., always decides to build the entire network). That the trivial algorithm is a $\frac{4}{3}$-approximation algorithm for LINEAR LATENCY NETWORK DESIGN is an easy corollary (essentially identical to Corollary 3.2.7) of Theorem 3.2.6, which states that in a network with linear latency functions a Nash flow has cost at most $\frac{4}{3}$ times that of any other feasible flow.

Corollary 5.3.1 *The trivial algorithm is a $\frac{4}{3}$-approximation algorithm for* LINEAR LATENCY NETWORK DESIGN.

Proof. Consider any instance (G, r, ℓ) with linear latency functions, with subgraph H minimizing $L(H, r, \ell)$. Let f and f^* denote flows at Nash equilibrium for (G, r, ℓ) and (H, r, ℓ), respectively. By Proposition 2.2.4, we may write $C(f) = r \cdot L(G, r, \ell)$ and $C(f^*) = r \cdot L(H, r, \ell)$. Since f^* is also feasible for (G, r, ℓ), Theorem 3.2.6 implies that $C(f) \leq \frac{4}{3} C(f^*)$ and hence $L(G, r, \ell) \leq \frac{4}{3} L(H, r, \ell)$. ∎

The main result of this section is that, unless $P = NP$, no better approximation is possible in polynomial time.

Theorem 5.3.2 *For any $\epsilon > 0$, there is no $(\frac{4}{3} - \epsilon)$-approximation algorithm for* LINEAR LATENCY NETWORK DESIGN *unless $P = NP$.*

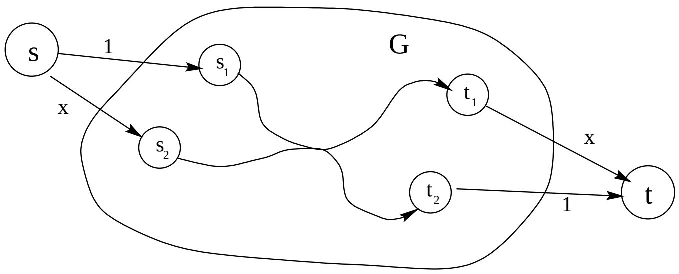


Figure 5.1: Proof of Theorem 5.3.2. In a "no" instance of 2DDP, existence of s_1-t_1 and s_2-t_2 paths implies the existence of an s_2-t_1 path.

Proof. We will make use of the problem 2 Directed Disjoint Paths (2DDP): given a directed graph $G = (V, E)$ and distinct vertices $s_1, s_2, t_1, t_2 \in V$, are there s_i-t_i paths P_i for $i = 1, 2$, such that P_1 and P_2 are vertex-disjoint? This problem was proved NP-complete by Fortune et al. [70]. We will show that a $(\frac{4}{3} - \epsilon)$-approximation algorithm for Linear Latency Network Design can be used to distinguish "yes" and "no" instances of 2DDP in polynomial time.

Consider an instance $\mathcal{I}$ of 2DDP, as above. Augment the vertex set V by an additional source s and sink t, and include directed edges $(s, s_1), (s, s_2), (t_1, t), (t_2, t)$ (see Figure 5.1). Denote the new network by $G' = (V', E')$ and endow the edges of E' with linear latency functions ℓ as follows: all edges of E are given the latency function $\ell(x) = 0$, edges (s, s_2) and (t_1, t) are given the latency function $\ell(x) = x$, and edges (s, s_1) and (t_2, t) are given the latency function $\ell(x) = 1$. This construction can clearly be done in polynomial time.

To complete the proof, it suffices to show the following two statements: (i) if $\mathcal{I}$ is a "yes" instance of 2DDP, then there is a subgraph H of G' satisfying $L(H, 1, \ell) = \frac{3}{2}$; (ii) if $\mathcal{I}$ is a "no" instance, then for any subgraph H of G', $L(H, 1, \ell) \geq 2$.

To prove (i), let P_1 and P_2 be vertex-disjoint s_1-t_1 and s_2-t_2 paths in G, respectively, and obtain H by deleting all edges of G not contained in some P_i. Then, H is a subgraph of G' with exactly two s-t paths, and routing half a unit of flow along each yields a flow at Nash equilibrium in which each path has latency $\frac{3}{2}$ (cf., Figure 1.2(a)).

For (ii), we may assume that H contains an s-t path. If H has an s-t path P containing an s_2-t_1 path, then define a flow f by routing a single unit of flow on P; this is a flow at Nash equilibrium, with respect to which every s-t path has latency 2 (cf., Figure 1.2(b)), so $L(H) = 2$. Otherwise, since $\mathcal{I}$ is a "no" instance, there are only two remaining possibilities (see Figure 5.1): either for precisely one $i \in \{1, 2\}$, H has an s-t path P containing an s_i-t_i path, or all s-t paths P in H contain an s_1-t_2 path of G. In either case, routing one unit of flow along such a path P provides a flow at Nash equilibrium showing that $L(H) = 2$. ■

Corollary 5.3.1 and Theorem 5.3.2 imply that efficiently detecting whether or not network performance is hampered by harmful extraneous edges in networks with linear latency functions is impossible, even in instances suffering from the most severe manifestations of this paradox. To make this statement precise, call an instance (G, r, ℓ) with linear latency functions *paradox-free* if $L(H, r, \ell) \geq L(G, r, \ell)$ for all subgraphs H of G (i.e., if the entire network is an optimal subnetwork) and *paradox-ridden* if for some subgraph H in G, $L(H, r, \ell) = \frac{3}{4}L(G, r, \ell)$. By Corollary 5.3.1, paradox-ridden instances are precisely those incurring a worst-possible loss in network performance due to detrimental extra edges. The construction in the proof of Theorem 5.3.2 then gives the following corollary.

Corollary 5.3.3 *Given an instance (G, r, ℓ) with linear latency functions that is either paradox-free or paradox-ridden, it is NP-hard to decide whether or not (G, r, ℓ) is paradox-ridden.*

5.4 General Latency Functions: An Approximability Threshold of $\lfloor n/2 \rfloor$

In this section we consider the problem of network design with the broadest possible class of latency functions (assuming we insist on the existence and uniqueness of flows at Nash equilibrium), the set of all continuous nondecreasing functions. We begin by proving in Subsection 5.4.1 that the trivial algorithm achieves an approximation ratio of $\lfloor n/2 \rfloor$, where n is the number of vertices in the network (in contrast to other sections, this performance guarantee does not trivially follow from our work bounding the price of anarchy). In Subsection 5.4.2, we introduce a new family of graphs generalizing the network of the original Braess's Paradox (Figure 1.2(b)); this family may be of independent interest, as (to the best of our knowledge) these networks give the first demonstration that the severity of Braess's Paradox can increase with the network size. We conclude in Subsection 5.4.3 by using this family to prove an optimal hardness result matching the upper bound provided by the trivial algorithm.

5.4.1 An $\lfloor n/2 \rfloor$-Approximation Algorithm

Our goal in this subsection is to prove that the trivial algorithm is an $\lfloor n/2 \rfloor$-approximation algorithm for General Latency Network Design, where n is the number of vertices in the network. Before embarking on the proof, it is important to contrast the settings of general and linear latency functions. In particular, we saw in the proof of Corollary 5.3.1 that a known result upper bounding the total latency of a Nash flow relative to any other feasible flow immediately yielded an identical upper bound on the performance of the trivial algorithm. Thus, if we knew that a Nash flow in a network with n vertices and general latency functions was at most $g(n)$ times as costly (with respect to the total latency measure) as any other feasible flow for some "nice" (e.g., linear) function $g(\cdot)$, we would be done.[1] Unfortunately, the nonlinear variant of Pigou's example (Subsection 2.4.4) shows that no such result can hold: with general latency functions, a Nash flow may be *arbitrarily* more costly than other feasible flows, even in networks with only two vertices and two edges.

However, this fact is not due cause for abandoning the goal of proving some kind of performance guarantee for the trivial algorithm; it merely indicates that a more delicate approach is required. In the example of Subsection 2.4.4, the flow with near-zero cost was far from at equilibrium: a few martyrs were routed on the upper edge (the edge with constant latency function $\ell(x) = 1$) for the benefit of the overwhelming majority of the flow (on the lower edge). Indeed, all (non-empty) subgraphs H of G satisfy $L(H) = 1$. Thus, while any subgraph provides an optimal solution to our network design problem, we have no way of proving any finite approximation ratio!

By comparing the output of the trivial algorithm only to feasible flows at equilibrium in a subgraph of G (rather than to *all* feasible flows), we obtain the main result of this subsection.

Theorem 5.4.1 *For any instance (G, r, ℓ) with $|V(G)| = n$, the trivial algorithm returns a solution of value at most $\lfloor \frac{n}{2} \rfloor$ times that of the optimal solution.*

Proof. Let f and f^* be flows at Nash equilibrium for (G, r, ℓ) and (H, r, ℓ), respectively, with H a subgraph of G containing an s-t path. By Propositions 2.5.1 and 2.6.2, we may assume that f is acyclic. Put $L = L(G, r, \ell)$ and $L^* = L(H, r, \ell)$; we wish to prove that $L \leq \lfloor n/2 \rfloor \cdot L^*$.

[1] Indeed, this argument will reoccur in Section 5.5.

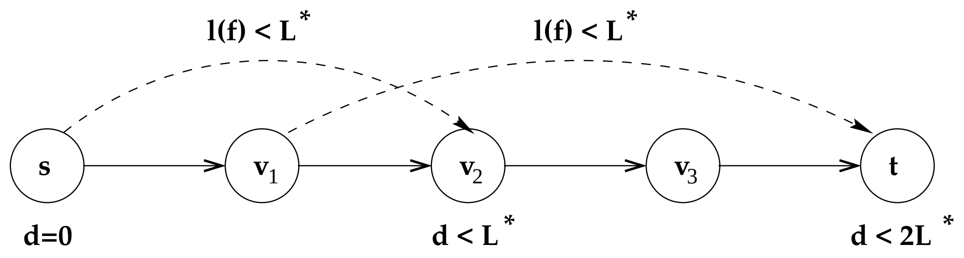


Figure 5.2: Proof of Theorem 5.4.1. If f is the flow sending one unit of flow on the four-hop path and f^* is the flow sending half a unit of flow on each of the other two paths, then the dashed edges are light.

The rest of the proof will make crucial use of Proposition 2.6.1. Accordingly, define $d(v)$ for $v \in V(G)$ as in Proposition 2.6.1, as the length (with respect to edge lengths $\ell_e(f_e)$) of a shortest s-v path. Assume for simplicity that n is odd and that every vertex of G is incident to an edge e with $f_e > 0$; extending the following argument to the general case is straightforward. Order the vertices $s = v_0, v_1, \ldots, v_{n-1} = t$ according to nondecreasing $d(v)$-value. If there is an edge $e = (v, w)$ with $f_e > 0$ and $\ell_e(f_e) = 0$ (so, by Proposition 2.6.1, $d(v) = d(w)$), break the tie by placing v before w in the ordering; this will always be possible since f is acyclic. Proposition 2.6.1 implies that this ordering is a topological one with respect to the flow f—that is, whenever $f_e > 0$, e is a forward edge with respect to our ordering. Our proof approach will be to show, by induction on i, that $d(v_{2i}) \leq i \cdot L^*$; the base case $i = 0$ is trivial.

Before considering the inductive step, we require a definition and a claim. Call an edge e *light* if $f_e \leq f_e^*$ and $f_e^* > 0$ (in particular, e must be present in H). Light edges are useful to us because they have latency at most L^* with respect to f^* (as every flow path of f^* has latency L^*) and hence latency at most L^* with respect to f (since latencies are nondecreasing); thus, vertices of G that are adjacent via a light edge differ in d-values by at most L^*. The next claim assures us of a healthy supply of light edges: every s-t cut consisting of a set of consecutive vertices (with respect to our topological ordering) contains a light edge (see Figure 5.2).

Claim: *Let $S = \{v_0, \ldots, v_k\}$ for some $k \in \{0, 1, \ldots, n-2\}$. Then some light edge has its tail in S and head outside of S.*

Proof. Let $\delta^+(S)$ denote the edges with tail inside S and head outside S, and $\delta^-(S)$ the edges with head inside S and tail outside S. Since S is an s-t cut and f is an

s-t flow of value r with no flow on edges in $\delta^-(S)$ (as the vertices are topologically sorted according to f), $\sum_{e\in\delta^+(S)} f_e = r$. Since S is an s-t cut and f^* is an s-t flow, $\sum_{e\in\delta^+(S)} f^*_e \geq r$. Hence, $f_e \leq f^*_e$ for some $e \in \delta^+(S)$ with $f^*_e > 0$. ■

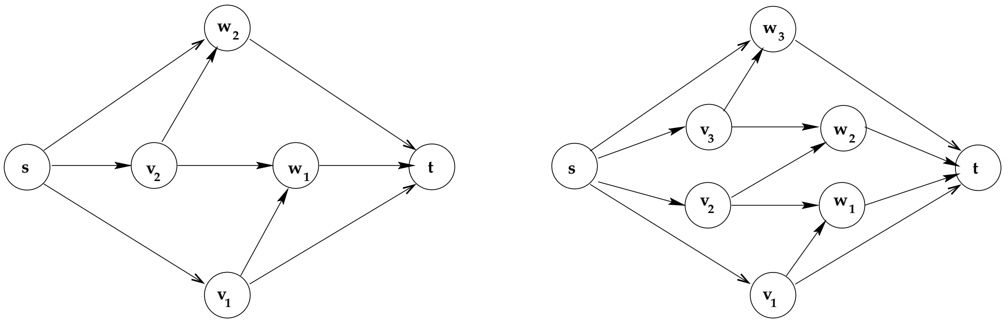
Now suppose $i \in \{1, \ldots, (n-1)/2\}$ and $d(v_{2(i-1)}) \leq (i-1)L^*$. Let k be the largest integer such that there is a path of light edges from v_j to v_k for some $j \leq 2(i-1)$; we will show that $k \geq 2i$. The previous claim immediately implies that k is well defined with $k > 2(i-1)$ (consider the head of a light edge in $\delta^+(\{v_0, \ldots, v_{2(i-1)}\})$). To see that $k \geq 2i$, observe that if $k = 2i-1$ then all light edges in $\delta^+(\{v_0, \ldots, v_{2(i-1)}\})$ (and there must be one) have head v_{2i-1} and no light edge in $\delta^+(\{v_0, \ldots, v_{2i-1}\})$ has tail v_{2i-1} (otherwise we would append such an edge to our maximal path), contradicting that $\delta^+(\{v_0, \ldots, v_{2i-1}\})$ must contain a light edge.

We have established the existence of a path P of light edges from v_j to v_k with $j \leq 2(i-1)$ and $k \geq 2i$. Inductively, we have $d(v_j) \leq d(v_{2(i-1)}) \leq (i-1)L^*$; since $d(v_{2i}) \leq d(v_k)$, we can finish the inductive step and the proof by showing that $d(v_k) - d(v_j) \leq L^*$ (informally, $d(v_{2(i-1)})$ and $d(v_{2i})$ are sandwiched between $d(v_j)$ and $d(v_k)$, so it suffices to upper bound the gap between the latter pair of numbers). Letting $d^*(v)$ denote the length of a shortest s-v path in H with respect to edge lengths $\ell_e(f^*_e)$, we can apply Proposition 2.6.1 to f^* in H to obtain $0 = d^*(s) \leq d^*(v_j) \leq d^*(v_k) \leq d^*(t) = L^*$. By Proposition 2.6.1, this implies that the latency of P with respect f^* is at most L^*; since all edges of P are light, it follows that the latency of P with respect to f is at most L^*. A final application of Proposition 2.6.1 then yields $d(v_k) - d(v_j) \leq L^*$, completing the inductive step and the proof. ■

5.4.2 The Braess Graphs

We seek to prove a lower bound on the approximability of network design (and in particular, on the performance of the trivial algorithm) that is linear in the number of vertices of the network. Toward this end, we will construct an infinite family of networks on which the trivial algorithm performs poorly (networks in which the value of a flow at Nash equilibrium can be vastly improved by removing some edges); we will prove hardness results in the next subsection via similar but more involved arguments.

We define the *kth Braess graph* B^k as follows: start with a set $V^k = \{s, v_1, \ldots, v_k, w_1, \ldots, w_k, t\}$ of $2k+2$ vertices and define E^k by $\{(s, v_i), (v_i, w_i), (w_i, t) : 1 \leq i \leq k\} \cup \{(v_i, w_{i-1}) : 2 \leq i \leq k\} \cup \{(v_1, t)\} \cup \{(s, w_k)\}$ (see Figure 5.3). We note



(a) B^2 (b) B^3

Figure 5.3: The second and third Braess graphs

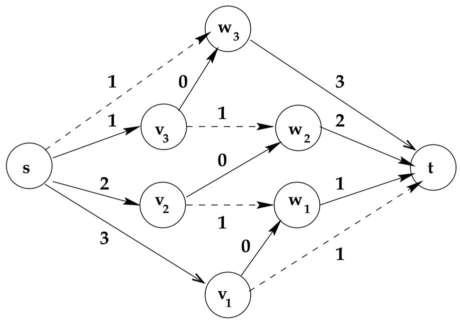
that B^1 is the graph in which Braess's Paradox was first discovered (Figure 1.2(b)).

We next define latency functions ℓ^k for the edges of B^k; these functions will prove useful in Proposition 5.4.2 below. For each edge of the form $e = (v_i, w_i)$, put $\ell^k_e(x) = 0$; for an edge e of the form (v_i, w_{i-1}), (s, w_k), or (v_1, t), put $\ell^k_e(x) = 1$; for $i \in \{1, 2, \ldots, k\}$ and an edge e of the form (w_i, t) or (s, v_{k-i+1}), put $\ell^k_e(x)$ equal to any nonnegative, continuous, and nondecreasing function satisfying $\ell^k_e(\frac{k}{k+1}) = 0$ and $\ell^k_e(1) = i$ (thus, ℓ^k_e may be chosen to be convex and infinitely differentiable, if desired).

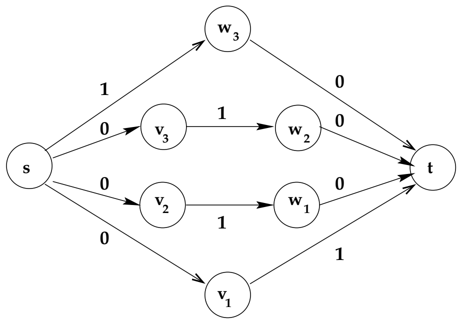
We can now show how to use the Braess graphs to construct instances on which the trivial algorithm for General Latency Network Design performs badly.

Proposition 5.4.2 *For any integer $n \geq 2$, there is an instance (G, r, ℓ) with $|V(G)| = n$ for which the trivial algorithm produces a solution with value at least $\lfloor \frac{n}{2} \rfloor$ times that of the optimal solution.*

Proof. We may suppose that n is even and at least four (for n odd, take a bad example for $n - 1$ and add an isolated vertex). Write $n = 2k + 2$ for $k \in \mathcal{N}$ and consider the instance (B^k, k, ℓ^k). For $i = 1, \ldots, k$, let P_i denote the path $s \to v_i \to w_i \to t$. For $i = 2, \ldots, k$, let Q_i denote the path $s \to v_i \to w_{i-1} \to t$; define Q_1 to be the path $s \to v_1 \to t$ and Q_{k+1} the path $s \to w_k \to t$. On one hand, routing one unit of flow on each of $P_1, \ldots, P_k$ yields a flow at Nash equilibrium for (B^k, k, ℓ^k) demonstrating that $L(B^k, k, \ell^k) = k + 1$ (see Figure 5.4(a) for an illustration when $k = 3$). On the other hand, if H is the subgraph obtained from B^k by deleting all edges of the form (v_i, w_i), routing $\frac{k}{k+1}$ units of flow on each of $Q_1, \ldots, Q_{k+1}$



(a) Nash flow for $(B^3, 3, \ell^3)$



(b) Nash flow in the optimal subgraph

Figure 5.4: Proof of Proposition 5.4.2, when $k = 3$. Solid edges carry flow in the flow at Nash equilibrium, dashed edges do not. Edge latencies are with respect to flows at Nash equilibrium.

yields a flow at Nash equilibrium for (H, k, ℓ^k) showing that $L(H, k, \ell^k) = 1$ (see Figure 5.4(b)). Thus, $L(G)/L(H) = k + 1 = n/2$, completing the proof. ■

5.4.3 Proof of Hardness

We begin with an informal description of the reduction. Recall that in an instance of the NP-hard problem Partition, we are given q positive integers $\{a_1, a_2, \ldots, a_q\}$ and seek a subset $S \subseteq \{1, 2, \ldots, q\}$ such that $\sum_{j \in S} a_j = \frac{1}{2} \sum_{j=1}^{q} a_j$ [77, SP12]. The idea of the reduction is to start with a Braess graph and replace the edges of the form (v_i, w_i) with a collection of parallel edges representing an instance $\mathcal{I} = \{a_1, \ldots, a_q\}$ of Partition. We will endow these edges with latency functions that simulate "capacities", with an edge representing an integer a_j of $\mathcal{I}$ receiving capacity a_j. Roughly speaking, if too many edges are removed from the network, there will be insufficient remaining capacity to send flow cheaply; if too few edges are removed, the excess of capacity results in a Nash flow similar to that of Figure 5.4(a); and if $\mathcal{I}$ is a "yes" instance of Partition and an appropriate collection of edges is removed, then the remaining network admits a Nash flow similar to that of Figure 5.4(b). (We can also obtain a nearly optimal inapproximability result using a strongly NP-complete problem; see Remark 5.4.6 below.)

Theorem 5.4.3 *For $\epsilon > 0$, there is no $(\lfloor n/2 \rfloor - \epsilon)$-approximation algorithm for* General Latency Network Design *unless $P = NP$.*

Proof. We prove that for any fixed $n \geq 2$, there is no $(\lfloor \frac{n}{2} \rfloor - \epsilon)$-approximation algorithm for General Latency Network Design restricted to (multi)graphs with n vertices. (We can also restrict our instances to be simple networks and derive a nearly optimal inapproximability result—see Remark 5.4.5 below.) As in the proof of Proposition 5.4.2, we may assume that n is even and at least four. Write $n = 2k + 2$ for $k \in \mathcal{N}$. We will show that an $(\frac{n}{2} - \epsilon)$-approximation algorithm for graphs with n vertices enables us to differentiate between "yes" and "no" instances of Partition in polynomial time.

Consider an instance $\mathcal{I} = \{a_j\}_{j=1}^{q}$ of Partition, with each a_j a positive integer. We may assume that each a_j is even, scaling if necessary. Put $A = \sum_{j=1}^{q} a_j$; the traffic rate of interest to us is $r = k\frac{A}{2} + k + 1$. Obtain a graph G from the kth Braess graph B^k by replacing each edge of the form (v_i, w_i) by q parallel edges, and denote these by $e_i^1, e_i^2, \ldots, e_i^q$.

We now specify the edge latency functions ℓ, which are more complicated than in the previous subsection. We require a sufficiently small constant δ ($1/A(q + k)$ is small enough) and a sufficiently large constant M ($n/2$ is large enough). In what follows, the constant M should be interpreted as a substitute for $+\infty$, and is used to penalize a flow for violating an edge capacity constraint. We require the constant δ to transform step functions (the type of function that would be most convenient for our argument) into continuous functions (which are allowable in our model); δ provides a small "window" in which to "smooth out" the discontinuities of a step function. For each edge e of the form (v_i, w_{i-1}), (s, w_k), or (v_1, t), define $\ell_e(x) = 1$ for $x \leq 1$ and $\ell_e(x) = M$ for $x \geq 1+\delta$ (ℓ_e may be defined arbitrarily on $(1, 1+\delta)$, subject to the usual continuity and monotonicity restrictions). We say that these edges have *capacity* 1. For an edge e of the form (w_i, t) or (s, v_{k-i+1}) (where $i \in \{1, \ldots, k\}$), define $\ell_e(x) = 0$ for $x \leq \frac{1}{2}A + 1$, $\ell_e(x) = i$ when $x = \frac{1}{2}A + \frac{k+1}{k}$, and $\ell_e(x) = M$ for $x \geq \frac{1}{2}A + \frac{k+1}{k} + \delta$; these edges have capacity $\frac{1}{2}A + \frac{k+1}{k}$. Finally, for an edge e of the form e_i^j, define $\ell_e(x) = 0$ for $x \leq a_j - \delta$, $\ell_e(a_j) = 1$, and $\ell_e(x) = M$ for $x \geq a_j + \delta$; thus e_i^j has capacity a_j. Each latency function can be described by a piecewise linear function with a small (constant) number of rational breakpoints and slopes, and the instance (G, r, ℓ) can be constructed from $\mathcal{I}$ in polynomial time.

Analogous to the proof of Theorem 5.3.2, it suffices to prove the following two statements: (i) if $\mathcal{I}$ is a "yes" instance, then G admits a subgraph H with $L(H, r, \ell) =$

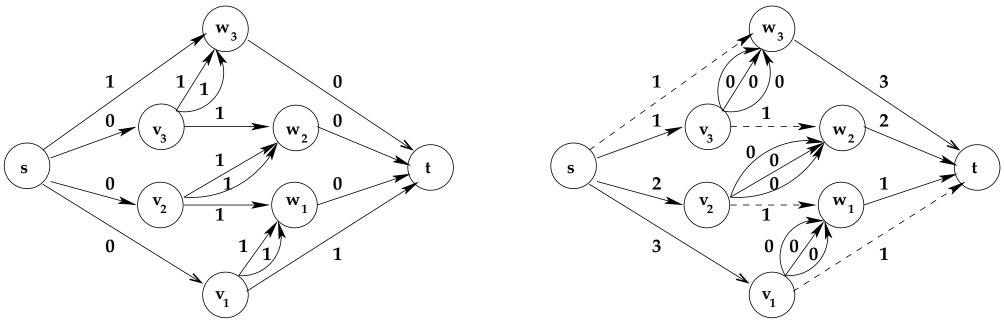
1; and (ii) if $\mathcal{I}$ is a "no" instance, then $L(H, r, \ell) \geq n/2$ for every subgraph H of G.

To prove (i), suppose that $\mathcal{I}$ admits a partition, and reindex the a_j's so that $\sum_{j=1}^{m} a_j = A/2$ for some $m \in \{1, 2, \ldots, q-1\}$. Obtain H from G by deleting all edges of the form e_i^j for $j > m$; thus, for each $i = 1, \ldots, k$, the remaining edges of the form e_i^j have total capacity $A/2$. Define the paths $Q_1, \ldots, Q_{k+1}$ as in the proof of Proposition 5.4.2: for $i = 2, \ldots, k$, Q_i denotes the path $s \to v_i \to w_{i-1} \to t$, Q_1 is the path $s \to v_1 \to t$, and Q_{k+1} is the path $s \to w_k \to t$. Define a feasible flow f as follows: for each $i = 1, \ldots, k$ and $j = 1, \ldots, m$, route a_j units of flow on the unique path containing edge e_i^j, and route 1 unit of flow on the path Q_i for $i = 1, 2, \ldots, k+1$. The flow f is at Nash equilibrium for (H, r, ℓ) and proves that $L(H, r, \ell) = 1$ (see Figure 5.5(a)).

In proving (ii), we first consider only subgraphs H that contains all edges not of the form e_i^j (i.e., H may be obtained from G by deleting only some of the parallel edges); as we will see, this case captures all of the difficulties of the proof. There are two subcases to consider.

Case 1: Suppose for each $i = 1, \ldots, k$, the total capacity A_i of edges of the form e_i^j in H is at least $A/2$. Since $\mathcal{I}$ is a "no" instance and each a_j is even, $A_i \geq A/2 + 2$ for each i. Then, define a flow f in G as follows: for each $i = 1, \ldots, k$ and $j = 1, \ldots, q$ such that e_i^j is present in H, route $\frac{a_j}{A_i}(\frac{A}{2} + \frac{k+1}{k})$ units of flow along the unique s-t path containing e_i^j. The flow f is at Nash equilibrium and proves that $L(H) = n/2$ (see Figure 5.5(b)).

Case 2: Suppose for some $i \in \{1, \ldots, k\}$, the total capacity A_i of edges of the form e_i^j in H is less than $A/2$ (and thus is at most $A/2 - 2$). Here, we will exploit the fact that all edges of the network are (essentially) capacitated to prove that a flow at Nash equilibrium must have large cost. Call an edge e *oversaturated* by a flow f if f_e exceeds the capacity of e by at least δ (and thus $\ell_e(f_e) = M \geq n/2$). A key observation is that if f is at Nash equilibrium for (H, r, ℓ) and oversaturates some edge, then $L(H, r, \ell) \geq n/2$. Now, since the total capacity of edges out of v_i is at most $A/2 - 1$ (recall (v_i, w_{i-1}) has capacity 1), any flow that places at least $\frac{A}{2} - 1 + q\delta$ units of flow on (s, v_i) will oversaturate some edge out of v_i. On the other hand, the total capacity of edges incident to s is $k\frac{A}{2} + k + 2 = r + 1$, so any feasible flow must either place at least $\frac{A}{2} - 1 + q\delta$ units of flow on (s, v_i) or oversaturate some other edge out of s (for δ sufficiently small). We conclude that any flow feasible for (H, r, ℓ) oversaturates at least one edge, and hence $L(H) \geq n/2$.



(a) A good Nash flow corresponding to a "yes" instance of PARTITION, with $m = 2$

(b) A bad Nash flow in a network with excess capacity

Figure 5.5: Proof of Theorem 5.4.3. Solid edges carry flow in the flow at Nash equilibrium, dashed edges do not. Edge latencies are with respect to flows at Nash equilibrium.

Finally, suppose H fails to contain an edge that is not of the form e_i^j. If for some $i \in \{1, 2, \ldots, k\}$, the total capacity of edges of the form e_i^j is at most $A/2$, then the argument of Case 2 still applies to show that $L(H) \geq n/2$—the previous argument merely required that any feasible flow oversaturates some edge, and this fact remains valid if we remove further edges. Also, if H fails to contain an edge of the form (s, v_i) or (w_i, t), then simple capacity considerations show that any feasible flow in H oversaturates some edge incident to s or t, respectively. If H contains all edges of the form (s, v_i) and (w_i, t) and the total capacity of edges of the form e_i^j in H is at least $A/2$ for each i, then the argument of Case 1 applies (by hypothesis, all edges used by the Nash flow in that case are present in H), showing that $L(H) = n/2$. This exhausts all possible cases, and the proof is complete. ∎

The matching upper and lower bounds of Theorems 5.4.1 and 5.4.3 have strong negative consequences for the problem of detecting harmful extraneous edges, as in the linear latency function setting (see Corollary 5.3.3). Defining an instance (G, r, ℓ) with general latency functions and n vertices to be *paradox-free* if $L(H, r, \ell) \geq L(G, r, \ell)$ for all subgraphs H of G and *paradox-ridden* if for some subgraph H in G, $L(H, r, \ell) = (\lfloor n/2 \rfloor)^{-1} L(G, r, \ell)$, we obtain the following corollary.

Corollary 5.4.4 *Given an instance (G, r, ℓ) with general latency functions that is* *either paradox-free or paradox-ridden, it is NP-hard to decide whether or not* (G, r, ℓ) *is paradox-ridden.*

Remark 5.4.5 The reduction of Theorem 5.4.3 also shows that, for any constant $\epsilon > 0$, there is no $O(n^{1-\epsilon})$-approximation algorithm for General Latency Network Design restricted to simple graphs (unless $P = NP$). To see why, choose a positive integer k satisfying $k > \frac{1}{\epsilon}$, and for a Partition instance $\mathcal{I}$ with q items, mimic the previous reduction beginning with the Braess graph B^{q^k} on $2q^k + 2$ vertices. Subdividing all parallel edges in the resulting multigraph yields a simple graph G (whose size is polynomial in that of $\mathcal{I}$) with $n = q^{k+1} + 2q^k + 2$ vertices. Defining r and ℓ as in the proof of Theorem 5.4.3, G has a subgraph H satisfying $L(H, r, \ell) = 1$ if $\mathcal{I}$ is a "yes" instance while $L(H, r, \ell) \geq q^k + 1$ for every subgraph H if $\mathcal{I}$ is a "no" instance. Thus, no $O(n^{(k-1)/k})$-approximation algorithm exists for General Latency Network Design restricted to simple graphs, unless $P = NP$.

Remark 5.4.6 The reduction of Theorem 5.4.3 makes use of the weakly NP-hard problem Partition [77], and it thus reasonable to ask for approximation algorithms with good performance guarantee but pseudopolynomial running time (running time polynomial in the network size and in the unary representation of the numbers used to describe the latency functions). Unfortunately, such an algorithm cannot exist (unless $P = NP$): the non-existence of an $O(n^{1-\epsilon})$-approximation algorithm for network design on simple graphs can also be derived from the more complicated construction in the proof of Theorem 5.5.6 below, and this construction relies only on the strongly NP-hard problem 2DDP of Section 5.3.

5.5 Polynomials of Bounded Degree: An Approximability Threshold of $\Theta(\frac{p}{\log p})$

In this section, we aim to show that the strong hardness results of Sections 5.3 and 5.4 extend beyond the particular classes of linear and general latency functions, and seem intrinsic to the problem of designing networks for selfish users. We will focus on networks with latency functions that are polynomials of bounded degree, but our techniques will also have consequences for networks possessing other types of well-behaved latency functions.

As in Section 5.3, we begin by observing that our previous work bounding the worst-case inefficiency of flows at Nash equilibrium yields an upper bound on the performance guarantee of the trivial algorithm. Recall from Proposition 3.5.5 that in an instance with polynomial latency functions of degree p (with all polynomial coefficients nonnegative), the total latency of a flow at Nash equilibrium is at most $[1 - p \cdot (p+1)^{-(p+1)/p}]^{-1}$ times that of any other feasible flow. For clarity, we will work with the following weaker form of Proposition 3.5.5.

Corollary 5.5.1 *There is a constant $c_1 > 0$ so that the following statement holds: if $p \geq 2$ and (G, r, ℓ) is an instance with polynomial latency functions of degree p for which f^* is feasible and f is a flow at Nash equilibrium, then $C(f) \leq c_1 \frac{p}{\ln p} \cdot C(f^*)$.*

As with linear latency functions (see Corollary 5.3.1), we immediately obtain an upper bound on the performance guarantee of the trivial algorithm for our network design problem restricted to networks with polynomial latency functions of degree p. We call this problem Polynomial(p) Latency Network Design.

Corollary 5.5.2 *There is a constant $c_1 > 0$ so that, for any $p \geq 2$, the trivial algorithm is a $c_1 \frac{p}{\ln p}$-approximation algorithm for* Polynomial(p) Latency Network Design.

We next work toward a proof of a matching hardness result. As in Section 5.4, we first give a family of networks (one network for each value of $p \geq 2$) on which the trivial algorithm performs poorly, and then describe how to obtain a general inapproximability result.

Proposition 5.5.3 *There is a constant $c_2 > 0$ so that, for any $p \geq 2$, the worst-case performance guarantee of the trivial algorithm is at least $c_2 \frac{p}{\ln p}$ for* Polynomial(p) Latency Network Design.

Proof. We will again make use of the Braess graphs of Subsection 5.4.2. In Section 5.4, we exploited the fact that general latency functions can be arbitrarily steep to construct a bad example for the trivial algorithm; here, we adapt the previous argument as best we can, given that only low-degree polynomials are available to us.

For a fixed integer p, define a set of latency functions ℓ^k for the edges of B^k as follows (where k is a parameter, depending on p, to be chosen later): for each edge of the form $e = (v_i, w_i)$, put $\ell^k_e(x) = 0$; for an edge e of the form (v_i, w_{i-1}), (s, w_k), or (v_1, t), put $\ell^k_e(x) = 1$; for an edge e of the form (w_i, t) or (s, v_{k-i+1}) put $\ell^k_e(x) = ix^p$. Next, consider the instance (B^k, k, ℓ^k) and define paths $P_1, \ldots, P_k$

and $Q_1, \ldots, Q_{k+1}$ as in Proposition 5.4.2. On one hand, routing one unit of flow on each of $P_1, \ldots, P_k$ yields a flow at Nash equilibrium for (B^k, k, ℓ^k) showing that $L(B^k, k, \ell^k) = k+1$ (as in Figure 5.4(a)). On the other hand, if H is the subgraph obtained from B^k by deleting all edges of the form (v_i, w_i), routing $\frac{k}{k+1}$ units of flow on each of $Q_1, \ldots, Q_{k+1}$ yields a flow at Nash equilibrium for (H, k, ℓ^k) showing that $L(H, k, \ell^k) = 1 + k(\frac{k}{k+1})^p$ (cf., Figure 5.4(b)). Thus,

$$L(H, k, \ell^k) = 1 + k(\frac{k}{k+1})^p \leq 1 + ke^{-p/(k+1)}.$$

For p sufficiently large, we may put $k = \lfloor \frac{p}{2 \ln p} \rfloor - 1 \geq 1$ to obtain $L(H, k, \ell^k) \leq 2$ and $L(B^k, k, \ell^k) = \lfloor \frac{p}{2 \ln p} \rfloor$; this completes the proof. ■

Remark 5.5.4 In the proof of Proposition 5.5.3, we have avoided optimizing constants for the sake of readability. We will make this tradeoff repeatedly in the rest of this section.

Finally, we extend our lower bound on the performance guarantee of the trivial algorithm to an inapproximability result. This task is more difficult than in Section 5.4; a crucial part of the hardness proof of that section leveraged the fact that general latency functions can model edge capacities. This is not entirely possible with low-degree polynomials, and we are forced instead to adapt the arguments of Section 5.3 to larger Braess graphs; in particular, our reduction is from the 2 Directed Disjoint Paths problem rather than from Partition. In essence, restricting the allowable class of latency functions forces us to encode the intractability of an NP-hard problem into the network topology of a network design instance rather than into the edge latency functions.

In preparation for the reduction, we require one preliminary result. The next proposition is a special case of a theorem of Hall [83].

Proposition 5.5.5 ([83]) *Let G be a network with polynomial latency functions ℓ and a single source-destination pair. Then $L(G, r, \ell)$ is a nondecreasing function of r.*

We can now prove that designing networks for selfish users is hard in networks with polynomial edge latency.

Theorem 5.5.6 *There is a constant $c_3 > 0$ so that the following statement holds: if $p \geq 2$ and $\epsilon > 0$, then no $(c_3 \frac{p}{\ln p} - \epsilon)$-approximation algorithm for* Polynomial(p) Latency Network Design *exists, unless $P = NP$.*

Proof. Fix a sufficiently large integer p, and put $k = \lfloor \frac{p}{16 \ln p} \rfloor - 1$ (which is at least 1 for large enough p). For any $\epsilon > 0$, we will show that a $(\frac{k}{5} - \epsilon)$-approximation algorithm for Polynomial(p) Latency Network Design enables us to differentiate between "yes" and "no" instances of the 2 Directed Disjoint Paths (2DDP) problem in polynomial time (for a definition of 2DDP, see the proof of Theorem 5.3.2).

Consider an instance $\mathcal{I} = \{G, s_1, s_2, t_1, t_2\}$ of 2DDP; we construct an instance of Polynomial(p) Latency Network Design (G', k, ℓ) as follows (illustrated in Figure 5.6). To define the graph G', we begin with k copies of G; call them $G_1, \ldots, G_k$ and denote the copy of s_i (t_i) in G_j by s_i^j (t_i^j). Next, add auxiliary vertices s, t, $v_1, \ldots, v_{k-1}$, and $w_1, \ldots, w_{k-1}$. The edge set of G' is as follows:

- each G_i inherits the edge set of G
- for $i = 1, \ldots, k-1$, we include edges from s to v_i, from v_i to s_2^i and s_1^{i+1}, from t_2^i and t_1^{i+1} to w_i, and from w_i to t
- we include edges (s, s_1^1), (s, s_2^k), (t_1^1, t), (t_2^k, t).

We define latency functions on the edges of G' as follows:

(A) for edges of the form (v_i, s_2^i) or (t_1^{i+1}, w_i), put $\ell(x) = 1$

(B) for edges (s, s_1^1) and (t_2^k, t), put $\ell(x) = 2 + (1 + \frac{1}{k})^p x^p$

(C) for (s, s_2^k) and (t_1^1, t), put $\ell(x) = 1 + k(\frac{4(k+1)}{4k+1})^p x^p$

(D) for $i = 1, \ldots, k-1$ and edges (s, v_i) and (w_{k-i}, t), put $\ell(x) = i(\frac{4(k+1)}{4k+1})^p x^p$

(E) for edges of the form (v_i, s_1^{i+1}) or (t_2^i, w_i), put $\ell(x) = 2 + (2 + \frac{2}{k})^p x^p$

(F) for edges in $G_1, \ldots, G_k$, put $\ell(x) = 0$.

We will call edges of the form (v_i, s_2^i) or (t_1^{i+1}, w_i) *type A edges*, and so forth. It is clear that (G', k, ℓ) can be constructed from $\mathcal{I}$ in polynomial time.

Next, we claim that if $\mathcal{I}$ is a "yes" instance of 2DDP, then there is a subgraph H of G' satisfying $L(H, k, \ell) \leq 5$. To see why, let P_1^* and P_2^* denote vertex-disjoint s_1-t_1 and s_2-t_2 paths in G. Deleting all edges in G' that lie in some copy G_i of G but not on (the corresponding copy of) either P_1^* or P_2^*, we obtain a subgraph H of G' that is the union of $2k$ distinct s-t paths. Routing $\frac{k}{k+1}$ units of flow on the path containing s_1^1 and t_1^1 and on the path containing s_2^k and t_2^k, and $\frac{k}{2(k+1)}$ units of flow

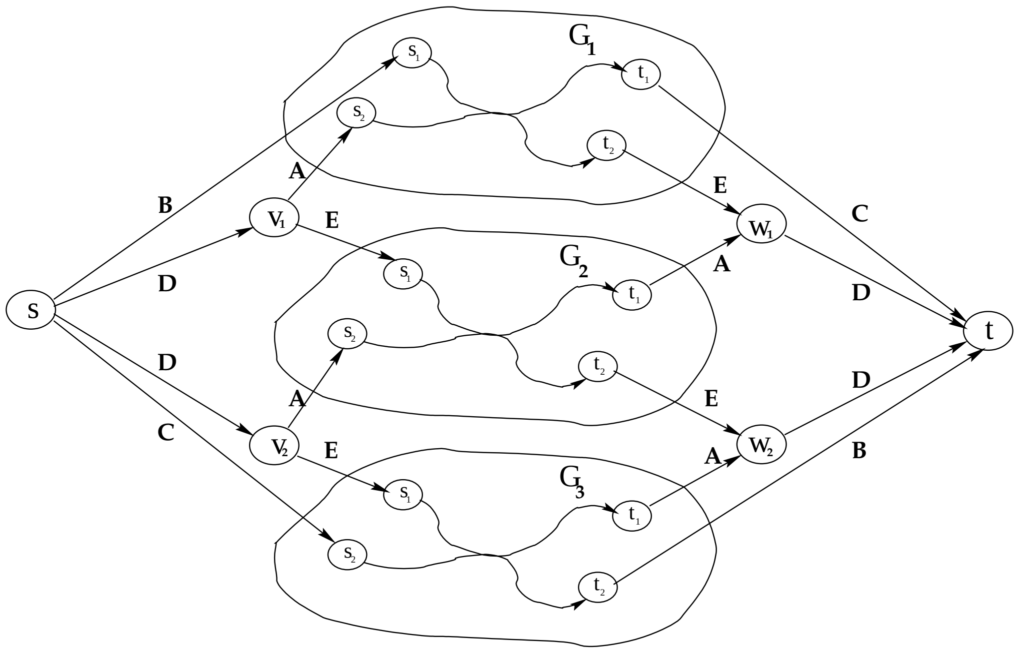


Figure 5.6: Proof of Theorem 5.5.6. Construction of (G', k, ℓ) when $k = 3$. Edges are labeled with their edge type.

on each of the other $2k-2$ paths, we obtain a flow at Nash equilibrium for (H, k, ℓ). This flow proves that

$$L(H, k, \ell) = 4 + k\left(\frac{4(k+1)}{4k+1}\frac{k}{k+1}\right)^p = 4 + k\left(1 - \frac{1}{4k+1}\right)^p \leq 4 + ke^{-p/(4k+1)} \leq 5,$$

with the picture of this Nash flow somewhat analogous to Figure 5.4(b).

Finally, we show that if $\mathcal{I}$ is a "no" instance of 2DDP, then $L(H, k, \ell) \geq k$ for all subgraphs H of G'. We will prove this in two steps. First, we will show that unless H contains most of the edges in G', "capacity considerations" (similar to those used in the proof of Theorem 5.4.3) imply that $L(H)$ is large. Second, we show that if H contains most of the edges in G', then the flow at Nash equilibrium in H is similar to the bad Nash flow of Proposition 5.5.3, again showing $L(H)$ to be large.

Fix a subgraph H of G' containing an s-t path, and let f be an acyclic Nash flow in (H, k, ℓ) (see Proposition 2.6.2). We claim that if some type A or C edge of G' does not carry flow in f (in particular, if some such edge is not in H), then $L(H) \geq k$. We will prove the claim for an edge of the form (v_i, s_2^i); the argument for an edge of the form (t_1^{i+1}, w_i) is symmetric, and the argument for type C edges is similar (and easier).

To prove this claim, we first observe that many edges of H are essentially capacitated, in the following sense. We assert that any of the following events forces $L(H) \geq p \geq k$ (using that $L(H) \geq \ell_e(f_e)$ for any edge e with $f_e > 0$):

(1) $f_e \geq \frac{8k+1}{8(k+1)}$ for a type B edge e

(2) $f_e \geq \frac{2k+1}{2(k+1)}$ for an edge e of type C or D

(3) $f_e \geq \frac{4k+1}{8(k+1)}$ for a type E edge e.

For example, we can derive

$$\left(\frac{4(k+1)}{4k+1}\frac{2k+1}{2(k+1)}\right)^p = \left(1 + \frac{1}{4k+1}\right)^p > \left[e^{1/(8k+2)}\right]^p = e^{p/(8k+2)} \geq p,$$

proving (2). The calculations for (1) and (3) are similar, so we omit them.

Now assume that edge (v_i, s_2^i) does not carry any flow in f. Then, either event (3) occurs (with edge (v_i, s_1^{i+1})) or else edge (s, v_i) carries at most $\frac{4k+1}{8(k+1)}$ units of flow; assume the latter. We claim that in this case, event (1) or event (2) must occur with some edge incident to s. For if not, edges incident to s carry at most

$$(k-1)\frac{2k+1}{2(k+1)} + \frac{4k+1}{8(k+1)} + \frac{8k+1}{8(k+1)} = \frac{8k^2+8k-2}{8(k+1)} < k$$

units of flow, contradicting that f is an s-t flow carrying k units of flow. We conclude that if edge (v_i, s_2^i) does not carry flow in f, then some event of the form (1), (2), or (3) occurs, proving that $L(H) \geq k$.

It remains to consider subgraphs H of G' in which all edges of type A or C carry flow in the Nash flow f of (H, k, ℓ), and to make use of our hypothesis that $\mathcal{I}$ is a "no" instance of 2DDP. The presence of these edges in H (all of which lie on s-t paths in H, since they carry flow in the acyclic flow f), together with the assumption that $\mathcal{I}$ is a "no" instance, imply that for each $i = 1, 2, \ldots, k$ there is an s-t path P_i in H containing the vertices s_2^i and t_1^i (cf., the proof of Theorem 5.3.2). Letting $r = \frac{k(4k+1)}{4(k+1)} < k$, the following flow is then at Nash equilibrium for (H, r, ℓ): for $i = 1, 2, \ldots, k$ route $\frac{4k+1}{4(k+1)}$ units of flow on P_i. This flow shows that $L(H, r, \ell) = k+3$ (this Nash flow is essentially the same as the bad Nash flow of Proposition 5.5.3). By Proposition 5.5.5, $L(H, \cdot, \ell)$ is an increasing function of the traffic rate (with H, ℓ fixed); hence, $L(H, k, \ell) \geq k + 3$.

We have shown that if $\mathcal{I}$ is a "no" instance of 2DDP, then $L(H, k, \ell) \geq k$ for all subgraphs H of G', and the proof is complete. ∎

Remark 5.5.7 The results of this section can be extended to networks with other types of latency functions with only minor modifications to the proofs. For example, in Sections A.1 and A.2 we introduce the notion of the *incline* of an instance (intuitively, a real number $\gamma \geq 1$ that measures the "steepness" of the network latency functions) and prove that the price of anarchy in instances with incline γ is at most γ. By the same argument as in Corollaries 5.3.1 and 5.5.2, this implies that the trivial algorithm is a γ-approximation algorithm for instances with incline at most γ. A straightforward modification of the proof of Theorem 5.5.6 shows that there is a constant $c > 0$ such that, for each $\gamma \geq 1$ and $\epsilon > 0$, there is no $(c \cdot \gamma - \epsilon)$-approximation algorithm for instances with incline at most γ (unless $P = NP$). We leave the details of the proof and further extensions of this sort to the interested reader.

Chapter 6

Stackelberg Routing

6.1 Introduction

In this chapter, we pursue a second approach to coping with selfishness. In many networks, there will be a mix of "selfishly controlled" and "centrally controlled" traffic—that is, the network is used by both selfish individuals and some central authority. For example, clients of a network may be charged at two different prices: clients paying the higher price are given access to the network and the ability to route their own traffic (presumably along a minimum-latency path), while clients paying only the "bargain rate" can use the network but have no control over how their traffic is routed (and thus this traffic qualifies as centrally controlled). Also, Korilis et al. [105] consider networks that allow a large customer to set up a so-called *virtual private network* of guaranteed and preassigned virtual paths for ongoing use [21], and argue that the bandwidth needed for a virtual private network may be viewed as centrally controlled (with the paths chosen by the network manager) while individual users of the network continue to behave in a selfish and independent fashion.

We investigate the following question: given a network with centrally and selfishly controlled traffic, *how should centrally controlled traffic be routed to induce "good" (albeit selfish) behavior from the noncooperative users?* This indirect approach to controlling selfish behavior has several appealing aspects: no communication is required between network users and an algorithm, no notion of currency is needed (cf., the approach of algorithmic mechanism design [136, 137, 155]), no resources need to be added to or removed from the network, and the routing of centrally controlled traffic is often easily modified as the amount of traffic evolves over time.

6.1.1 Summary of Results

We consider a game in which the roles of different players are asymmetric. One player (responsible for routing the centrally controlled traffic and interested in minimizing total latency) acts as a *leader*, in that it may hold its routing (its *strategy*) fixed while all other players (the *followers*) react independently and selfishly to the leader's strategy, reaching a Nash equilibrium relative to it. These types of games, called *Stackelberg games*, and the resulting *Stackelberg equilibria* have been well studied in the game theory literature (see Subsection 6.1.3 below).

We study the following questions:

(1) among all leader strategies for a given network, can we characterize and/or compute the strategy inducing the *Stackelberg equilibrium*—that is, the equilibrium of minimum total latency?

(2) what is the worst-case ratio between the total latency of the Stackelberg equilibrium and that of the optimal routing of all of the traffic?

For networks with arbitrary edge latency functions but consisting only of two nodes and a collection of parallel links, we give a simple algorithm for computing a leader strategy that induces an equilibrium with total latency no more than $\frac{1}{\beta}$ times that of the minimum-latency flow, where β denotes the fraction of traffic that is centrally controlled. This algorithm runs in polynomial time provided the network latency functions are standard (see Definition 2.3.5). We also show that no stronger guarantee is possible in networks of parallel links, and that this guarantee cannot be achieved in general networks. Thus, in a network of parallel links, a manager controlling a constant fraction of the network traffic can induce an equilibrium with total latency at most a constant-factor larger than that incurred by the minimum-latency flow. This result stands in sharp contrast to our results about *Nash* equilibria, as the nonlinear variant of Pigou's example (Subsection 2.4.4) shows that the total latency of a flow at Nash equilibrium can be arbitrarily larger than that of a minimum-latency flow, even in networks of parallel links.

For networks of parallel links in which every edge latency function is linear in the edge congestion, we give a simple algorithm that runs in $O(m^2)$ time and computes a strategy inducing an equilibrium with total latency no more than $\frac{4}{3+\beta}$ times that of the minimum-latency flow, where β is the fraction of centrally controlled traffic and m is the number of edges. We again show that no stronger guarantee is possible.

Finally, we consider the optimization problem of computing the strategy inducing the Stackelberg equilibrium and show that it is NP-hard, even in the special case of networks of parallel links with linear latency functions.

6.1.2 Comparison to the Price of Anarchy

The results of this chapter give a (sharp) trade-off between a minimum-latency flow and a flow at Nash equilibrium (as a function of the fraction of the traffic that is centrally controlled) in networks of parallel links, in the following sense. In Chapter 3 we showed that a flow at Nash equilibrium can be arbitrarily more costly than the minimum-latency flow, but if every edge latency function is linear then the total latency of a Nash flow is no more than $\frac{4}{3}$ times that of a minimum-latency flow. Thus, the results of this chapter reduce to those of Chapter 3 when $\beta = 0$, give the trivial result that the Stackelberg equilibrium for $\beta = 1$ is the minimum-latency flow, and quantify the worst possible ratio between the cost of the Stackelberg equilibrium (in some sense, a "mixture" of a Nash flow and a minimum-latency flow) and the cost of a minimum-latency flow for all intermediate values of β.

Our approach also adds an algorithmic dimension to our work in Part II bounding the price of anarchy, in that one aspect of our analysis of Stackelberg equilibria is the design of algorithms for efficiently computing good Stackelberg strategies. Further, while optimal and Nash flows can be characterized and computed efficiently via convex programming (see Propositions 2.3.1 and 2.5.1)—a fact that was crucial for our work bounding the price of anarchy—the hardness result of this chapter implies that no such characterization of Stackelberg equilibria is possible. With the central approach of Part II ruled out, we will require new techniques for bounding the inefficiency of Stackelberg equilibria.

6.1.3 Related Work

Stackelberg games and Stackelberg equilibria have been thoroughly studied in the game theory literature (see, for example, [75] or [13, §3.6] for an introduction and [184] for their origin) and have previously been applied to problems in competitive facility location [153], networking [53, 54, 58, 104], and more general continuous-time systems (see the surveys of Cruz [41, 42], the book of Bagchi [12], and the references therein). With the exception of [53, 104], however, the leader/follower hierarchy has been used to model classes of selfish users with different priority levels; this setting differs from ours in that no user is interested in optimizing system performance.[1] The note of Douligeris and Mazumdar [53] is concerned only with experimental results on the effectiveness of Stackelberg strategies. The paper of Korilis et al. [104], while more similar in spirit to ours, focuses on deriving necessary and sufficient conditions (on the number of selfish users, the fraction of the traffic that is centrally controlled, etc.) for the existence of a leader strategy inducing an optimal routing of all of the traffic; moreover, only one type of latency function is considered. By contrast, we are interested in simple leader strategies that *always* induce optimal or near-optimal behavior from the network users for *any* set of latency functions.

6.1.4 Organization

In Section 6.2 we extend our basic traffic model to accommodate Stackelberg equilibria. In Section 6.3 we introduce three simple algorithms for computing Stackelberg strategies in networks of parallel links. In Sections 6.4 and 6.5, we prove that our third algorithm achieves the best-possible worst-case performance guarantee for networks of parallel links with general and linear latency functions, respectively. In Section 6.6, we prove that computing the optimal strategy is NP-hard, even in networks of parallel links with linear latency functions.

6.2 Stackelberg Strategies and Induced Equilibria

In this section we define our notion of a Stackelberg game and consider two examples. We will focus on networks in which all traffic shares a common source and destination, although the definitions of this section are easily extended to a multicommodity setting.

Recall we desire a *hierarchical* game, where a *leader* routes centrally controlled traffic and, holding this strategy fixed, the network users react in a noncooperative and selfish manner. This idea is formalized in the next two definitions. By a *Stackelberg instance* (G, r, ℓ, β), we mean a single-commodity instance (G, r, ℓ) in the sense of Section 2.1 together with a parameter $\beta \in [0, 1]$ specifying the fraction of the network traffic that is centrally controlled.

[1] Traditionally, Stackelberg games model *selfish* users with asymmetric roles; our use of them is somewhat unconventional.

Definition 6.2.1 A *(Stackelberg) strategy* for the Stackelberg instance (G, r, ℓ, β) is a flow feasible for $(G, \beta r, \ell)$.

Definition 6.2.2 Let f be a strategy for Stackelberg instance (G, r, ℓ, β), and define $\tilde{\ell}_e$ by $\tilde{\ell}_e(x) = \ell_e(f_e + x)$ for each edge $e \in E$. An *equilibrium induced by strategy* f is a flow g at Nash equilibrium for the instance $(G, (1-\beta)r, \tilde{\ell})$. We then say that $f + g$ is a *flow induced by* f for (G, r, ℓ, β).

Existence and essential uniqueness of induced equilibria follow easily from Propositions 2.2.4 and 2.5.1.

Proposition 6.2.3 *Let f be a strategy for a Stackelberg instance with continuous, nondecreasing latency functions. Then there exists a flow induced by f, and any two such induced flows have equal cost.*

The following simple observation will be useful in Sections 6.4 and 6.5.

Lemma 6.2.4 *Let f be a strategy for Stackelberg instance (G, r, ℓ, β) inducing equilibrium g, where G is a network of parallel links. Let G' denote the subgraph induced by the edges e on which $g_e > 0$. Then $f + g$, restricted to the edges of G', is a flow at Nash equilibrium for the instance (G', r', ℓ), where $r' = \sum_{e \in G'}(f_e + g_e)$. In particular, all edges e on which $g_e > 0$ have a common latency with respect to $f + g$.*

We next consider two examples that demonstrate both the usefulness and the limitations of Stackelberg strategies. First consider Pigou's example (Subsection 1.2.1). Recall that in the absence of centrally controlled traffic, a Nash flow incurs total latency $\frac{4}{3}$ times that of the optimal flow. Suppose instead that half of the traffic is controlled by the network manager (i.e., that $\beta = \frac{1}{2}$) and consider the strategy f of routing all centrally controlled traffic on the top edge (the edge with latency function $\ell(x) = 1$). Then, as all remaining traffic will be routed on the lower edge in the equilibrium induced by f, the flow induced by f is precisely the minimum-latency flow. Thus, in this particular instance, total latency can be minimized via a Stackelberg strategy.

Now consider a small modification to Pigou's example, in which we replace the latency function of the lower edge with the latency function $\ell(x) = 2x$. The flow at Nash equilibrium puts half of the traffic on each edge for a cost of 1, while the optimal flow routes only $\frac{1}{4}$ of the traffic on the lower edge, for a cost of $\frac{7}{8}$. On the other hand, if we again allow the network manager to route half of the traffic, we see that for *any* strategy f, the flow induced by f is the flow at Nash equilibrium and hence is not optimal. In this example, there is no available strategy by which the network manager can improve network performance.

6.3 Three Stackelberg Strategies

6.3.1 Two Natural Strategies

We begin our investigation of Stackelberg strategies for networks of parallel links by considering two natural approaches that provide suboptimal performance guarantees. To motivate our results in the simplest possible way, throughout this subsection we will consider examples with linear latency functions in which half of the traffic is centrally controlled ($\beta = \frac{1}{2}$).

First consider the following strategy for an instance $(G, r, \ell, \frac{1}{2})$: if f^* is the minimum-latency flow for instance $(G, \frac{1}{2}r, \ell)$, put $f = f^*$. In words, we choose the strategy of minimum cost (ignoring the existence of traffic that is not centrally controlled). We call this the *Aloof* strategy since it refuses to acknowledge the rest of the traffic in the network. Pigou's example (Subsection 1.2.1) shows that this strategy performs quite poorly: in that network, the Aloof strategy routes all flow on the bottom edge (the edge with latency function $\ell(x) = x$) and in the induced flow all traffic is routed on the bottom edge. Thus, the Aloof strategy induces the (inefficient) Nash flow while the strategy that routes all centrally controlled traffic on the top edge induces the optimal flow. Applying this type of argument to the nonlinear variant of Pigou's example (Subsection 2.4.4), it is also easy to see that the Aloof strategy can perform arbitrarily badly in networks with general latency functions.

A second attempt at a good strategy might be as follows: if f^* is the minimum-latency flow for (G, r, ℓ), put $f = \frac{1}{2}f^*$. We call this the *Scale* strategy, since it is simply the optimal flow, suitably scaled. To understand why we should be dissatisfied with the Scale strategy, consider the two-node, two-link example with latency functions $\ell_1(x) = 1$ and $\ell_2(x) = \frac{3}{2}x$ and traffic rate 1. The minimum-latency flow routes $\frac{2}{3}$ of the flow on the first link and the rest on the second link (with total cost $\frac{5}{6}$), so the Scale strategy will route $\frac{1}{3}$ units of flow on the first link and $\frac{1}{6}$ units on the second link. All selfish traffic is then routed on the second link, inducing the flow with $\frac{1}{3}$ units of flow on the first link and the rest on the second, having cost

1. It would appear that the Scale strategy routed too much flow on the second link (since selfish traffic flocked to it, anyways); indeed, the strategy that instead routes all centrally controlled traffic on the first link induces a flow with the superior cost of $\frac{7}{8}$.[2]

6.3.2 The Largest Latency First (LLF) Strategy

Intuitively, both the Aloof and Scale strategies suffer from a common flaw: both route traffic on edges that will subsequently be inundated in any induced equilibrium while routing too little traffic on edges that selfish users are prone to ignore. This observation suggests that a good strategy should give priority to the edges that are least appealing to selfish users—edges with relatively high latency. With this intuition in mind, the following strategy for a Stackelberg instance (G, r, ℓ, β) defined on a network of m parallel links (which we call the *Largest Latency First* or *LLF* strategy) should seem natural:

(1) Compute a minimum-latency flow f^* for (G, r, ℓ).

(2) Label the edges of G from 1 to m so that $\ell_1(f_1^*) \leq \cdots \leq \ell_m(f_m^*)$.

(3) Let $k \leq m$ be minimal with $\sum_{i=k+1}^{m} f_i^* \leq \beta r$.

(4) Put $f_i = f_i^*$ for $i > k$, $f_k = \beta r - \sum_{i=k+1}^{m} f_i^*$, and $f_i = 0$ for $i < k$.

We will say that an edge i is *saturated* by a strategy f if $f_i = f_i^*$. Thus, the LLF strategy saturates edges one-by-one (in order from the largest latency with respect to f^* to the smallest) until there is no centrally controlled traffic remaining. Note that as long as all link latency functions are standard (see Definition 2.3.5), Fact 2.3.6 implies that the LLF strategy can be computed in polynomial time (the bottleneck is step (1)); in Section 6.5 we will see that it can be computed in $O(m^2)$ time when every latency function is linear.[3]

The next two sections are devoted to proving that the LLF strategy always induces a flow with near-optimal total latency.

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[2] In addition, a slightly more complicated example shows that the Scale strategy can perform arbitrarily badly in networks with general latency functions.

[3] We ignore the fact that an optimal flow can only be computed up to an arbitrarily small additive factor (see Fact 2.3.6), as this does not affect our results in any significant way.

6.4 Arbitrary Latency Functions: A Performance Guarantee of $1/\beta$

In this section we prove that the LLF strategy induces a near-optimal flow for networks of parallel links with arbitrary latency functions. We note that *no* performance guarantee is possible for such networks in the absence of centrally controlled traffic: without additional restrictions on edge latency functions, the Nash flow may incur arbitrarily more latency than the optimal flow (as shown by the nonlinear variant of Pigou's example). Thus, the benefit of a leader (and of a carefully chosen leader strategy) is particularly striking in this general setting.

A simple variation on previous examples demonstrates the limits of Stackelberg strategies. In a two-node, two-link instance with $\beta = \frac{1}{2}$ and latency functions $\ell_1(x) = 1$ and $\ell_2(x) = 2^p x^p$ for $p \in \mathcal{Z}^+$, any Stackelberg strategy induces the Nash flow (half of the flow on each link, with cost 1) while the minimum-latency flow routes $\frac{1}{2} + \delta_p$ units of flow on the first link and the rest on the second link, for a cost of $\frac{1}{2} + \epsilon_p$ with $\delta_p, \epsilon_p \to 0$ as $p \to \infty$. Thus the best induced flow may be (arbitrarily close to) twice as costly as the optimal flow. Similar examples show that for any $\beta \in (0, 1)$, the best induced flow may be $\frac{1}{\beta}$ times as costly as the optimal flow.

The main result of this section is that the LLF strategy always induces a flow of cost no more than $\frac{1}{\beta}$ times that of the minimum-latency flow. A rough outline of the proof is as follows. Our goal is to exploit the iterative structure of the LLF strategy and proceed by induction on the number of edges. If the LLF strategy first saturates the mth edge (with the ordering of edges as in the description of the LLF strategy), a natural idea is to apply the inductive hypothesis to the remainder of the LLF strategy on the first $m-1$ edges to derive a performance guarantee. This idea nearly succeeds, but there are two difficulties. First, it is possible that the LLF strategy fails to saturate any edges; we will see below that this case is easy to analyze and causes no trouble. Second, in order to obtain a clean application of the inductive hypothesis to the first $m-1$ edges, we require that the optimal and LLF-induced flows route the same total amount of flow on these edges—i.e., that the LLF-induced equilibrium eschews the mth edge.[4] We resolve this difficulty with the following lemma, which states that if the LLF strategy saturates the mth edge, then *some* induced equilibrium routes all traffic on the first $m-1$ edges—this

[4]This can fail in trivial cases, such as in a two-node, two-link network in which both edges have latency function $\ell(x) = 1$.

suffices for our purposes, since different induced flows have equal cost.

Lemma 6.4.1 *Let (G, r, ℓ, β) denote a Stackelberg instance on a network of m parallel links with optimal flow f^*, and label the edges of G from 1 to m so that $\ell_m(f_m^*) \geq \ell_i(f_i^*)$ for all $i \in \{1, 2, \ldots, m\}$. If f is a strategy with $f_m = f_m^*$, then there exists an induced equilibrium g with $g_m = 0$.*

Proof. Consider an arbitrary induced equilibrium g and suppose $g_m > 0$. Roughly speaking, the idea is to prove that this scenario only occurs when several latency functions (that of the mth edge, and others) are locally constant; then, traffic routed on edge m in the induced equilibrium can be evacuated to other edges with locally constant latency functions to provide a new induced equilibrium.

Formally, let $L = \ell_m(f_m + g_m) = \ell_m(f_m^* + g_m)$ denote the common latency with respect to $f + g$ of every edge with $g_i > 0$ (see Lemma 6.2.4). We must have $\ell_m(f_m^*) \geq L$; otherwise $\ell_i(f_i^*) < L$ for all i yet $\ell_i(f_i + g_i) \geq L$ for all i, contradicting that f^* and $f + g$ are flows at the same rate. Thus, since ℓ_m is nondecreasing, ℓ_m is locally constant: ℓ_m is equal to L on $[f_m^*, f_m^* + g_m]$.

Next, let E' denote the subgraph of edges on which $f_i + g_i < f_i^*$; since $f_m + g_m > f_m^*$, E' is non-empty. For $i \in E'$, we have $\ell_i(f_i + g_i) \geq L$, $\ell_i(f_i^*) \leq \ell_m(f_m^*) = L$, and ℓ_i nondecreasing; hence, ℓ_i is equal to L on $[f_i + g_i, f_i^*]$. Since f^* and $f + g$ are flows at the same rate, we must have $\sum_{i \in E'}[f_i^* - (f_i + g_i)] \geq g_m$. Finally, consider modifying g as follows: move all traffic previously routed on edge m to edges in E', subject to the constraint $f_i + g_i \leq f_i^*$. We have already observed that there is sufficient "room" on edges in E' for this operation, and that all latency functions are constant in the domain of our modifications. We have thus exhibited a new induced equilibrium with no traffic routed on edge m, completing the proof. ■

We are now prepared to prove the main result of this section.

Theorem 6.4.2 *Let $\mathcal{I} = (G, r, \ell, \beta)$ denote a Stackelberg instance on a network of parallel links. If f is an LLF strategy for $\mathcal{I}$ inducing equilibrium g and f^* is a minimum-latency flow for the instance (G, r, ℓ), then $C(f + g) \leq \frac{1}{\beta} C(f^*)$.*

Proof. We proceed by induction on the number of edges m (for each fixed m, we will prove the theorem for arbitrary ℓ, r, and β). The case of a single edge is trivial.

Fix a Stackelberg instance $\mathcal{I} = (G, r, \ell, \beta)$ with at least two edges, and let f^* denote a minimum-latency flow for the instance (G, r, ℓ) and f the corresponding LLF strategy. Label the edges 1 to m so that $\ell_1(f_1^*) \leq \ell_2(f_2^*) \leq \cdots \leq \ell_m(f_m^*)$.

By scaling, we may assume that $r = 1$ (use latency functions $\tilde{\ell}$ with $\tilde{\ell}_i(x) = \ell_i(rx)$ otherwise). Let L denote the common latency with respect to $f + g$ of every edge with $g_i > 0$ (see Lemma 6.2.4).

Case 1: Suppose $g_k = 0$ for some edge k. Let E_1 denote the edges i for which $g_i = 0$ and E_2 the edges for which $g_i > 0$; both of these sets are non-empty. Let G_1 and G_2 denote the corresponding subgraphs of G. For $j = 1, 2$ let β_j denote the amount of centrally controlled traffic routed on edges in E_j and C_j the cost incurred by $f + g$ on edges in E_j. By Lemma 6.2.4, $C_2 = (1 - \beta_1)L$ and $C_1 \geq \beta_1 L$. Now, f^* restricted to E_2 is an optimal flow for $(G_2, 1 - \beta_1)$ and hence f restricted to E_2 is an LLF strategy for the instance $\mathcal{I}_2 = (G_2, 1 - \beta_1, \beta')$ where $\beta' = \frac{\beta_2}{1-\beta_1}$. Applying the inductive hypothesis to $\mathcal{I}_2$ and using the fact that $f_i^* \geq f_i = f_i + g_i$ for all $i \in E_1$, we obtain

$$C(f^*) \geq C_1 + \beta' C_2.$$

Proving that $C(f + g) \leq \frac{1}{\beta} C(f^*)$ thus reduces to showing

$$\beta(C_1 + C_2) \leq C_1 + \beta' C_2.$$

Since $\beta \leq 1$ and $C_1 \geq \beta_1 L$, it suffices to prove this inequality with C_1 replaced by $\beta_1 L$. Writing $C_2 = (1 - \beta_1)L$ and $\beta' = \frac{\beta_2}{1-\beta_1}$ and dividing through by L, we need only check that

$$\beta(\beta_1 + (1 - \beta_1)) \leq \beta_1 + \frac{\beta_2}{1 - \beta_1}(1 - \beta_1)$$

which clearly holds (both sides are equal to β).

Case 2: Suppose $g_i > 0$ for every edge i, so $C(f + g) = L$. We may assume that the LLF strategy failed to saturate edge m (otherwise, by Proposition 6.2.3, we can finish by applying the previous case to the better-behaved induced flow guaranteed by Lemma 6.4.1). Thus, $\beta < f_m^*$.

As in the proof of Lemma 6.4.1, we must have $\ell_m(f_m^*) \geq L$; otherwise, $\ell_i(f_i^*) < L$ for all edges i while $\ell_i(f_i + g_i) = L$ for all i, contradicting that f^* and $f + g$ are flows at the same rate. Having established that edge m has large latency with respect to f^* and that f_m^* is fairly large, it is now a simple matter to lower bound $C(f^*)$:

$$C(f^*) \geq f_m^* \ell_m(f_m^*) \geq \beta L = \beta C(f + g).$$

■

Remark 6.4.3 Theorem 6.4.2 applies only to networks of parallel links, and the proof makes crucial use of the "decomposable" nature of such networks. On the other hand, the guarantee of Theorem 6.4.2 cannot be extended to general network topologies (cf., our work in Sections 3.3–3.4); even in the four-node network of Braess's Paradox (Figure 2.2), a network manager controlling a β fraction of the traffic cannot in general induce a flow with cost at most a $\frac{1}{\beta}$ factor times that of the minimum-latency flow (see Section B.3).

6.5 Linear Latency Functions: A Performance Guarantee of $4/(3+\beta)$

6.5.1 Properties of the Nash and Optimal Flows

In this subsection we undertake a deeper study of Nash and optimal flows for instances on networks of parallel links with linear latency functions. The results of this subsection will be instrumental in proving a stronger performance guarantee for the LLF strategy for these instances.

Fix a network G of m parallel links with linear latency functions (so that $\ell_e(x) = a_e x + b_e$ for each $e \in E$ with $a_e, b_e \geq 0$) and label them from 1 to m so that $b_1 \leq b_2 \leq \cdots \leq b_m$. We may assume that at most one edge has a constant latency function ($a_i = 0$) since all but the one with smallest b-value can be safely discarded; under this assumption, the Nash and optimal flows are always unique. We may similarly assume that an edge with a constant latency function is the mth edge.

Our first goal is to understand the structure of the Nash flow $\bar{f}$ as a function of the rate r. It is useful to imagine r increasing from 0 to a large value, with the corresponding Nash flow changing in a continuous fashion; an intuitive description of this process is as follows. Initially, when r is nearly zero, all traffic is routed on the edge having the smallest constant term. Once the first edge is sufficiently congested, the second edge looks equally attractive (this occurs when $a_1\bar{f}_1 + b_1 = b_2$—i.e., when the amount of flow on the first edge is $\frac{b_2 - b_1}{a_1}$). Subsequent traffic will be routed on both of the first two edges, at rates proportional to $\frac{1}{a_1}$ and $\frac{1}{a_2}$ (traffic will be routed so that these two edges continue to have equal latency). Once $(b_3 - b_2)(\frac{1}{a_1} + \frac{1}{a_2})$ further units of traffic have been routed on the first two edges, the third edge will be equally attractive and new traffic will be spread out among the first three edges, and so on. We may thus envision the Nash flow as being constructed in phases: within phase i traffic is routed on the first i edges according to fixed relative proportions and at the end of the phase (after enough new flow has been routed) an additional edge is put into use.

We now formalize this intuitive description of the Nash flow $\bar{f}$. For $i = 1, \ldots, m$, let v_i denote the m-vector $(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_i}, 0, 0, \ldots, 0) \in \mathcal{R}^m_+$; if $a_m = 0$ put $v_m = (0, 0, \ldots, 1)$. The vector v_i should be interpreted as a specification of the way traffic is routed on the first i edges during the ith phase. Next, define δ_i for $i = 0, 1, \ldots, m-1$ inductively by $\delta_0 = 0$ and $\delta_i = \min\{(b_{i+1} - b_i)\|v_i\|_1, r - \sum_{j=0}^{i-1} \delta_i\} \geq 0$ (where $\|\cdot\|_1$ denotes the L_1-norm of a vector). We also put $\delta_m = r - \sum_{j=0}^{m-1} \delta_i$. The scalar δ_i should be interpreted as the total amount of traffic routed in the ith phase. We can then describe $\bar{f}$ as follows (where we interpret the m-vector $\bar{f}$ as a function on the edges of G in the obvious way).

Lemma 6.5.1 *Let $\mathcal{I}$ be an instance on a network of m parallel links with linear latency functions, as above. Then the Nash flow for $\mathcal{I}$ is given by*

$$\bar{f} = \sum_{i=1}^{m} \delta_i \frac{v_i}{\|v_i\|_1}.$$

Our characterization of optimal flows (Proposition 2.3.1 and Corollary 2.3.2) yields an analogous result for computing them by an explicit formula. Note that when a latency function has the form $\ell(x) = ax + b$, the corresponding marginal cost function (see Section 2.3) is $\ell^*(x) = 2ax + b$. Recalling from Corollary 2.3.2 that an optimal flow is simply a flow at Nash equilibrium with respect to latency functions ℓ^*, we see that the optimal flow is created by the same process as the Nash flow, except that new edges are incorporated at a more rapid pace so as to spread traffic over a wider range of edges (and thus achieve a smaller total latency).

Formally, let v_i be as above and define δ^*_i inductively by $\delta^*_0 = 0$, $\delta^*_i = \min\{\frac{1}{2}(b_{i+1} - b_i)\|v_i\|_1, r - \sum_{j=0}^{i-1} \delta^*_i\}$, and $\delta^*_m = r - \sum_{j=0}^{m-1} \delta^*_i$. Letting f^* denote the minimum-latency flow for (G, r, ℓ), the analogue of Lemma 6.5.1 is as follows.

Lemma 6.5.2 *Let $\mathcal{I}$ be an instance on a network of m parallel links with linear latency functions, as above. Then the optimal flow for $\mathcal{I}$ is given by*

$$f^* = \sum_{i=1}^{m} \delta^*_i \frac{v_i}{\|v_i\|_1}.$$

Lemmas 6.5.1 and 6.5.2 have several useful corollaries. We summarize them below.

Corollary 6.5.3 *Let G be a network of m parallel links with linear latency functions, with at most one edge having constant latency. Label the edges of G from 1 to m so that if $\ell_i(x) = a_i x + b_i$, then b_i is nondecreasing in i. Then:*

(a) If f^* and $\bar{f}$ denote optimal and Nash flows for (G, r, ℓ), then $f^*_m \geq \bar{f}_m$.

(b) If f^* and $\bar{f}$ denote optimal and Nash flows for (G, r, ℓ), then $f^*_1 \leq \bar{f}_1 \leq 2f^*_1$.

(c) If f^* is the optimal flow for (G, r, ℓ) and $\tilde{f}^*$ is the optimal flow for $(G, \tilde{r}, \ell)$ with $r \geq \tilde{r}$, then $f^*_i \geq \tilde{f}^*_i$ for each edge i.

(d) For any rate r, the optimal and Nash flows for (G, r, ℓ) can be computed in $O(m^2)$ time.

Proof. Parts (c) and (d) are immediate from Lemmas 6.5.1 and 6.5.2. For the remaining parts, fix an instance (G, r, ℓ) and define v_i, δ_i, and δ^*_i as in Lemmas 6.5.1 and 6.5.2. Our first observation is that, for any $i \in \{1, 2, \ldots, m\}$, the Nash flow routes at least as much traffic in the first i phases as the optimal flow—formally, that $\sum_{k=1}^{i} \delta_k \geq \sum_{k=1}^{i} \delta^*_k$ for all i. Since $\sum_{k=1}^{m} \delta_k = \sum_{k=1}^{m} \delta^*_k = r$, we obtain $\delta^*_m \geq \delta_m$ and hence $f^*_m \geq \bar{f}_m$, proving (a).

It remains to prove part (b) of the corollary. We may assume that $m > 1$ (otherwise $\bar{f}_1 = f^*_1 = r$). Letting m' equal m if there is no edge with constant latency function and $m - 1$ otherwise, Lemmas 6.5.1 and 6.5.2 give

$$f^*_1 = \frac{1}{a_1} \sum_{i=1}^{m'} \frac{\delta^*_i}{\|v_i\|_1}$$

and

$$\bar{f}_1 = \frac{1}{a_1} \sum_{i=1}^{m'} \frac{\delta_i}{\|v_i\|_1}.$$

By the definitions of δ_i and δ^*_i, we have $\delta_i \leq 2\delta^*_i$ for $i = 1, 2, \ldots, m$ and hence $\bar{f}_1 \leq 2f^*_1$. For the other inequality, we recall that $\sum_{k=1}^{i} \delta^*_k \leq \sum_{k=1}^{i} \delta_k$ for each i and observe that $\|v_i\|_1$ is increasing in i (for $i \in \{1, 2, \ldots, m'\}$); it follows that $f^*_1 \leq \bar{f}_1$. ■

Corollary 6.5.3(d) implies that the LLF strategy can be computed in $O(m^2)$ time for instances with linear latency functions.

6.5.2 Proof of Performance Guarantee

In Section 6.2 we saw an example with linear latency functions and $\beta = \frac{1}{2}$ in which *no* strategy can induce a flow with cost less than $\frac{8}{7}$ times that of the minimum-latency flow. This example is easily modified (by giving the second edge the latency function $\ell_2(x) = \frac{1}{1-\beta}x$) to show that, for any $\beta \in (0, 1)$, the minimum-cost induced flow for a Stackelberg instance (G, r, ℓ, β) with linear latency functions may be $\frac{4}{3+\beta}$ times as costly as the minimum-latency flow for (G, r, ℓ). The main result of this section is a matching upper bound for the LLF strategy.

Before proving this result, we give an alternative description of LLF that is more convenient for our analysis. This description is based on the following lemma.

Lemma 6.5.4 *Let f^* be an optimal flow for instance (G, r, ℓ) defined on a network G of parallel links with $\ell_e(x) = a_e x + b_e$ for each edge e. Let i and j be two edges of G. Then $\ell_i(f_i^*) \geq \ell_j(f_j^*)$ if and only if $b_i \geq b_j$.*

Proof. The lemma is clear when $f_i^* = f_j^* = 0$. Otherwise, we will make use of our characterization of optimal flows via marginal cost functions (Proposition 2.3.1). If exactly one of f_i^*, f_j^* is 0 (say f_i^*), then by Proposition 2.3.1 we know that the marginal cost $\ell_i^*(f_i^*) = \ell_i(0) = b_i$ of edge i is at least the marginal cost $\ell_j^*(f_j^*) = 2a_j f_j^* + b_j$ of edge j. Thus we necessarily have both $\ell_i(f_i^*) \geq \ell_j(f_j^*)$ and $b_i \geq b_j$. Finally, if $f_i^*, f_j^* > 0$ then by Proposition 2.3.1 we have $2a_i f_i^* + b_i = 2a_j f_j^* + b_j = L^*$ for some L^*; thus $b_i \geq b_j$ if and only if $a_i f_i^* \leq a_j f_j^*$. The lemma follows by writing $\ell_i(f_i^*) = L^* - a_i f_i^*$ and $\ell_j(f_j^*) = L^* - a_j f_j^*$. ■

Lemma 6.5.4 gives the following equivalent description of the LLF strategy: saturate edges one-by-one, in decreasing order of constant terms, until no centrally controlled traffic remains. It may seem surprising that the LLF strategy makes no use of the a_i-values in ordering the edges; however, this is consistent with our observation in Subsection 6.5.1 that the order in which edges are used by the minimum-latency flow (if we think of the rate as increasing from 0 to some large value) depends only on the constant terms of the edges' latency functions.

We are finally prepared to prove a $\frac{4}{3+\beta}$ performance guarantee for the LLF strategy for networks of parallel links with linear latency functions. The general approach is similar to that of Theorem 6.4.2 and is again by induction on the number of edges. However, new difficulties arise in proving a stronger performance guarantee. The case in which there is some edge k on which the induced equilibrium routes no flow (i.e., $g_k = 0$ for some edge k) is nearly identical to the first case of Theorem 6.4.2, and the desired performance guarantee can easily be extracted from the inductive guarantee for the smaller instance induced by the edges on which $g_i > 0$. The second case, where the induced equilibrium routes traffic on all edges, is substantially more complicated. In particular, the simple approach in the proof of Theorem 6.4.2 does *not* use any inductive guarantee in this case and is thus not strong enough to prove a guarantee better than $\frac{1}{\beta}$. For this reason, much of the proof is devoted to defining an appropriate smaller instance that allows for clean application of the inductive hypothesis and to extending the inductive guarantee into one for the original instance.

Theorem 6.5.5 *Let $\mathcal{I} = (G, r, \ell, \beta)$ denote a Stackelberg instance on a network of m parallel links with linear latency functions. If f is an LLF strategy for $\mathcal{I}$ inducing equilibrium g and f^* is a minimum-latency flow for (G, r, ℓ), then $C(f + g) \leq \frac{4}{3+\beta} C(f^*)$.*

Proof. We proceed by induction on the number of edges m; for each fixed m, we will prove the theorem for arbitrary (linear) ℓ, r, and β. The case of one edge is trivial.

Fix a Stackelberg instance $\mathcal{I} = (G, r, \ell, \beta)$ with at least two edges, with edges labeled 1 to m in the order that they are considered by the LLF strategy; by Lemma 6.5.4, we may write $\ell_i(x) = a_i x + b_i$ with b_i nondecreasing in i. Let f^* denote a minimum-latency flow for (G, r, ℓ). We begin with several simplifying assumptions, each made with no loss of generality. As in Theorem 6.4.2, we may assume that $r = 1$. We assume (as usual) that there is at most one edge with a constant latency function. It will also be convenient to assume that the first edge has constant term 0 (i.e., that $b_1 = 0$). To enforce this assumption we may subtract b_1 from every latency function before applying our argument: assuming $r = 1$, this modification decreases the cost of every feasible flow by precisely b_1 and only increases the ratio in costs between any two feasible flows. Finally, we assume that $a_1 > 0$; otherwise, the first edge has latency function $\ell(x) = 0$ and the instance is trivial.

Let f denote an LLF strategy for $\mathcal{I}$ and g the induced equilibrium. Let $L > 0$ denote the common latency of every edge i on which $g_i > 0$ (see Lemma 6.2.4). We will need to apply the inductive hypothesis in two different ways, and our analysis breaks into two cases.

Case 1: Suppose $g_k = 0$ for some edge k. As in the proof of Theorem 6.4.2, let E_1 denote the edges on which $g_i = 0$ and E_2 the edges on which $g_i > 0$. Let G_1 and G_2 denote the corresponding (non-empty) subgraphs of G. For $j = 1, 2$, let β_j denote the amount of centrally controlled traffic routed on edges in E_j. For $j = 1, 2$ let C_j denote the cost incurred by $f + g$ on edges in E_j. Observe that $C_1 \geq \beta_1 L$ and $C_2 = (1 - \beta_1)L$. Since f^* restricted to E_2 is an optimal flow for $(G_2, 1 - \beta_1, \ell)$, f restricted to E_2 is an LLF strategy for $\mathcal{I}_2 = (G_2, 1 - \beta_1, \ell, \beta')$, where $\beta' = \frac{\beta_2}{1-\beta_1}$. The inductive hypothesis (applied to $\mathcal{I}_2$) and the fact that $f_i^* \geq f_i = f_i + g_i$ for all $i \in E_1$ imply that

$$C(f^*) \geq C_1 + \frac{3 + \beta'}{4} C_2.$$

Proving that $C(f + g) \leq \frac{4}{3+\beta} C(f^*)$ thus reduces to showing

$$(3 + \beta)(C_1 + C_2) \leq 4C_1 + (3 + \beta')C_2.$$

Since $\beta \leq 1$ and $C_1 \geq \beta_1 L$, it suffices to prove this inequality with C_1 replaced by $\beta_1 L$. Writing $C_2 = (1 - \beta_1)L$, $\beta' = \frac{\beta_2}{1-\beta_1}$, and dividing through by L verifies the result.

Case 2: Suppose $g_i > 0$ for every edge $i \in E$. This implies that $f + g$ is a Nash flow for $(G, 1, \ell)$. By Corollary 6.5.3(a) we have $f_m < f_m + g_m \leq f_m^*$ and in particular $f_m^* > 0$. It follows that the LLF strategy f failed to saturate edge m, so $f_m = \beta$ and $f_i = 0$ for $i < m$.

Our first goal is to show that f is an LLF strategy not only for $\mathcal{I}$ but also for $\mathcal{I}' = (G', 1 - g_1, \ell, \frac{\beta}{1-g_1})$, where G' is the graph induced by the last $m - 1$ edges of G; we may then apply the inductive hypothesis to f restricted to this smaller instance. Toward this end, let $\tilde{f}^*$ denote the optimal flow for the instance $(G', 1 - g_1, \ell)$. Since $f + g$ restricted to G' is a Nash flow for $(G', 1 - g_1, \ell)$, we must have $\tilde{f}_m^* \geq f_m + g_m$ (see Corollary 6.5.3(a)); since $\beta = f_m < f_m + g_m \leq \tilde{f}_m^*$, the LLF strategy for $\mathcal{I}'$ is precisely f (restricted to the edges of G').

Let C_1^*, C_2^* denote the total latency incurred by f^* on the first edge and in G', respectively. The next claim gives a lower bound on C_2^*, as a function of the amount of traffic routed on edges of G' in the optimal flow.

Claim: *If $r' \geq 1 - g_1$, then the cost of the optimal flow for (G', r', ℓ) is at least*

$$\frac{3 + \beta'}{4}(1 - g_1)L + (r' - 1 + g_1)L$$

where $\beta' = \frac{\beta}{1-g_1}$.

Proof. The claim is proved for $r' = 1 - g_1$ by applying the inductive hypothesis to the instance $\mathcal{I}' = (G', 1 - g_1, \ell, \beta')$ and using the fact that f is an LLF strategy for G' inducing a flow of cost $(1 - g_1)L$. Suppose now that $r' > 1 - g_1$. We again denote the optimal flow for $(G', 1 - g_1, \ell)$ by $\tilde{f}^*$. Since $\tilde{f}^*$ and $f + g$ (restricted to the edges of G') are flows at the same rate (namely, $1 - g_1$) and the common latency of every edge with respect to $f + g$ is L, there is some edge i with $\tilde{f}_i^* > 0$ and $\ell_i(\tilde{f}_i^*) \geq L$. Since the marginal cost of an edge is at least its latency, Proposition 2.3.1 implies that the marginal cost of every edge in G' is at least L with respect to $\tilde{f}^*$. By Corollary 6.5.3(c) and the observation that marginal costs are nondecreasing functions of the edge congestion, extending $\tilde{f}^*$ from an optimal flow for $(G', 1 - g_1, \ell)$ to an optimal flow for (G', r', ℓ) involves the routing of $r' - (1 - g_1)$ units of flow, all routed at a marginal cost of at least L. Thus, the overall cost of an optimal flow to (G', r', ℓ) must be at least $C(\tilde{f}^*) + (r' - 1 + g_1)L \geq \frac{3+\beta'}{4}(1 - g_1)L + (r' - 1 + g_1)L$. ■

Our goal is to prove that $(3 + \beta)C(f + g) \leq 4C(f^*) = 4(C_1^* + C_2^*)$; with the claim in hand, we have reduced the proof of the theorem to proving the inequality

$$(3 + \beta)L \leq (3 + \beta')(1 - g_1)L + 4(g_1 - f_1^*)L + 4a_1(f_1^*)^2$$

where $\beta' = \frac{\beta}{1-g_1}$ (recall that $\ell_1(x) = a_1 x$ and hence $C_1^* = a_1(f_1^*)^2$). For any fixed value of g_1, $f_1^* \in [\frac{1}{2}g_1, g_1]$ (see Corollary 6.5.3(b)). Using the identity $a_1 g_1 = L$ and differentiating, we find that the right-hand side is minimized by $f_1^* = \frac{1}{2}g_1$. Since the left-hand side is independent of f_1^*, it suffices to prove that

$$(3 + \beta)L \leq (3 + \beta')(1 - g_1)L + 2g_1 L + a_1 g_1^2.$$

Substituting for β', using the identity $a_1 g_1 = L$, and dividing by L gives

$$3 + \beta \leq (3 + \frac{\beta}{1 - g_1})(1 - g_1) + 3g_1$$

which clearly holds, proving the theorem. ■

6.6 Complexity of Computing Optimal Strategies

Thus far, we have measured the performance of a Stackelberg strategy by comparing the cost of the corresponding induced flow to the cost of the minimum-latency flow. Another natural approach for evaluating a strategy is to compare the cost of the induced flow to that of the *least costly flow induced by some Stackelberg strategy*, that is, to the cost of the flow induced by the *optimal* strategy. Motivated by the latter measure, in this section we study the optimization problem of computing the optimal Stackelberg strategy.

We have seen that in networks of parallel links the LLF strategy provides the best possible (worst-case) performance guarantee relative to the cost of the minimum-latency flow. In particular, the algorithm of Subsection 6.3.2 may be viewed as a $\frac{1}{\beta}$-approximation algorithm for computing the optimal Stackelberg strategy in networks of parallel links and a $\frac{4}{3+\beta}$-approximation algorithm for such instances with linear latency functions. Simple examples show that the LLF strategy is *not* always the optimal strategy, and thus our algorithm fails to solve this optimization problem exactly (see Section B.4). Our main result of this section is strong evidence that no such polynomial-time algorithm exists.

Theorem 6.6.1 *The problem of computing the optimal Stackelberg strategy is NP-hard, even for instances in networks of parallel links with linear latency functions.*

Proof. We reduce from a problem we call $\frac{1}{3}$-$\frac{2}{3}$ Partition: given n positive integers $a_1, a_2, \ldots, a_n$, is there a subset $S \subseteq \{1, 2, \ldots, n\}$ satisfying $\sum_{i \in S} a_i = \frac{1}{3} \sum_{i=1}^{n} a_i$? The canonical reduction from the NP-complete problem Subset Sum to Partition is easily modified to show that $\frac{1}{3}$-$\frac{2}{3}$ Partition is NP-hard (see Garey and Johnson [77] for problem definitions and Karp [95] or Kozen [109, P.129] for the canonical reduction). We will show that deciding the problem $\frac{1}{3}$-$\frac{2}{3}$ Partition reduces to deciding whether or not a given Stackelberg instance on a network of parallel links with linear latency functions admits a Stackelberg strategy inducing a flow with a given cost.

Given an arbitrary instance $\mathcal{I}$ of $\frac{1}{3}$-$\frac{2}{3}$ Partition specified by positive integers $a_1, \ldots, a_n$, put $A = \sum_{i=1}^{n} a_i$ and define a Stackelberg instance $\mathcal{I}' = (G, 2A, \ell, \frac{1}{4})$ on a network G of parallel links with edge set $\{1, 2, \ldots, n+1\}$ and with linear latency functions $\ell_i(x) = \frac{x}{a_i} + 4$ for $i = 1, \ldots, n$ and $\ell_{n+1}(x) = \frac{3x}{A}$. It is clear that $\mathcal{I}'$ can be constructed from $\mathcal{I}$ in polynomial time.[5] We claim that $\mathcal{I}$ is a "yes instance" (that is, admits a $\frac{1}{3}$-$\frac{2}{3}$ partition) if and only if there is a Stackelberg strategy for instance $\mathcal{I}'$ inducing a flow with cost at most $\frac{35}{4}A$.

First suppose $\mathcal{I}$ is a "yes instance" of $\frac{1}{3}$-$\frac{2}{3}$-Partition, with $S \subseteq \{1, 2, \ldots, n\}$ satisfying $\sum_{i \in S} a_i = \frac{2}{3}A$, and consider the strategy defined by $f_i = \frac{3}{4}a_i$ for $i \in S$

[5] We are assuming that a linear latency function is represented in the problem instance $\mathcal{I}'$ by the binary encodings of its two coefficients. See Section 5.2 for more details on the encoding of an instance.

and $f_i = 0$ otherwise (since $\sum_{i=1}^{n+1} f_i = \frac{3}{4}\sum_{i\in S} a_i = \frac{3}{4}\frac{2}{3}A = \frac{1}{2}A$, this defines a Stackelberg strategy). The induced equilibrium is then $g_i = 0$ for $i \in S$, $g_i = \frac{1}{4}a_i$ for $i \in \{1, 2, \ldots, n\} \setminus S$, and $g_{n+1} = \frac{17}{12}A$. In the induced flow $f + g$, the $A/2$ units of traffic on edges corresponding to S experience $\frac{19}{4}$ units of latency, while the other $3A/2$ units of traffic experience $\frac{17}{4}$ units of latency. The cost of $f + g$ is thus

$$\frac{A}{2}\frac{19}{4} + \frac{3A}{2}\frac{17}{4} = \frac{35}{4}A.$$

Now suppose that $\mathcal{I}$ is a "no instance" of $\frac{1}{3}$-$\frac{2}{3}$-Partition, and consider any Stackelberg strategy f for $\mathcal{I}'$, inducing equilibrium g. We need to show that $C(f + g) > \frac{35}{4}A$. Call edge $i \in \{1, 2, \ldots, n + 1\}$ *heavy* if $g_i = 0$ and *light* otherwise. Our first observation is that edge $n + 1$ must be light (even if all centrally controlled traffic is routed on edge $n+1$, some selfish traffic will use it). Next, we note that for $i, j \in \{1, 2, \ldots, n\}$, the marginal cost $\frac{2(f_i+g_i)}{a_i} + 4$ of edge i is at most the marginal cost $\frac{2(f_j+g_j)}{a_j} + 4$ of edge j if and only if the latency $\ell_i(f_i + g_i)$ of edge i is at most $\ell_j(f_j + g_j)$. We may assume that all heavy edges have the same marginal cost with respect to $f + g$, as rerouting some centrally controlled traffic from a heavy edge with large marginal cost to a heavy edge with small marginal cost does not affect the induced equilibrium and can only decrease the cost of the induced flow. We can therefore also assume that all heavy edges have equal latency with respect to $f + g$. Naturally, Lemma 6.2.4 implies that all light edges possess a common latency with respect to $f + g$. That all edges have one of two latencies will make the cost of $f + g$ easy to compute.

With the induced flow $f + g$ still fixed, let $S \subseteq \{1, 2, \ldots, n\}$ denote the set of heavy edges. If $S = \emptyset$ then $f + g$ is the Nash flow for $\mathcal{I}'$ with $f_i + g_i = \frac{a_i}{2}$ for $i \in \{1, 2, \ldots, n\}$ and $f_{n+1} + g_{n+1} = \frac{3}{2}A$, satisfying $C(f + g) = 9A > \frac{35}{4}A$. So suppose S is non-empty and define $\lambda \in (0, 1]$ by the equation $\sum_{i\in S} a_i = \lambda A$. Define $\mu \in (0, \frac{1}{2}]$ by the equation $\sum_{i\in S} f_i = \mu A$. Our aim is to lower bound the cost of $f + g$ as a function of the parameters λ and μ.[6]

Since all heavy edges have equal latency, for i heavy we must have $f_i + g_i = f_i = \frac{\mu}{\lambda}a_i$ with $\ell_i(f_i + g_i) = 4 + \frac{\mu}{\lambda}$. Since all light edges have equal latency, we must have

$$f_i + g_i = a_i\frac{2 - 3\mu}{4 - 3\lambda}$$

[6]Not all joint values of $\lambda \in (0, 1]$ and $\mu \in (0, \frac{1}{2}]$ are achievable with Stackelberg strategies, but this will not hinder our analysis.

with

$$\ell_i(f_i + g_i) = 4 + \frac{2 - 3\mu}{4 - 3\lambda}$$

for $i \in \{1, 2, \ldots, n\} \setminus S$ and

$$f_{n+1} + g_{n+1} = A\left(\frac{4}{3} + \frac{2 - 3\mu}{12 - 9\lambda}\right)$$

with

$$\ell_{n+1}(f_{n+1} + g_{n+1}) = 4 + \frac{2 - 3\mu}{4 - 3\lambda}.$$

The total cost of this solution is

$$\begin{aligned} C(f+g) &= \mu A\left(4 + \frac{\mu}{\lambda}\right) + (2-\mu)A\left(4 + \frac{2-3\mu}{4-3\lambda}\right) \\ &= A\left(8 + \frac{(4-3\lambda)\mu^2 + \lambda(2-\mu)(2-3\mu)}{\lambda(4-3\lambda)}\right). \end{aligned}$$

Holding λ fixed and differentiating with respect to μ, we find that this expression has unique minimizer $\mu = \lambda$ when $\lambda \leq \frac{1}{2}$ and $\mu = \frac{1}{2}$ when $\lambda \geq \frac{1}{2}$ (subject to the condition $\mu \in (0, \frac{1}{2}]$). There are now two cases to analyze. First suppose that $\lambda \leq \frac{1}{2}$. Setting $\mu = \lambda$ we obtain

$$C(f+g) = A\left(8 + \frac{4 - 4\lambda}{4 - 3\lambda}\right);$$

differentiating with respect to λ, we see that the expression has a unique minimizer $\lambda = \frac{1}{2}$ (subject to the condition $\lambda \in (0, \frac{1}{2}]$) yielding cost $A(8 + \frac{4}{5}) > \frac{35}{4}A$. Finally, assume that $\lambda \geq \frac{1}{2}$. Setting $\mu = \frac{1}{2}$ we find that the cost of $f + g$ is given by

$$C(f+g) = A\left(8 + \frac{1}{\lambda(4 - 3\lambda)}\right);$$

differentiating with respect to λ, we find that this expression has unique minimizer $\lambda = \frac{2}{3}$, at which point the equation reads $C(f+g) = \frac{35}{4}A$. However, since $\mathcal{I}$ is a "no instance" of $\frac{1}{3}$-$\frac{2}{3}$ Partition, we must have $\lambda \neq \frac{2}{3}$ and hence $C(f+g) > \frac{35}{4}A$. We have exhausted all possible cases, and the reduction is complete. ■

Chapter 7

Recent and Future Work

In this thesis, we have studied the loss in network performance due to selfish routing. We have both quantified the worst-possible ratio between flows at Nash equilibrium and the best coordinated outcome (the minimum-latency flow) and investigated how to control the inefficiency inherent in a Nash flow via network design and Stackelberg strategies. While we have succeeded in answering a few basic questions about selfish routing, we feel that our work is only a small step toward understanding the complex interactions between selfish behavior and a desire for global optimization in networks. In the hope of conveying this sentiment to the reader, in this final chapter we suggest some open questions and unexplored directions that beckon for further research. We also summarize recent work related to these topics. Our list is not exhaustive and is meant only to indicate the wide array of possibilities for future work; the imaginative reader will doubtless discover further interesting lines of inquiry.

How Common is Braess's Paradox?

In Chapter 5 we studied the worst-possible increase in total latency due to harmful extraneous edges in a network, thereby determining the extent to which Braess's Paradox generalizes to and becomes more severe in large networks. Since we examined this issue through the lens of worst-case analysis, we were naturally led to "extremal" bad examples (the Braess graphs of Subsection 5.4.2) unlikely to occur in practice. The following important question is as yet poorly understood: to what extent does Braess's Paradox occur in "typical" networks?

Open Question 1 For a single-commodity instance (G, r, ℓ), let $L(G, r, \ell)$ denote the common latency of every flow path of a Nash flow for (G, r, ℓ) (as in Chapter 5). Let $\tau(G, r, \ell) \geq 1$ denote the largest ratio between $L(G, r, \ell)$ and $L(H, r, \ell)$ for a subgraph H of G. What can be said about the distribution of $\tau(G, r, \ell)$ for some "reasonable" distribution on single-commodity instances (G, r, ℓ)? How often can removing edges improve the flow at Nash equilibrium—that is, for what fraction of instances is $\tau(G, r, \ell) > 1$?

Remark 7.0.1 As an example, we can obtain a simple yet nontrivial distribution on instances (G, r, ℓ) by adapting the classical random graph model $\mathcal{G}(n, p)$ of Erdös and Rényi [23, 61]. Fix parameters $n \in \mathcal{N}$ and $p \in (0, 1)$. Define an instance (G, r, ℓ) by the following random process: set the vertex set of G to be $V = \{1, 2, \ldots, n\}$; independently for each ordered pair (i, j) of distinct vertices, include (one copy of) edge (i, j) with probability p; independently for each included edge e, assign e the latency function $\ell(x) = 1$ or $\ell(x) = x$, chosen uniformly at random; set vertex 1 to be the source, vertex 2 to be the destination, and the traffic rate r to be 1. What is $E[\tau(G, r, \ell)]$? Does this expectation tend to a limit as $n \to \infty$ for a fixed choice of p (say, $p = \frac{1}{2}$)?

Remark 7.0.2 It is a "folklore" belief that instances with τ-value greater than 1 are fairly common, and thus Braess's Paradox fails to qualify as a "pathological" example. Some headway in this direction has been made by Steinberg and Zangwill [176] (whose approach was later extended to more general traffic models by Dafermos and Nagurney [48]), who argue that "Braess's Paradox is about as likely to occur as not occur" [176, P.312]. The analysis of [176] leaves much to be done, however: Steinberg and Zangwill [176] restrict attention to subgraphs H with one less edge than G, assume that every edge used by the Nash flow in H is also used by the Nash flow in G (an assumption that fails in our version of Braess's Paradox—see Subsections 1.2.2 and 2.4.2), and do not specify a probability distribution on problem instances.

A related (and easier) question is the following: for what networks G is there a traffic rate r and a set of edge latency functions ℓ such that $\tau(G, r, \ell) > 1$? Let us call such a network *vulnerable*. Vulnerable networks are therefore the networks that, under an adversarial choice of latency functions and traffic rate, suffer degradation in network performance due to undesirable extra edges. Confining our study to networks (always assumed to possess a worst-case choice of latency functions) rather than to instances (networks already endowed with an arbitrary set of latency functions) simplifies matters considerably. This fact is illustrated by the following characterization of vulnerable networks, asserted by Murchland [128] and proved in detail by Milchtaich [125].

Fact 7.0.3 ([125, 128]) *Let G be a directed graph with source vertex s, destination vertex t, and with every vertex lying on some s-t path. Then the following are equivalent:*

(1) G is vulnerable

(2) G contains a subdivision of the network of Braess's Paradox (Figure 2.2) as a subgraph.

By a well-known forbidden subgraph characterization of series-parallel graphs [57, 182], Fact 7.0.3 implies that the vulnerable graphs are precisely those for which the subgraph induced by the vertices lying on some s-t path fails to be *two-terminal series-parallel*. This in turn implies that vulnerable graphs can be recognized in linear time [182]. Fact 7.0.3 also shows that vulnerable graphs are ubiquitous (the class of two-terminal series-parallel directed graphs is a restrictive one), a fact that could be useful in proving that "most" instances have τ-value greater than 1; see Open Question 1 above.

This characterization of vulnerable graphs stands in stark contrast to the problem of identifying the instances (G, r, ℓ) satisfying $\tau(G, r, \ell) > 1$; our hardness results of Chapter 5 imply that, assuming $P \neq NP$, such instances have no similarly simple (to be precise, polynomial-time checkable) characterization. Given the simplicity of Fact 7.0.3 and its proof, it is natural to seek generalizations. Toward this end, we will say that a network G is *c-vulnerable* if there is a traffic rate r and a set of latency functions ℓ such that $\tau(G, r, \ell) > c$. Recalling the Braess graphs of Subsection 5.4.2, we pose the following problem.

Open Question 2 Prove or disprove: there is a function g such that every $g(c)$-vulnerable network contains a subdivision of the cth Braess graph B^c as a subgraph.

Remark 7.0.4 Fact 7.0.3 implies that the trivial algorithm (the network design heuristic of building the whole network) is optimal for networks that exclude subdivisions of the first Braess graph (Figure 2.2) as subgraphs. Similarly, a positive resolution to Open Question 2 would prove that the trivial algorithm has constant approximation ratio for networks that exclude subdivisions of sufficiently large Braess graphs.

The Average Price of Anarchy

Throughout Chapter 3, we were of a single mind: to compute the price of anarchy, defined as the worst-possible ratio $\rho(G, r, \ell)$ between the costs of a Nash and of an optimal flow for an instance (G, r, ℓ). As with Braess's Paradox, little is known about the value of ρ in "typical" instances.

Open Question 3 What can be said about the distribution of $\rho(G, r, \ell)$ for some "reasonable" distribution on instances (G, r, ℓ)?

Progress on this question for any nontrivial class of instances (such as the setting outlined in Remark 7.0.1) would be of interest.

Friedman [74] recently proved an interesting result related to Open Question 3, stating that in a network with arbitrary latency functions, for "most" traffic rate vectors the cost of selfish routing is much smaller than the worst-case value. To state his result more precisely, fix a network G with latency functions ℓ, and let $N(r)$ be the cost of a Nash flow for instance (G, r, ℓ). Friedman uses the ratio $\Lambda(r) = N(r)/N(r/2)$ as a sensitivity measure of the problem instance (G, r, ℓ). Applying Theorem 3.6.1 to $(G, r/2, \ell)$ shows that the ratio $\rho(G, r, \ell)$ between the cost of the Nash and optimal flows for (G, r, ℓ) is bounded above by $\Lambda(r)$, and this bound can be achieved. Friedman [74] shows that for "most" traffic rate vectors r' in $[r/2, r]$, the ratio $\rho(G, r', \ell)$ is only $O(\log \Lambda(r))$.

Stackelberg Routing

In Chapter 6 we studied the problem of indirectly controlling selfish network users via Stackelberg strategies—that is, by routing a small fraction of the overall traffic centrally. Our work concerned only networks of parallel links, leaving the important generalization to arbitrary networks open. While the bad example of Section B.3 shows that the guarantee of Theorem 6.4.2 for networks of parallel links cannot be extended to arbitrary networks, a guarantee with worse dependence on β (the fraction of traffic that is centrally controlled) may be possible.

Open Question 4 Is there a function $g(\cdot)$ such that the following statement holds: for any single-commodity Stackelberg instance (G, r, ℓ, β) with standard latency functions, there is an efficiently computable Stackelberg strategy that induces a flow with cost at most $g(\beta) \cdot C(f^*)$, where f^* is an optimal flow for (G, r, ℓ)?

We emphasize that the function g can have arbitrary dependence on β, but is independent of the size of the network G.

Remark 7.0.5 We confine our question to single-commodity instances because multicommodity instances can be largely immune to Stackelberg strategies. Precisely, there are instances with n vertices and $k = \Theta(n)$ commodities such that any Stackelberg strategy routing half of the traffic of each commodity induces a flow with cost $\Omega(k)$ times that of the optimal routing of all of the traffic.

We noted in Section 6.6 that we evaluate a Stackelberg strategy by comparing the cost of the flow it induces to that of an optimal routing of all of the traffic, rather than to the minimum-latency flow induced by some Stackelberg strategy. Outside of our hardness result for computing Stackelberg strategies (Theorem 6.6.1), we have not considered the complexity of the optimization problem of computing the best Stackelberg strategy. Recent work by Kumar and Marathe [112] resolves this problem for networks of parallel links with a fully polynomial-time approximation scheme[1] (FPTAS) for the problem under mild conditions on the network latency functions. The results of [112] also apply to networks slightly more general than those of parallel links, but the problem of approximating the optimal Stackelberg strategy in general networks remains open.

Admission Control

Throughout this work, we have assumed that all traffic rates are given and immutable. What if rates are under control of the network manager—that is, what if *admission control* is permitted? Several algorithmic questions arise in this setting; we will describe one in detail. Let us consider a network in which the amount of traffic between each source-destination pair is easy to control, but centralized routing is infeasible. The network manager wishes to maximize the amount of traffic routed (e.g., in order to maximize revenue). To make the problem nontrivial, we impose *quality of service (QoS)* constraints: to each commodity i we associate a threshold L_i^{max} representing the maximum amount of latency that network users corresponding to commodity i are willing to tolerate. The manager's problem is then

[1]A *fully polynomial-time approximation scheme* for a minimization problem is an algorithm A with the following property for some polynomial $p(\cdot,\cdot)$: given error parameter $\epsilon > 0$ and problem instance $\mathcal{I}$ with size $|\mathcal{I}|$, A returns a solution to $\mathcal{I}$ with objective function value at most $1+\epsilon$ times that of optimal in time at most $p(|\mathcal{I}|, \epsilon^{-1})$.

to maximize the amount of traffic routed subject to the QoS constraints, assuming selfish routing.

Open Question 5 Design a good approximation algorithm for the following optimization problem: given a network G with latency functions ℓ, a vector r^{max} of maximum allowable traffic rates, and a vector L^{max} of QoS constraints, find a traffic rate vector r maximizing $\sum_i r_i$ subject to $r_i \leq r_i^{max}$ and $L_i(f) \leq L_i^{max}$ for each commodity i, where $L_i(f)$ is the common latency of every s_i-t_i flow path in a Nash flow f for (G, r, ℓ).

Remark 7.0.6 The optimization problem posed in Open Question 5 completely ignores the issue of *fairness*, in that the optimal solution may route an enormous amount of one commodity and none of another. Addressing fairness concerns such as this is yet another wide open area for future work.

The Price of Selfishness in Other Games

While open questions about the traffic model studied in this dissertation abound, an even more exciting direction for future research is the study of the inefficiency of selfish behavior in other games (in networks and otherwise). Before elaborating on this point, we briefly mention some recent efforts along these lines. Two papers that generalize models previously mentioned in this work and then study the price of selfishness are Schulz and Stier Moses [167], who extend the traffic routing model of Chapter 2 to networks with explicit edge capacity constraints, and Czumaj et al. [43], who augment the load-balancing model of Koutsoupias and Papadimitriou [108] by allowing arbitrary (nonlinear) objective functions. Vetta [183] departs more significantly from previous work and studies the inefficiency of Nash equilibria in a broad class of profit-maximization problems that includes auctions, facility location games, as well as games related to the selfish routing problems of this thesis. To connect Vetta's work with ours, define a game in a multicommodity flow network where players correspond to commodities, and each player controls both the routing of its flow (cf., the finite splittable instances of Section 4.2) and its traffic rate. Player i receives revenue π_i for each unit of flow it sends, and experiences cost equal to the total latency incurred by flow of commodity i; player i's objective function is to maximize its profit (revenue minus cost), and the global objective function is defined as the sum of all profits (equivalently, total revenue minus total latency). A consequence of Vetta's work is that, under certain conditions, a Nash equilibrium of this game will obtain at least half of the profit enjoyed by the best coordinated outcome; see [183] for further details.

Given the prevalence of game-theoretic analysis in the networking literature (illustrated by, for example, the survey of Altman et al. [6] and the many references therein), we expect the idea of quantifying the inefficiency arising from selfish behavior to find numerous applications beyond those of this dissertation and of the papers mentioned above. Moreover, we believe that a key contribution of our work is the identification of several questions about the inefficiency of Nash equilibria (or of other game-theoretic solution concepts) that are likely to have clean and nontrivial solutions. We conclude by making this assertion concrete and offering a set of questions that should constitute a general and useful paradigm for analyzing selfish behavior in future research:

- What is the worst-case ratio between the objective function value (perhaps the sum or the minimum of player utilities) of a selfish equilibrium and that of the best coordinated outcome? (Due originally to Koutsoupias and Papadimitriou [108].)

- Are there other types of comparisons that bound the price of selfishness in a meaningful way? (Cf., our bicriteria bound in Section 3.6 and Friedman's "average-case" price of anarchy result mentioned above.)

- What are the "sources of inefficiency" for selfish equilibria? Do simple games suffer from the worst-possible consequences of uncoordinated behavior? (For example, we saw in Sections 3.3–3.4 that the complexity of the underlying network topology in essence fails to contribute to the inefficiency of flows at Nash equilibrium.)

- Are there natural design and/or management principles that ensure that the price of selfishness is reasonable?

Part IV

Appendices

Appendix A

Odds and Ends

This appendix gathers together some results about selfish routing that may be of interest but do not fall within the scope of the main text. We begin in Section A.1 with a "quick and dirty" upper bound on the price of anarchy that follows relatively easily from our work in Chapter 2. In Section A.2 we describe different methods of quantifying the "steepness" of network latency functions. In Section A.3 we apply one of these methods to quantify the potential "unfairness" of optimal flows.

A.1 A "Quick and Dirty" Upper Bound on the Price of Anarchy

The proof of Proposition 2.5.1 provides a fairly general method for upper-bounding the ratio ρ between the cost of a flow at Nash equilibrium and of a minimum-latency flow. Specifically, we have the following theorem.

Theorem A.1.1 *Suppose the instance (G, r, ℓ) and the constant $\gamma \geq 1$ satisfy*

$$x \cdot \ell_e(x) \leq \gamma \cdot \int_0^x \ell_e(t)dt$$

for all edges e and all positive real numbers x. Then

$$\rho(G, r, \ell) \leq \gamma.$$

Proof. Roughly speaking, the theorem holds since a flow at Nash equilibrium for (G, r, ℓ) optimizes an objective function (the objective function of $(NLP2)$ in the proof of Proposition 2.5.1) that is at most a factor γ away from the true objective function $C(\cdot)$. More formally, let f and f^* denote Nash and optimal flows for (G, r, ℓ), respectively; we can then derive

$$\begin{aligned} C(f) &= \sum_{e\in E} \ell_e(f_e) f_e \\ &\leq \gamma \sum_{e\in E} \int_0^{f_e} \ell_e(t)dt \\ &\leq \gamma \sum_{e\in E} \int_0^{f_e^*} \ell_e(t)dt \\ &\leq \gamma \sum_{e\in E} \ell_e(f_e^*) f_e^* \\ &= \gamma \cdot C(f^*) \end{aligned}$$

where the first inequality follows from the hypothesis, the second inequality from the fact that the Nash flow f optimizes the objective function $\sum_e \int_0^{f_e} \ell_e(t)dt$ (see Proposition 2.5.1), and the third inequality from the assumption that every latency function ℓ_e is nondecreasing. ■

Remark A.1.2 Theorem A.1.1 and its proof do not make use of the combinatorial structure possessed by a network, and therefore apply more generally to the nonatomic congestion games of Section 4.4.

While the hypothesis of Theorem A.1.1 is somewhat opaque, it nevertheless gives a nontrivial upper bound on the cost of selfish routing for many instances, such as instances with latency functions that are polynomials with nonnegative coefficients.

Corollary A.1.3 *Suppose every latency function of instance (G, r, ℓ) is a polynomial with nonnegative coefficients and degree at most p. Then,*

$$\rho(G, r, \ell) \leq p + 1.$$

Remark A.1.4 A comparison of Table 3.1 and Corollary A.1.3 shows that the more sophisticated approach to bounding the price of anarchy presented in Chapter 3 (in particular, Theorem 3.3.8) can give a better guarantee than that of Theorem A.1.1. On the other hand, simple two-node, two-link examples show that the conclusion of Theorem A.1.1 cannot be improved without refining the hypothesis.

A.2 Notions of Steepness

A.2.1 Incline

A theme of this dissertation is the dependence of the price of anarchy (as well as other quantities of interest) on the class of allowable edge latency functions; the intuition afforded by the nonlinear version of Pigou's example (Subsection 2.4.4) suggests that the price of anarchy grows with the "steepness" of the network latency functions. Because of this phenomenon, much of Chapter 3 can be seen as a struggle to formulate an appropriate notion of "steepness" that makes this dependence precise (culminating in the definition of the anarchy value of a latency function in Subsection 3.3.1). In stating and proving Theorem A.1.1, we made another attempt at quantifying the steepness of a latency function. We record this attempt in the following definition.

Definition A.2.1 The *incline* $\Gamma(\ell)$ of a latency function ℓ is

$$\Gamma(\ell) = \sup_{x>0} \frac{x \cdot \ell(x)}{\int_0^x \ell(t)dt},$$

with the interpretation $\frac{0}{0} = 1$. The *incline* $\Gamma(G, r, \ell)$ of instance (G, r, ℓ) is

$$\Gamma(G, r, \ell) = \max_{e \in E} \Gamma(\ell_e).$$

Since latency functions are nondecreasing, the incline of any latency function (and hence of any instance) is at least 1. If instance (G, r, ℓ) has incline at most γ, then we will call (G, r, ℓ) γ-*inclined*. Thus Theorem A.1.1 can be succinctly stated: the price of anarchy in γ-inclined instances is at most γ.

A.2.2 Steepness

Definition A.2.1 is not aesthetically appealing and can be motivated only via the proof of Theorem A.1.1: the incline of an instance measures the discrepancy between the different objective functions minimized by Nash and optimal flows. There is also a slightly weaker yet somewhat more intuitive version of incline, which we introduce next. To do so, we must recall two notions from Section 2.3. The first is that of a *standard* latency function (see Definition 2.3.5), and the second is that of a *marginal cost function*, which for a differentiable latency function ℓ is defined by $\ell^*(x) = \frac{d}{dy}(y \cdot \ell(y))(x) = \ell(x) + x \cdot \ell'(x)$.

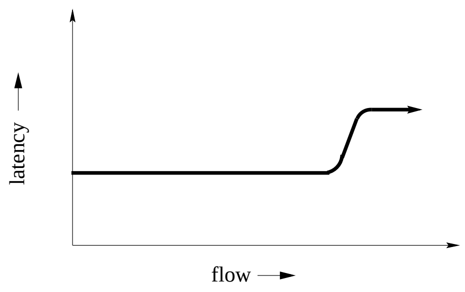

Figure A.1: A latency function with large steepness but moderate incline

Definition A.2.2 The *steepness* $\Sigma(\ell)$ of a standard latency function ℓ is

$$\Sigma(\ell) = \sup_{x>0} \frac{\ell^*(x)}{\ell(x)},$$

with the interpretation $\frac{0}{0} = 1$. The *steepness* $\Sigma(G, r, \ell)$ of an instance (G, r, ℓ) with standard latency functions is

$$\Sigma(G, r, \ell) = \max_{e \in E} \Sigma(\ell_e).$$

Remark A.2.3

(a) If instance (G, r, ℓ) has steepness at most σ, we will call (G, r, ℓ) σ-*steep*.

(b) The steepness of a latency function is bounded below by its incline.

(c) From the previous observation, a σ-steep instance is σ-inclined; by Theorem A.1.1, it follows that the price of anarchy in σ-steep instances is at most σ.

(d) Some latency functions (such as polynomials) have equal steepness and incline; in general, however, the steepness of a latency function can far exceed its incline. This fact is illustrated in Figure A.1, which shows a latency function with large steepness but moderate incline. It is this picture that inspires our terminology; very roughly speaking, a latency function that increases sharply even at a single point is steep (by our definition), while only a latency function whose graph has a large (global) increase in "elevation" can have large incline.

Why bother defining steepness, which seems similar to but weaker than the notion of incline? First, we believe the steepness of an instance to have a more natural interpretation than the incline. Recall from Corollary 2.3.2 that the optimal flow in a network with standard latency functions is nothing more than a flow at Nash equilibrium with respect to the marginal cost functions ℓ^*; thus, the steepness of an instance simply measures the worst-case discrepancy between how a Nash and an optimal flow evaluates the cost of increasing flow on an edge (cf., the hard-to-interpret objective function minimized by a flow at Nash equilibrium). Second, we will see in Section A.3 that the steepness of an instance controls the potential "unfairness" of an optimal flow (recall the example of Subsection 2.4.5).

A.3 How Unfair is Optimal Routing?

We saw in Subsection 2.4.5 that optimal flows, while minimizing the total latency, may lack desirable *fairness* properties—specifically, that some traffic in a minimum-latency flow may be routed on paths with larger latency than that incurred by all traffic in a Nash flow. This drawback of routing traffic optimally has inspired practitioners to find traffic assignments that minimize total latency subject to explicit *length constraints* [90], which require that no network users experience much more latency than in a flow at Nash equilibrium. The question we study in this section is the following: *how much worse off can network users be in an optimal flow than in one at Nash equilibrium?*.

For the rest of this section, we will confine ourselves to instances in which all traffic shares a common source and destination. Define the *unfairness* of such an instance (G, r, ℓ) as the maximum ratio between the latency of a flow path of an optimal flow for (G, r, ℓ) and that of a flow path of a Nash flow for (G, r, ℓ). We denote the unfairness of instance (G, r, ℓ) by $u(G, r, \ell)$.

Our first observation is that $u(G, r, \ell)$ can be arbitrarily large if we do not place additional restrictions on the class of allowable latency functions. To see this, modify the example of Subsection 2.4.5 as follows: for any positive integer p, define the latency of the first edge as the constant function $\ell(x) = (p + 1)(1 - \epsilon)$ and that of the second edge as $\ell(x) = x^p$. In this example, $u(G, r, \ell) = (p + 1)(1 - \epsilon)$, which tends to $+\infty$ with p.

In the spirit of our work bounding the price of anarchy, we aim to quantify the worst possible unfairness as a function of the class of allowable latency functions. We have already formulated the appropriate notion of "steepness" for quantifying the unfairness of optimal flows in instances with standard latency functions in the previous section; namely, the notion of *steepness* given in Definition A.2.2.

Theorem A.3.1 *If (G, r, ℓ) is an instance with a single source-destination pair and standard latency functions, then*

$$u(G, r, \ell) \leq \Sigma(G, r, \ell).$$

Proof. Let (G, r, ℓ) be an instance with source s, destination t, standard latency functions, and steepness σ. Suppose f and f^* are Nash and optimal flows for (G, r, ℓ), respectively. We need to show that the maximum latency of a flow path of f^* is at most σ times the latency of a flow path of f.

Suppose for contradiction that P_1, P_2 are paths s-t satisfying $f_{P_1} > 0$, $f^*_{P_2} > 0$, and $\ell_{P_2}(f^*) > \sigma \cdot \ell_{P_1}(f)$. Since f is at Nash equilibrium for (G, r, ℓ), by Proposition 2.2.2 all flow paths of f have a common latency L with respect to latency functions ℓ. Similarly, by Corollary 2.3.2 all flow paths of f^* have a common latency L^* with respect to latency functions ℓ^*.

Now, as every latency function is nondecreasing, we have $\ell_e(x) \leq \ell^*_e(x)$ for all e and x. Thus, we may derive

$$L = \ell_{P_1}(f) < \frac{1}{\sigma}\ell_{P_2}(f^*) \leq \frac{1}{\sigma}\ell^*_{P_2}(f^*) = \frac{L^*}{\sigma}.$$

By Proposition 2.2.4, the cost of the flow f is $C(f) = rL$. The cost of the optimal flow f^* is not so easy to compute, as flow paths have equal latency with respect to functions ℓ^* but not with respect to ℓ. However, since every latency function has steepness at most σ, we obtain $\ell^*_P(f^*) \leq \sigma \cdot \ell_P(f^*)$ for every path P and hence

$$C(f^*) \geq \frac{1}{\sigma} \sum_{P \in \mathcal{P}} \ell^*_P(f^*) f^*_P = \frac{1}{\sigma} r L^* > rL = C(f),$$

which contradicts the optimality of f^*. ■

For example, an instance whose latency functions are polynomials with nonnegative coefficients of degree at most p has unfairness at most $p + 1$.

Remark A.3.2 Theorem A.3.1 is not sharp on all instances (for a trivial case, take G to be a single link with latency function $\ell(x) = x$). However, the theorem is best possible in the following sense: for any real number $c \geq 1$, there is an instance (G, r, ℓ) with standard latency functions satisfying $\Sigma(G, r, \ell) \leq c$ (namely, the example given at the beginning of this section with p everywhere replaced by $c - 1$) with unfairness arbitrarily close to c.

Remark A.3.3 Theorem A.3.1 and the previous remark provide an analogue of our work in Sections 3.3–3.4 showing that the price of anarchy is independent of the network topology. Let $\mathcal{L}$ denote a standard class of latency functions including the constant functions, and define the *steepness* $\Sigma(\mathcal{L})$ by $\Sigma(\mathcal{L}) = \sup_{\ell \in \mathcal{L}} \Sigma(\ell)$. Then $\sup_{(G,r,\ell)} u(G,r,\ell)$ (where the supremum ranges over instances with a single source-destination pair and latency functions in $\mathcal{L}$) is precisely $\Sigma(\mathcal{L})$, with worst-case examples furnished by networks of two parallel links. In fact, it is not difficult to see that this statement remains true with the weaker assumptions that $\mathcal{L}$ is standard and is diverse in the sense that $\{\ell(0) : \ell \in \mathcal{L}\} = (0, \infty)$ (cf. Theorem 3.4.4, where more than two links are required for worst-case examples of the inefficiency of Nash flows).

A further generalization (in the spirit of Section 3.5) is the following: if $\mathcal{L}$ is standard and contains a latency function that is positive when evaluated with zero congestion, then the worst-case unfairness of optimal flows (with respect to $\mathcal{L}$) is achieved (modulo an arbitrarily small additive factor) in subdivisions of a two-node, two-link network. Some assumption on $\mathcal{L}$ is necessary for this sort of result; indeed, any network with latency functions drawn from $\mathcal{L}_p = \{ax^p : a > 0\}$ has steepness $p+1$ but unfairness 1 (by a straightforward generalization of Corollary 3.2.2).

Appendix B

A Collection of Counterexamples

B.1 Necessity of Continuous, Nondecreasing Latency Functions for Nash Flows

In this section, we provide several examples demonstrating that the useful properties of flows at Nash equilibrium presented in Chapter 2, such as existence and uniqueness, fail if we allow edge latency functions to be discontinuous or nonmonotone.

The next two propositions study networks with latency functions that are nondecreasing but not continuous. We first show that Nash flows need not exist in such networks.

Proposition B.1.1 *There is a network G with nondecreasing discontinuous latency functions ℓ and a traffic rate r such that (G, r, ℓ) fails to admit a feasible flow at Nash equilibrium.*

Proof. Let G denote a two-node two-link network. Define one latency function by $\ell_1(x) = 1$ and another by

$$\ell_2(x) = \begin{cases} x & \text{if } x < 1 \\ 2 & \text{if } x \geq 1. \end{cases}$$

It is evident that both latency functions are nondecreasing and that no flow feasible for $(G, 1, \ell)$ meets the definition of a Nash flow set forth in Definition 2.2.1. ■

The next proposition demonstrates that networks with discontinuous latency functions can admit Nash flows with distinct costs.

Proposition B.1.2 *There is a network G with nondecreasing discontinuous latency functions ℓ and a traffic rate r such that (G, r, ℓ) admits two feasible flows at Nash equilibrium with different costs.*

Proof. Define a network G as in the previous proposition. Define the first latency function ℓ_1 by

$$\ell_1(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{3}] \\ 1 & \text{if } x > \frac{1}{3} \end{cases}$$

and the second by

$$\ell_2(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{3}] \\ 1 & \text{if } x > \frac{1}{3}. \end{cases}$$

Again set the traffic rate r to be 1. One flow at Nash equilibrium routes $\frac{1}{3}$ of the flow on the first edge and the rest on the second, for a cost of $\frac{2}{3}$; another routes $\frac{2}{3}$ of the the flow on the first edge and the rest on the second, for a cost of $\frac{5}{6}$. ■

Remark B.1.3 The previous example also shows that the characterization of Nash flows given in Proposition 2.2.2 fails when network latency functions need not be continuous: Nash flows in such networks need not route all flow on paths having minimum-latency.

We next consider networks in which latency functions are continuous but are not assumed to be nondecreasing. While Nash flows still exist (as the proof of Proposition 2.5.1 shows), they are no longer unique in any sense. This is demonstrated in the next proposition.

Proposition B.1.4 *There is a network G with nonmonotone continuous latency functions ℓ and a traffic rate r such that (G, r, ℓ) admits two feasible flows at Nash equilibrium with different costs.*

Proof. Let G denote a network with two nodes and three parallel links. Endow the first two links with the latency function $\ell(x) = (x - \frac{1}{2})^2 + 1$ and the third with latency function $\ell(x) = \max\{2 - 2x, 0\}$. Then $(G, 1, \ell)$ admits a Nash flow that routes half the traffic on each of the first two links (for a cost of 1) and another Nash flow that routes all traffic on the third link (for a cost of 0). ■

Remark B.1.5 A variant on the previous example shows that the characterization of Nash flows given in Proposition 2.2.2 fails in networks with nonmonotone latency functions. To see this, replace the latency function on the third edge of the network

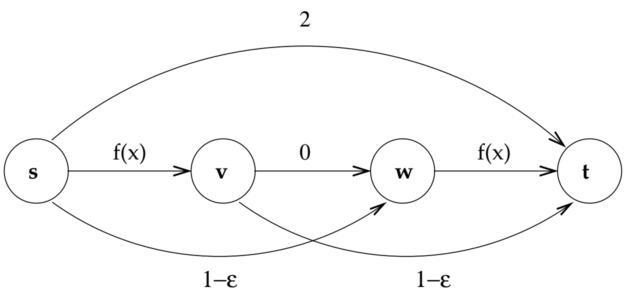

Figure B.1: Theorem 4.1.3 is sharp

above by the less severe latency function $\ell(x) = \max\{1-x, 0\}$. The flow that routes half of the traffic on each of the first two edges equalizes the latency of all three edges at 1 but is not at Nash equilibrium; any traffic would be better off by rerouting itself on the third link.

B.2 Theorem 4.1.3 is Sharp

In Section 4.1 we defined the notion of a flow at ϵ-approximate Nash equilibrium and showed in Theorem 4.1.3 that if f is at ϵ-approximate Nash equilibrium for the instance (G, r, ℓ) and f^* is feasible for $(G, 2r, \ell)$, then $C(f) \leq \frac{1+\epsilon}{1-\epsilon} C(f^*)$. We now show that the factor of $\frac{1+\epsilon}{1-\epsilon}$ cannot be improved in general network topologies.

Fix $\epsilon \in (0,1)$ and consider the network G shown in Figure B.1 (with topology identical to that of Figure 4.1, namely the Braess Paradox graph of Figure 2.2 with a direct s-t edge added). Four of the edges have constant latency functions, as shown, and by $f(x)$ we mean a nondecreasing, continuous function equal to 0 on $[0, 1-\delta]$ and to $1+\epsilon$ on $[1, \infty)$ (where $\delta > 0$ is arbitrarily small). The flow f routing 1 unit of flow on the three-hop path $s \to v \to w \to t$ is at ϵ-approximate Nash equilibrium for $(G, 1, \ell)$ and has cost $2(1+\epsilon)$. On the other hand, the flow f^* routing $1-\delta$ units of flow on each of the two-hop paths and 2δ units of flow on the s-t edge is feasible and has cost approaching $2(1-\epsilon)$ as $\delta \to 0$.

On the other hand, the factor of $\frac{1+\epsilon}{1-\epsilon}$ can be improved to $1+\epsilon$ in networks of parallel links.

Proposition B.2.1 *Let G be a network of parallel links and f at ϵ-approximate Nash equilibrium for (G, r, ℓ). If f^* is feasible for $(G, 2r, \ell)$, then $C(f) \leq (1+\epsilon)C(f^*)$.*

Proof. Let L be the minimum latency of any edge with respect to f; since G is a network of parallel links and f is at ϵ-approximate Nash equilibrium, we have $C(f) \leq (1+\epsilon)rL$. Let E_1 be the edges e of G for which $f^*_e < f_e$, and E_2 the rest of the edges. Since G is a network of parallel links, f^* routes less than $\sum_{e \in E_1} f_e \leq r$ units of flow on edges in E_1. Thus, f^* routes at least $2r - r = r$ units of flow on edges of E_2. Since $f^*_e \geq f_e$ for all edges $e \in E_2$, we have $\ell_e(f^*_e) \geq \ell_e(f_e) \geq L$ for all $e \in E_2$. We have shown that at least r units of f^* experience at least L units of latency; hence $C(f^*) \geq rL \geq \frac{C(f)}{1+\epsilon}$. ■

B.3 Stackelberg Routing in General Networks

In Section 6.4, we proved that in networks of parallel links a carefully chosen Stackelberg strategy induces a flow with cost no more than $\frac{1}{\beta}$ times that of the minimum-latency flow, where β is the fraction of the traffic that is centrally controlled. In this section we show that this guarantee cannot be extended to more general network topologies. Specifically, we have the following bad example in the graph of Braess's Paradox (Figure 2.2).

Proposition B.3.1 *There is a Stackelberg instance (G, r, ℓ, β) in which no flow induced by a Stackelberg strategy has cost at most $C(f^*)/\beta$, where f^* is an optimal flow for (G, r, ℓ).*

Proof. Let G be the graph of Braess's Paradox (see Figure B.2). Let the latency functions of edges (s, w) and (v, t) be $\ell(x) = 1$ and of (v, w) be $\ell(x) = 0$ (as in Braess's Paradox). Define the latency function of the remaining two edges by $\ell(x) = f(x)$, where $f(x) = 0$ on $[0, \frac{3}{4} - \epsilon]$, $f(x) = 1 - \epsilon$ on $[\frac{3}{4}, \infty)$, and $f(x)$ is defined arbitrarily on $(\frac{3}{4} - \epsilon, \frac{3}{4})$ subject to the usual continuity and monotonicity restrictions (where $\epsilon > 0$ is arbitrarily small). The flow feasible for $(G, 1, \ell)$ routing $\frac{1}{2} - 2\epsilon$ units of flow on the three-hop path and $\frac{1}{4} + \epsilon$ units of flow on each of the two-hop paths has cost approaching $\frac{1}{2}$ as $\epsilon \to 0$.

Now consider the Stackelberg instance $(G, 1, \ell, \frac{1}{2})$, and any Stackelberg strategy f. We must show that the flow induced by f has large cost. We first observe that, for any strategy f, all selfish traffic will be routed on the three-hop path $s \to v \to w \to t$. All that remains is a simple case analysis.

Case 1: Suppose f routes at least $\frac{1}{4}$ units of flow both on edge (s, v) and on edge (w, t). By the previous observation, the flow induced by f routes at least $\frac{3}{4}$ units of

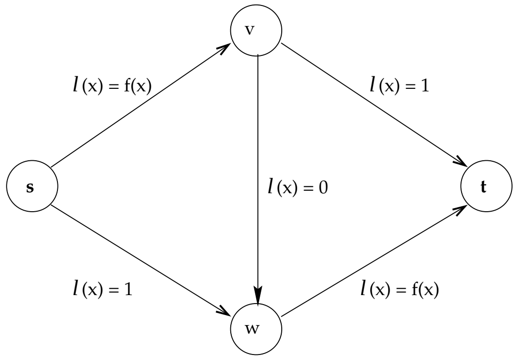

Figure B.2: A bad example for Stackelberg routing

traffic on each of these edges and thus its cost is at least $\frac{3}{2}(1-\epsilon)$.

Case 2: Suppose f routes less that $\frac{1}{4}$ units of flow on edge (w,t); then at least $\frac{1}{4}$ units of flow are routed on the path $s \to v \to t$. This implies that the congestion on edge (s,v) is at least $\frac{3}{4}$. The cost of the flow induced by f must then be at least $\frac{5}{4} - \epsilon$ (with at least $1-\epsilon$ latency incurred on arcs (s,v) and (s,w) and at least $\frac{1}{4}$ latency incurred on arc (v,t)).

Case 3: If neither case 1 nor case 2 occurs, then f routes less than $\frac{1}{4}$ units of flow on the edge (s,v) and hence at least $\frac{1}{4}$ units of flow on the path $s \to w \to t$. Symmetric to the previous case, this implies that the cost of the flow induced by f is at least $\frac{5}{4} - \epsilon$.

We have shown that, as $\epsilon \to 0$, the cost of the minimum-latency flow for $(G,1,\ell)$ tends to (at most) $\frac{1}{2}$ while the total latency of the min-cost flow induced by some Stackelberg strategy tends to $\frac{5}{4} > \frac{1}{\beta}\frac{1}{2}$. The proof is complete. ■

Remark B.3.2 With only a little more work, the counterexample of Proposition B.3.1 can be modified to possess standard (or even convex) latency functions.

B.4 LLF is Not Optimal

In this section we demonstrate that the LLF strategy of Chapter 6 need not be the optimal Stackelberg strategy, even in networks of parallel links with linear latency functions.

To see this, consider a network G with two nodes and three edges, with latency functions $\ell_1(x) = x$, $\ell_2(x) = 1 + x$, and $\ell_3(x) = 1 + x$. In the instance $(G, 1, \ell, \frac{1}{6})$, the optimal flow routes $\frac{2}{3}$ of the traffic on the first edge and splits the remaining traffic equally between the last two edges. The LLF strategy thus routes the $\frac{1}{6}$ units of centrally controlled flow on the third edge, inducing a flow with $\frac{5}{6}$ of the flow on the first edge and the rest on the third, for a cost of $\frac{8}{9}$. On the other hand, the Stackelberg strategy that routes $\frac{1}{12}$ units of flow on each of last two edges induces a flow with cost $\frac{7}{8}$.

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