Equilibrium Efficiency and Price Complexity in Sponsored Search Auctions

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Abstract

Modern sponsored search auctions are derived from the Generalized Second Price (GSP) auction. Although the GSP auction is not truthful, results by Edelman, Ostrovsky, and Schwarz [7] and Varian [13] give senses in which its outcome is equivalent to that of the truthful and welfare-maximizing VCG mechanism. The first main message of this paper is that these properties are not unique to the GSP auction: there is a large class of payment rules that, when coupled with the rank-by-bid allocation rule, induce sponsored search auctions with comparable guarantees. The second main message is that the GSP auction is "optimally simple", subject to possessing a welfare-maximizing Nash equilibrium, when the complexity of a payment rule is measured using the dependencies between bids and slot prices.

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1 Introduction

In the standard formulation of a one-shot sponsored search auction, n advertisers vie for k ad slots on a search results page for some keyword. Each advertiser has a private valuation v_i for a click, each slot j has a "click-through-rate (CTR)" α_j, and if advertiser i is placed in slot j at a price of p_j per click, then its utility is defined as $\alpha_j(v_i - p_j)$. Renaming the slots so that $\alpha_1 \geq \cdots \geq \alpha_k$ and the advertisers so that $v_1 \geq \cdots \geq v_n$, the welfare-maximizing solution assigns the ith advertiser to the ith slot for $i = 1, 2, \ldots, k$. Modern sponsored search auctions are derived from the Generalized Second Price (GSP) auction, which assigns the ith highest bidder to the ith slot for $i = 1, 2, \ldots, k$ — the "rank-by-bid" allocation rule — and charges the $(i+1)$th highest bid for a click in that slot.

The practical importance of the GSP auction justifies studying it from a theoretical perspective; Edelman, Ostrovsky, and Schwarz [7] and Varian [13] were the first to do so. The main results in [7, 13] give senses in which the outcome of the GSP auction is equivalent to that of the truthful and welfare-maximizing VCG mechanism. For example, even though the GSP auction is not truthful, it always has a full-information Nash equilibrium in which the allocations and payments are the same as in the VCG mechanism (under truthful reporting) [7, 13]. This equilibrium also has an ascending implementation [7].[1]

The first main message of this paper is that the properties singled out in previous work [7, 13] are *not at all unique to the GSP auction*. Rather, there is a *large class* of payment rules that, when coupled with the rank-by-bid allocation rule, induce sponsored search auctions in which the VCG outcome is a full-information Nash equilibrium that also admits an ascending implementation. Mathematically, our result is a near-characterization of the anonymous payment rules in which slot prices depend only on lower bids that there are "efficiency-inducing" in this sense.[2] Our sufficient conditions are relatively weak and demonstrate that a wide range of payment rules are efficiency-inducing.[3] One interpretation of this result is that the previously identified attractive properties of the GSP auction — "a result of evolution of inefficient market institutions, which were gradually replaced by increasingly superior designs" [7, page 253] — are perhaps not so surprising in retrospect.

The second main message of this paper is that the intuitive "simplicity" of the GSP auction — which presumably has played a signficiant role in its original design and enduring appeal — can and should be formalized.[4] Toward this end, we measure the complexity of a payment rule using the dependencies of slot prices on the bids. In the GSP auction, the price of every slot j depends only on a single bid (the $(j+1)$th highest). An easy argument shows that the total number of dependencies cannot be smaller than k in any payment rule that guarantees an efficient Nash equilibrium. More interestingly, for a large class of efficiency-inducing payment rules (including

[1]The focus on full-information Nash equilibria is justified in [7] by the repeated nature of sponsored search auctions. There are generally multiple such equilibria; the selection of the one corresponding to the VCG outcome is justified in [7, 13] by proving that it is the "locally envy-free" equilibrium that is the best for the advertisers and worst for the search engine revenue, and that it admits a natural ascending implementation [7]. It was also justified later by Cary et al. [6] as the unique fixed point of myopic best-response dynamics under the "Balanced Bidding" strategy. Of course, other equilibrium selection rules can also be considered; see Hashimoto [10] for an alternative. While both the full-information assumption and the equilibrium selection rule in [7, 13] are worth questioning, we adopt them here and focus on other research directions.

[2]We consider ex post ascending implementations; Edelman et al. [7] provided somewhat stronger justifications of the VCG outcome in the GSP auction.

[3]One simple example is a VCG mechanism that is implemented using incorrect geometric CTRs: while not truthful, it has an efficient Nash equilibrium if (and only if) the incorrect estimates are "more spread out" than the actual CTRs.

[4]One obvious point is that the GSP payment rule is independent — and hence robust to incorrect estimates — of the CTRs. This "detail-free" property is not shared by the VCG payment rule.

all efficiency-inducing linear rules, and many others), we show that the price of each slot j must depend on the $(j+1)$th highest bid. Thus the GSP payment rule has *minimal complexity* in a strong sense: its dependencies are precisely the intersection of those of all rules in the class.

1.1 Further Related Work

See [3, 11] for surveys about sponsored search auctions; we discuss here only the papers most related to the present work.

The paper of Yenmezy [14] is related to our first result, as he presents some sufficient conditions for a sponsored search auction payment rule to always induce efficient equilibria. The third "regularity condition" in [14] is quite restrictive, however, and is not even satisfied by the VCG mechanism. In addition, no necessary conditions are presented in [14].

The necessary and sufficient conditions in our first result are similar but incomparable to those discovered by Ashlagi, Monderer, and Tennenholtz [4] for a completely different problem — characterizing the sponsored search auction payment rules that admit an efficient and individually rational (IR) mediator. The mediator plays on behalf of the players, and can effectively coordinate responses to a deviating player. However, there is no fundamental relationship between our goal and theirs: there are payment rules that always induce efficient equilibria yet do not admit an IR and efficient mediator, and conversely.[5]

Finally, we note that a number of authors have asked whether the GSP auction remains efficiency-inducing in broader contexts, such as when bidders have multi-parameter types, with decidely mixed results [1, 2, 8, 9].

2 Model and Preliminaries

2.1 Sponsored Search Auctions

We first recall the standard theoretical model of a sponsored search auction. There are k slots and $n = k + 1$ agents. Agent $i \in [n]$ has a private nonnegative valuation $v_i \geq 0$ per click. We typically relabel the players' names so that valuations are non-increasing: $v_1 \geq v_2 \geq \cdots \geq v_{k+1}$.[6] We write V for the space of valuation profiles (nonnegative and non-increasing n-vectors) and $\mathbf{v}$ for a generic profile. Each slot i has a positive click-through-rate (CTR) α_i, and we assume these are strictly decreasing in the slot number. For convenience, we sometimes make use of a slot $k+1$ with $\alpha_{k+1} = 0$. We assume that player i's utility is $v_i \cdot \alpha_j$ minus its payments to the search engine, with the semantics that an impression in slot j has probability α_j of leading to a click. We do not explicitly discuss the seemingly more general model of separable CTRs, which also include an agent-specific multiplier, but they cab be accommodated easily in what follows by rescaling agents' bids appropriately. We focus on direct-revelation mechanisms, which accept a nonnegative bid-per-click b_i from each player and outputs an allocation (an assignment of bidders to slots) and payments (a price-per-click for each slot).

[5]The rough intuition is as follows. First, the collective response to a deviator afforded by a mediator makes implementation via a mediator possible in some cases where a full-information equilibrium does not exist. For the other direction, consider a would-be efficient equilibrium and an outcome that differs from it by a single unilateral deviation. If the deviation causes high payments for all agents, then the would-be equilibrium is indeed self-enforcing, but it is hard to enforce with an IR mediator. A detailed write-up of these arguments is available from the authors.

[6]We break ties lexicographically according to some fixed permutation on the bidders' names. Almost everything in this paper is independent of the choice of a tie-breaking rule.

2.2 Two Fundamental Examples

The next two examples are the most well-studied mechanisms in the sponsored search literature.

Example 2.1 (The VCG Mechanism) In the present context, the Vickrey-Clarke-Groves (VCG) mechanism works as follows (for known CTRs α): it accepts a bid from each bidder; and for $j = 1, 2, \ldots, k$, it assigns the jth highest bidder to the jth slot at a per-click price of

$$q_j(\mathbf{b}) = \frac{\sum_{l=j+1}^{k+1} b_l(\alpha_{l-1} - \alpha_l)}{\alpha_j}, \tag{1}$$

where b_l denotes the lth highest bid (the $(k+1)$th bidder gets nothing and pays $q_{k+1}(\mathbf{b}) = 0$).

It is well known that the VCG mechanism is *truthful*: for every bidder, setting the bid equal to the private valuation is a dominant strategy. For this reason, we often use bids and valuations interchangeable in the VCG mechanism, and in particular denote the price q_j in (1) as a function $q_j(\mathbf{v})$ of the valuation profile $\mathbf{v}$ (rather than the bid profile $\mathbf{b}$). Note that $q_j(\mathbf{v})$ is also a function of the CTRs α, whose dependence we usually leave implicit. When $\mathbf{v}$ is clear from the context we abuse notation and denote $q_j(\mathbf{v})$ by q_j.

Example 2.2 (The GSP Auction) The *generalized second-price (GSP) auction* differs from the VCG mechanism in only one respect: the price-per-click charged to a bidder $j \leq k$ is the next-highest bid b_{j+1}, rather than the quantity in (1).

Simple examples show that the GSP auction is *not* truthful [7].

2.3 Efficiency-Inducing Payment Rules

A direct-revelation mechanism comprises two parts: an *allocation rule* and a *payment rule*, which are functions from bids to allocations (i.e., slots) and to prices, respectively. The VCG and GSP mechanisms have identical *rank-by-bid* allocation rules, and distinct payment rules that share several properties. First, both are *anonymous* in the sense that the price paid by a bidder depends only on its bid and the set of the other bids, and is independent of the names of the bidders.[7] Second, both are *upper triangular*, meaning that the price of slot j is a function only of the smaller bids $b_{j+1}, \ldots, b_{k+1}$.[8] Third, both are *efficiency-inducing*, or simply *efficient* for short, in the following sense.

Definition 2.3 (Efficient Payment Rule) Let $\mathbf{x}$ denote the rank-by-bid allocation rule. A payment rule $\mathbf{p}$ is *efficiency-inducing*, or *efficient* for short, if for every valuation profile $\mathbf{v}$ there is a full-information Nash equilibrium bid profile $\mathbf{b}$ such the equilibrium allocation $\mathbf{x}(\mathbf{b})$ is the efficient allocation $\mathbf{x}(\mathbf{v})$ and the equilibrium prices $\mathbf{p}(\mathbf{b})$ are the VCG prices $\mathbf{q}(\mathbf{v})$.

In Definition 2.3, the condition that $\mathbf{x}(\mathbf{b})$ is efficient is equivalent to requiring that the bids in $\mathbf{b}$ are ordered according to the valuations $\mathbf{v}$.

The VCG payment rule is efficiency-inducing (with $\mathbf{b} = \mathbf{v}$) because the corresponding mechanism is truthful. A non-trivial and important fact is that the GSP payment rule is also efficient

[7] As noted earlier, agent-specific CTR multipliers, which violate anonymity, can be accommodated in our model by rescaling agents' bids accordingly.

[8] The name is motivated by the special case of linear payment rules — like those in the VCG mechanism and the GSP auction — which, when expressed in matrix form as a linear map from bids $b_2, \ldots, b_{k+1}$ to per-click prices $p_1, \ldots, p_k$, is upper-triangular.

in the sense of Definition 2.3 [7, 13]. We now have the language to phrase formally the first main question of this paper: *Which anonymous and upper-triangular payment rules are efficient?*[9]

2.4 Useful Properties of the VCG Payments

Many of our arguments rely on well-known properties of the VCG prices (1); we recall these next and include a proof for completeness.

Proposition 2.4 For every valuation profile $\mathbf{v}$ and CTRs α:

1. For every slot $j \in [k]$,

$$q_j(\mathbf{v}) = \frac{\alpha_{j+1}}{\alpha_j} q_{j+1}(\mathbf{v}) + \left(1 - \frac{\alpha_{j+1}}{\alpha_j}\right) v_{j+1},$$

   with the convention that $\alpha_{k+1} = q_{k+1}(\mathbf{v}) = 0$. That is, the VCG price for slot j is a convex combination of the $(j+1)$th valuation and VCG price for the $(j+1)$th slot.

2. The vector $\mathbf{q}(\mathbf{v})$ is non-increasing: $q_1(\mathbf{v}) \geq \cdots \geq q_k(\mathbf{v})$.

3. The VCG prices are *envy free*: for every $i, j \in [k+1]$,

$$\alpha_j(v_j - q_j(\mathbf{v})) \geq \alpha_i(v_j - q_i(\mathbf{v})).$$

   That is, given a choice from all slots at the VCG prices $\mathbf{q}(\mathbf{v})$, each slot j is an optimal choice for the corresponding bidder j.

4. The VCG prices satisfy *local indifference*: for every bidder $j \in \{2, 3, \ldots, k, k+1\}$,

$$\alpha_j(v_j - q_j(\mathbf{v})) = \alpha_{j-1}(v_j - q_{j-1}(\mathbf{v})).$$

Proof: For part (1), we use the definition (1) to derive

$$q_j = \frac{\sum_{l=j+2}^{k+1} v_l(\alpha_{l-1} - \alpha_l)}{\alpha_j} + v_{j+1}\frac{(\alpha_j - \alpha_{j+1})}{\alpha_j} = \frac{\alpha_{j+1}}{\alpha_j} q_{j+1} + \left(1 - \frac{\alpha_{j+1}}{\alpha_j}\right) v_{j+1}.$$

Also, rearranging this equation proves part (4).

Next, since a bidder in the VCG mechanism can always obtain a nonnegative utility with a zero bid, truthfulness implies that $q_{j+1}(\mathbf{v}) \leq v_{j+1}$ for every j. Part (2) now follows immediately from part (1).

To prove part (3), consider valuations $\mathbf{v}$ and an agent with value v_j that gets slot j and has utility $\alpha_j(v_j - q_j)$ in the truthful VCG outcome. For every lower slot $i \geq j$, $\alpha_i(v_j - q_i)$ is precisely the utility that bidder j would get by bidding for slot i. Since the VCG mechanism is truthful, $\alpha_j(v_j - q_j) \geq \alpha_i(v_j - q_i)$.

Finally, consider a higher slot $i < j$. For every $l \leq j$, part (4) implies that $\alpha_l(v_l - q_l) = \alpha_{l-1}(v_l - q_{l-1})$. Since $\alpha_{l-1} > \alpha_l$ and $v_l \geq v_j$, $\alpha_l(v_j - q_l) \geq \alpha_{l-1}(v_j - q_{l-1})$. Chaining inequalities of this form yields $\alpha_j(v_j - q_j) \geq \alpha_i(v_j - q_i)$, as desired. ■

[9] Our qualitative interpretation of our results in Section 3 — that numerous payment rules share the previously noted theoretical properties of the GSP auction — is only strengthened by our restriction to anonymous and upper-triangular payment rules.

3 Necessary and Sufficient Conditions for Efficient Payment Rules

3.1 Necessary Conditions for an Efficient Payment Rule

An obvious necessary condition for a payment rule $\mathbf{p}$ to satisfy Definition 2.3 for a given vector α of CTRs is that its range includes the space $Q(\alpha) = \{\mathbf{q}(\mathbf{v}) \mid \mathbf{v} \in V\}$ of all realizable VCG prices. Precisely, let S denote the set of all nonnegative and non-increasing n-vectors, which represents all sorted bid profiles. Call the bid profile $\mathbf{b}$ *feasible for* $\mathbf{v}$ if $\mathbf{b} \in S$ and $\mathbf{p}(\mathbf{b}) = \mathbf{q}(\mathbf{v})$ — these are precisely the candidate efficient equilibria with respect to $\mathbf{v}$. A bid profile is *feasible* if it is feasible for some valuation profile $\mathbf{v}$. In other words, the feasible bid profiles are the set $\mathbf{p}^{-1}(Q(\alpha))$.

Definition 3.1 (Onto $Q(\alpha)$) For given CTRs α, an anonymous payment rule $\mathbf{p}$ *is onto* $Q(\alpha)$ if for every valuation profile $\mathbf{v} \in V$ there is a bid profile feasible for it.

Example 3.2 The GSP payment rule $\mathbf{p}$ is onto $Q(\alpha)$ for every α. To see why, fix $\mathbf{v}$ and recall that $\mathbf{q}(\mathbf{v})$ is non-increasing (Proposition 2.4(2)). Thus, setting $b_i = q_{i-1}(\mathbf{v})$ for each $i = 2, 3, \ldots, k+1$ (and $b_1 = b_2$, say) yields a profile in S with $\mathbf{p}(\mathbf{b}) = \mathbf{q}(\mathbf{v})$.

For a non-example, take $k = 2$ and consider the rule $p_1(\mathbf{b}) = p_2(\mathbf{b}) = b_3$. Choosing a profile $\mathbf{v}$ with $q_1(\mathbf{v}) > q_2(\mathbf{v})$ — and such a profile exists for every strictly decreasing vector α — shows that $\mathbf{p}$ is not onto $Q(\alpha)$.

We call our second necessary condition *local monotonicity*.

Definition 3.3 (Local Monotonicity) An anonymous and upper-triangular payment rule $\mathbf{p}$ is *locally monotone* if for every valuation profile $\mathbf{v} \in V$ there is a feasible bid profile $\mathbf{b}$ for $\mathbf{v}$ with

$$p_{j-1}(b_j, b_{j+1}, ..., b_{k+1}) \leq p_{j-1}(b_{j-1}, b_{j+1}, ..., b_{k+1})$$

for every $j \in \{2, ..., k, k+1\}$.

We now formally prove that these two conditions are necessary for a payment rule to be efficient.

Theorem 3.4 (Necessary Conditions for Efficiency) *Let α be a vector of CTRs. An anonymous and upper-triangular payment rule is efficiency-inducing only if it is onto $Q(\alpha)$ and locally monotone.*

Proof: Fix α and let $\mathbf{p}$ be an anonymous, upper-triangular, and efficient payment rule. First, by the definitions, $\mathbf{p}$ must be onto $Q(\alpha)$. Second, assume for contradiction that $\mathbf{p}$ is not locally monotone. Then there exists a non-decreasing valuation profile $\mathbf{v}$ such that for every corresponding feasible bid vector $\mathbf{b}$ for $\mathbf{v}$, there exists an index $j \in \{2, ..., k, k+1\}$ for which $p_{j-1}(b_j, b_{j+1}, ..., b_{k+1}) > p_{j-1}(b_{j-1}, b_{j+1}, ..., b_{k+1})$. We prove that $\mathbf{p}$ is not efficiency-inducing by showing that the profile $\mathbf{b}$ cannot be an equilibrium. By the local indifference of the VCG prices (Proposition 2.4(4)) and the fact that $\mathbf{q}(\mathbf{v}) = \mathbf{p}(\mathbf{b})$, $\alpha_j(v_j - p_j(\mathbf{b})) = \alpha_{j-1}(v_j - p_{j-1}(\mathbf{b}))$. Since $\mathbf{p}$ is upper triangular, $q_{j-1}(\mathbf{v}) = p_{j-1}(\mathbf{b}) = p_{j-1}(b_j, b_{j+1}, ..., b_{k+1})$.

First suppose that b_{j-2} is strictly larger than b_{j-1}. (If $j = 2$, we use the convention that $b_{j-2} = \infty$.) Consider a deviation by agent j from $\mathbf{b}$, bidding up to get the slot $j-1$ via some bid strictly between b_{j-1} and b_{j-2}. After bidder j's deviation, the price of slot $j-1$ becomes $p_{j-1}(b_{j-1}, b_{j+1}, ..., b_{k+1})$ which is less than $p_{j-1}(b_j, b_{j+1}, ..., b_{k+1})$ by assumption. Thus, after bidder j's deviation its utility is

$$\alpha_{j-1}(v_j - p_{j-1}(b_{j-1}, b_{j+1}, ..., b_{k+1})) > \alpha_{j-1}(v_j - p_{j-1}(\mathbf{b})) = \alpha_j(v_j - p_j(\mathbf{b})),$$

which shows that $\mathbf{b}$ is not an equilibrium.

Finally, if $b_{j-2} = b_{j-1}$, we consider the deviation in which bidder j bids b_{j-1}. There are two cases, depending on the details of the tie-breaking rule: either bidder j is assigned to slot $j-1$ or some other bidder (with the same bid) is assigned to it. If a different bidder is assigned to that slot we can replace the valuations $\mathbf{v}$ that we started with by a profile in which the valuations of these two bidders are exchanged and consider the other bidder instead. By anonymity, the feasible bid vectors remain the same for this permuted valuation profile. With the same (feasible) bid vector, a deviation by the bidder that is now assigned to slot j with bids $\mathbf{b}$ to the bid b_{j-1} will cause that bidder to be assigned to slot $j-1$. The argument in the previous paragraph now applies and shows that $\mathbf{b}$ is not an equilibrium. ■

Remark 3.5 Theorem 3.7 provides sufficient conditions for a payment rule to be efficient that are "close" to the necessary conditions in Theorem 3.4. However, the twin conditions of being onto $Q(\alpha)$ and locally monotone are not, by themselves, always sufficient (Example A.1). Also, the two necessary conditions in Theorem 3.4 are logically independent — neither one implies the other in general.

3.2 Sufficient Conditions for an Efficient Payment Rule

We now show that an anonymous payment rule that is onto $Q(\alpha)$ and that satisfies a somewhat stronger monotonicity condition than Definition 3.3 is efficiency-inducing.

Definition 3.6 An anonymous and upper-triangular payment rule $\mathbf{p}$ is *monotone* if for every slot $j \in [k]$, $p_j(\mathbf{b}') \geq p_j(\mathbf{b})$ whenever $b_i{}' \geq b_i$ for every $i > j$.

The GSP and VCG payment rules are monotone, as is every linear payment rule that corresponds to a nonnegative matrix.

Theorem 3.7 (Sufficient Conditions for Efficiency) *Let α be a vector of CTRs. An anonymous and upper-triangular payment rule is efficiency-inducing if it is onto $Q(\alpha)$ and monotone.*

Proof: Let $\mathbf{p}$ satisfy the hypotheses of the theorem and consider a sorted valuation profile $\mathbf{v}$. Since $\mathbf{p}$ is onto $Q(\alpha)$, there is a sorted vector of bids $\mathbf{b} \in S$ with $\mathbf{q}(\mathbf{v}) = \mathbf{p}(\mathbf{b})$. In our candidate efficient equilibrium, bidder j bids b_j.

To show that this bid profile $\mathbf{b}$ is an equilibrium, consider a deviation by the bidder j — currently assigned to the jth slot with utility $\alpha_j(v_j - q_j(\mathbf{v}))$ — that results in assignment to the slot i. By upper-triangularity, if $i = j$ then its utility is unchanged. If $i > j$ then it gets slot i at price $q_i(\mathbf{v})$ and its payoff is $\alpha_i(v_j - q_i(\mathbf{v}))$, which is at most $\alpha_j(v_j - q_j(\mathbf{v}))$ by Proposition 2.4(3). If $i < j$, then the new bid profile $\mathbf{b}'$ satisfies $b'_h \geq b_h$ for every $h > i$. By the monotonicity of $\mathbf{p}$, bidder j gets slot i at some price that is at least $q_i(\mathbf{v})$, and Proposition 2.4(3) again implies that its new utility is at most $\alpha_j(v_j - q_j(\mathbf{v}))$. Since no deviating bid can improve bidder j's utility, the profile $\mathbf{b}$ is an equilibrium. ■

While Theorem 3.7 only guarantees that the VCG outcome arises as one (out of many) Nash equilibria, we show in Theorem 3.11 that, under a slightly different monotonicity condition, this equilibrium also admits a natural ascending implementation.

Example 3.8 (Interpreting Theorem 3.7) The two sufficient conditions in Theorem 3.7 are fairly weak, in that numerous payment rules satisfy them. For example, consider only linear payment rules $\mathbf{p}$, where $p_j(\mathbf{b})$ has the form $\sum_{\ell > j} \lambda_{j\ell} b_\ell$ for each j. We have already noted that if all of the λ's are nonnegative then the rule satisfies Definition 3.6. If they also satisfy $\lambda_{j,j+1} > 0$ for every $j \in [k-1]$ and $\sum_{\ell=h}^{k+1} \lambda_{j,\ell} \leq \sum_{\ell=h}^{k+1} \lambda_{j+1,\ell}$ for every $j \in [k-1]$ and $h \geq j+1$, then the rule is onto $Q(\alpha)$ for every α. To give one concrete example, the payment rule $p_1(\mathbf{b}) = b_2/2$ and $p_2(\mathbf{b}) = b_3$ is efficient, with equilibrium bids $b_1 = b_2 = 2q_1(\mathbf{v})$ and $b_3 = q_2(\mathbf{v}) \leq q_1(\mathbf{v})$.

Remark 3.9 (Weakening the Monotonicity Condition) By inspection of its proof, Theorem 3.7 continues to hold if the monotonicity condition in Definition 3.6 applies only to pairs of bid profiles that differ in a single unilateral deviation by a bidder to a higher slot. While these weakened sufficient conditions are similar to the necessary conditions in Theorem 3.4, even these are not always necessary — even certain linear rules with some *negative* coefficients $\lambda_{i,j}$ can be efficiency-inducing (Example A.2).

Remark 3.10 (Implementing Envy Free Outcomes) The proof of Theorem 3.7 immediately shows the following more general statement: for every anonymous upper-triangular monotone payment rule $\mathbf{p}$, every CTR vector α and valuation profile $\mathbf{v}$, and *every* vector $\mathbf{y}$ of slot prices that are envy-free with respect to $\mathbf{v}$ and α — VCG prices or otherwise — there is an equilibrium with the efficient allocation and the prices $\mathbf{y}$ if and only if there is a bid profile $\mathbf{b} \in S$ with $\mathbf{p}(\mathbf{b}) = \mathbf{y}$. This generalizes the fact that, for every α and $\mathbf{v}$, every envy-free price vector arises at an equilibrium of the GSP auction [7, 13].

3.3 Extension: An Ascending Implementation

An anonymous and upper-triangular payment rule is *strongly locally monotone* if the price of each slot is strictly increasing in the next bid.[10] This condition is slightly stronger than local monotonicity (Definition 3.3) and incomparable to monotonicity (Definition 3.6). We next sketch an argument that, if a payment rule satisfies this condition and is onto $Q(\alpha)$, then the VCG outcome in the corresponding sponsored search auction always has an ascending implementation similar to the one presented in [7].

We consider the Generalized English Auction of Edelman et al. [7], but also allow non-GSP payment rules $\mathbf{p}$. By definition, a strategy of an advertiser assigns the choice of dropping out or not for every history of the game, given that the advertiser has not already dropped out. We consider the following strategy for each advertiser i. If j slots remain unfilled and the previous bidders dropped out at times $b_{k+1}, \ldots, b_{j+1}$, then bidder i drops out — thereby receiving the jth slot at a price of $p_j(b_{j+1}, \ldots, b_{k+1})$ — at the time b_j equal to the supremum of all times $t \geq b_{j+1}$ for which

$$\alpha_j(v_i - p_j(b_{j+1}, \ldots, b_{k+1})) \leq \alpha_{j-1}(v_i - p_{j-1}(t, b_{j+1}, \ldots, b_{k+1})), \tag{2}$$

unless some other bidder drops out first. If no such times exist — because the left-hand side is bigger than the right-hand side for all $t \geq b_{j+1}$ — then bidder i drops out immediately with $b_j = b_{j+1}$.

Theorem 3.11 (Ascending Implementation) *Let α be a vector of CTRs. For every anonymous, upper-triangular, and strictly locally monotone payment rule that is onto $Q(\alpha)$, the strategies described above are an ex post equilibrium in which the allocation and payments coincide with the VCG outcome.*

[10] For example, every linear rule that is onto $Q(\alpha)$ is also strongly locally monotone — this follows from our proof of Proposition 4.2.

Proof (sketch): We first claim that if the players follow the suggested strategies, then the resulting allocation and payments coincide with the VCG outcome. Fix valuations $\mathbf{v}$. Since $\mathbf{p}$ is onto $Q(\alpha)$, there is a sorted bid vector $\mathbf{b}$ such that $\mathbf{p}(\mathbf{b}) = \mathbf{q}(\mathbf{v})$. By local indifference (Proposition 2.4(4)),

$$\alpha_j(v_j - p_j(b_{j+1}, \ldots, b_{k+1})) = \alpha_{j-1}(v_j - p_{j-1}(b_j, b_{j+1}, \ldots, b_{k+1})) \tag{3}$$

for every $j = 2, 3, \ldots, k+1$. Suppose we have proved inductively that the bidders with (sorted) valuations $v_{j+1}, \ldots, v_{k+1}$ drop out first, at the respective times $b_{j+1}, \ldots, b_{k+1}$. Equation (3) and strict local monotonicity imply that $b_j \geq b_{j+1}$ is the unique value of t that satisfies (2) with equality for bidder j — the left- and right-hand sides of (2) are independent of and strictly decreasing in t, respectively. Since the value of the largest t that satisfies (2) is strictly increasing in the valuation v_i, bidder j will be the next one to drop out, at the time b_j. Thus, the drop-out times are $\mathbf{b}$, resulting in the same allocation and prices as in the VCG mechanism.

We now prove that the suggested strategies form an equilibrium. Consider a bidder i, who would receive payoff $\alpha_i(v_i - q_i(\mathbf{v}))$ in the Generalized English Auction if it did not deviate. This is the same payoff bidder i would receive in the VCG mechanism if all bidders reported their true valuations. Consider a unilateral deviation by bidder i that causes it to be assigned the slot $j \neq i$, and let S denote the bidders assigned to the slots $j+1, \ldots, k+1$ after the deviation. The strategies are such that, after bidder i's deviation, the bidders of S behave (and drop out) exactly as if all other bidders have higher valuations and are playing the suggested strategies. Thus, the previous paragraph implies that bidder i receives slot j at the VCG price $q_j(\mathbf{v}_S)$ of a valuation profile in which the lowest $k - j + 1$ valuations are $\mathbf{v}_S$ (which is well defined by (1)). This payoff $\alpha_j(v_i - q_j(\mathbf{v}_S))$ is the same that bidder i would receive in the VCG mechanism if it misreported its valuation as larger than all bidders of S and smaller than all other bidders. Since the VCG mechanism is truthful, bidder i can only decrease its payoff in the Generalized English Auction with this deviation. ■

3.4 Application: The VCG Mechanism with Wrong Click-Through Rates

For every vector α of CTRs, the corresponding VCG mechanism is truthful. However, since the payment rule of the VCG mechanism depends on α, *wrong estimates of the CTRs destroy truthfulness*. But perhaps there is still an efficient equilibrium, even when the wrong CTRs are used? We can use our necessary and sufficient conditions (Theorems 3.4 and 3.7) to shed light on this question.

We focus on the special case in which the CTRs form a geometric series. It will be evident from the proofs that more general results are possible, but this restriction permits a crisp characterization of exactly when the VCG mechanism with incorrect CTRs gives an efficient outcome at equilibrium. Precisely, we assume there there is a constant *CTR ratio* γ such that $\alpha_{j+1}/\alpha_j = \gamma$ for each $j \in [k-1]$. We now prove that, when there are at least 3 slots, the VCG mechanism with incorrect geometric CTRs is efficiency inducing *if and only if* the estimated CTR ratio γ' is an *underestimate* of the correct value γ.[11]

Proposition 3.12 (VCG with Incorrect CTRs) Assume that $k \geq 3$ and that the true CTRs are a geometric series with ratio $\gamma < 1$. The VCG mechanism which uses CTRs as a geometric series with ratio $\gamma' \leq \gamma$ is efficiency inducing, while the mechanism with estimated CTR ratio $\gamma' > \gamma$ is *not* efficiency inducing.

Proof: Consider the VCG prices with geometric series of CTRs with ratio γ. Fix a valuation profile $\mathbf{v}$. By Proposition 2.4(1) it holds that $q_k = v_{k+1}$ and that $q_j = \gamma q_{j+1} + (1-\gamma)v_{j+1}$ for

[11] With only two slots, it turns out that *every* incorrect estimate is efficiency-inducing — we omit the simple proof.

$j \in [k-1]$. Let $\mathbf{p}$ be the payment rule defined by VCG with ratio γ'. For bids $\mathbf{b}$ it holds that $p_k = b_{k+1}$ and that $p_j = \gamma' p_{j+1} + (1-\gamma')b_{j+1}$ for $j \in [k-1]$.

By Theorem 3.7, to prove that $\mathbf{p}$ is efficiency inducing it is sufficient to show that it is monotone and onto $Q(\alpha)$. Monotonicity is obvious from (1). We are left to show that if $\gamma' \leq \gamma$ then $\mathbf{p}$ is onto $Q(\alpha)$.

The unique bid profile $\mathbf{b}$ that satisfies meets the condition $p_j(\mathbf{b}) = q_j(\mathbf{v})$ for all $j \in [k]$ satisfies $b_{k+1} = q_k = v_{k+1}$ and $b_{j+1} = \frac{q_j - \gamma' q_{j+1}}{1-\gamma'}$ for $j \in [k-1]$. To prove that this profile is indeed feasible we need to show that it is sorted. This holds if and only if $q_j - \gamma' q_{j+1} \geq q_{j+1} - \gamma' q_{j+2}$ for every j. We substitute $q_j = \gamma q_{j+1} + (1-\gamma)v_{j+1}$ and $q_{j+2} = \frac{q_{j+1} - (1-\gamma)v_{j+2}}{\gamma}$ and simplify to get that $b_{j+1} \geq b_{j+2}$ holds if and only if

$$\frac{\gamma'}{\gamma}(v_{j+2} - q_{j+1}) \leq v_{j+1} - q_{j+1}.$$

If $v_{j+2} = q_{j+1}$ this always holds as $v_{j+1} \geq q_{j+1}$. Otherwise we can divide by $v_{j+2} - q_{j+1}$ and derive the equivalent statement

$$\frac{\gamma'}{\gamma} \leq 1 + \frac{v_{j+1} - v_{j+2}}{v_{j+2} - q_{j+1}}.$$

This holds when $\gamma' \leq \gamma$ since the left-hand side is at most 1 and the right-hand side is at least 1 (since $v_{j+1} \geq v_{j+2}$ and $v_{j+2} \geq q_{j+1}$).

For the converse, consider the VCG payment rule $\mathbf{p}$ corresponding to a CTR ratio γ' that is strictly larger than the actual CTR ratio γ. To prove that $\mathbf{p}$ is not efficient, we only need to show $\mathbf{p}$ is not onto $Q(\alpha)$. Consider a valuation profile $\mathbf{v}$ in which $v_{k+1} = 1$ and $v_k = v_{k-1} = \frac{2-\gamma}{1-\gamma} > 1$; the larger valuations can be set arbitrarily (subject to monotonicity). For a geometric series of CTRs it holds that $q_j = \gamma q_{j+1} + (1-\gamma)v_{j+1}$. For these valuations it holds that $q_k = 1$, $q_{k-1} = \gamma \cdot 1 + (1-\gamma)\frac{2-\gamma}{1-\gamma} = 2$ and $q_{k-2} = \gamma \cdot 2 + (1-\gamma)\frac{2-\gamma}{1-\gamma} = 2 + \gamma$.

Now there is a unique bid vector candidate $\mathbf{b}$ (up to the irrelevant choice of $b_1 \geq b_2$) that can possibly be a bid vector that corresponds to $\mathbf{v}$, and $\mathbf{b}$ can be computed easily. Clearly $b_{k+1} = v_{k+1} = 1$. As $\mathbf{p}$ is the VCG payment rule with ratio γ' it holds that $p_j = \gamma' p_{j+1} + (1-\gamma')b_{j+1}$ or equivalently $b_{j+1} = \frac{p_j - \gamma' p_{j+1}}{1-\gamma'}$. This implies that $b_k = \frac{q_{k-1} - \gamma' q_k}{1-\gamma'} = \frac{2-\gamma'}{1-\gamma'}$. It also implies that $b_{k-1} = \frac{q_{k-2} - \gamma' q_{k-1}}{1-\gamma'} = \frac{2+\gamma-2\gamma'}{1-\gamma'} = \frac{2-\gamma'}{1-\gamma'} - \frac{\gamma'-\gamma}{1-\gamma'} < b_k$ as $\gamma' - \gamma > 0$. This proves that $\mathbf{p}$ is not onto $Q(\alpha)$. ■

4 The Simplicity of the GSP Auction

This section provides one way to formulate "payment rule simplicity" and thereby formalize the intuitive "minimal complexity" of the GSP auction. To get started, let $\mathcal{P}(\alpha)$ denote the payment rules that are anonymous, upper-triangular, and efficiency-inducing with respect to the CTRs α. For $\mathbf{p} \in \mathcal{P}(\alpha)$, let $\chi^{\mathbf{p}}_{i,j}$ denote 0 if the slot price $p_i(\mathbf{b})$ is independent of the bid b_j and 1 otherwise. By upper triangularity, $\chi^{\mathbf{p}}_{i,j} = 0$ whenever $j \leq i$. We start with the simple observation that, for every α, the GSP payment rule minimizes the total number of dependencies over all payment rules in $\mathcal{P}(\alpha)$.

Proposition 4.1 For every α and $\mathbf{p} \in \mathcal{P}(\alpha)$, $\sum_{i,j} \chi^{\mathbf{p}}_{i,j} \geq k = \sum_{i,j} \chi^{GSP}_{i,j}$.

Proof: Suppose $\mathbf{p}$ has less than k dependencies in all. Then for some j, p_j is a constant function. Then $\mathbf{p}$ is not onto $Q(\alpha)$ and does not belong to $\mathcal{P}(\alpha)$. ■

To obtain much stronger minimality statements, we restrict the class of payment rules in one of two ways. First, let $\mathcal{L}(\alpha)$ denote the set of all *linear* payment rules in $\mathcal{P}(\alpha)$, where a linear rule has the form $p_j(b_{j+1}, \ldots, b_{k+1}) = \sum_{\ell>j} \lambda_{j\ell} b_\ell$ for each j. We next show that, for every α, every rule in $\mathcal{L}(\alpha)$ has dependencies that are a superset of those in the GSP auction.

Proposition 4.2 For every α and every $\mathbf{p} \in \mathcal{L}(\alpha)$, $\chi^{\mathbf{p}}_{i,j} \geq \chi^{GSP}_{i,j}$ for every $i, j \in [k]$. That is, the GSP auction has the minimal set of dependencies over all payment rules in $\mathcal{L}(\alpha)$.

Proof: Consider CTRs α and a linear rule $\mathbf{p} \in \mathcal{L}(\alpha)$, with $p_j(b_{j+1}, \ldots, b_{k+1}) = \sum_{\ell>j} \lambda_{j\ell} b_\ell$ for each j. We prove the proposition by showing that $\lambda_{i,i+1} > 0$ for every $i \in [k]$.

We proceed by backward induction on i. Consider a generic sorted valuation profile $\mathbf{v}$ with $v_{k+1} > 0$. Since $\mathbf{p}$ is efficiency-inducing, there is a sorted bid profile $\mathbf{b} \in S$ with $\mathbf{p}(\mathbf{b}) = \mathbf{q}(\mathbf{v})$. For the base case, we have $q_k(\mathbf{v}) = v_{k+1}$ and $p_k(\mathbf{b}) = \lambda_{k,k+1} b_{k+1}$. Since $q_k(\mathbf{v}) = p_k(\mathbf{b})$ and $v_{k+1} > 0$ we have that $\lambda_{k,k+1}$ is non-zero — and positive, since $b_{k+1} \geq 0$ — and the equilibrium bid b_{k+1} is uniquely defined (as $v_{k+1}/\lambda_{k,k+1}$, independent of the higher valuations). For a general slot $i < k$, fix positive and monotone values for the valuations $v_{i+2}, \ldots, v_{k+1}$. By the inductive hypothesis, this uniquely fixes the corresponding equilibrium bids $b_{i+2}, \ldots, b_{k+1}$, independent of the higher valuations $v_1, \ldots, v_{i+1}$. Since $\mathbf{p}$ is efficiency-inducing (and hence onto $Q(\alpha)$), for each value of $v_{i+1} \geq v_{i+2}$ we must be able to choose a bid $b_{i+1} \geq b_{i+2}$ satisfying $p_j(b_{j+1}, \ldots, b_{k+1}) = q_j(\mathbf{v})$. Since $q_i(\mathbf{v})$ is a strictly increasing function of v_{i+1} with $v_{i+2}, \ldots, v_{k+1}$ fixed, independent of $v_1, \ldots, v_i$ (recall (1)), the linearity of $\mathbf{p}$ implies that such bid choices are only possible if $\lambda_{i,i+1} > 0$, and for each choice of v_{i+1} there is only one candidate choice of b_{i+1}. This completes the inductive step and the proof. ■

In our second class of payment rules, equilibrium bids are independent of higher valuations.

Definition 4.3 A rule $P \in \mathcal{P}$ has *unique equilibrium bids that are independent of higher valuations* if:

- For every vector of values $\mathbf{v}$ there is a unique corresponding vector of bids $\mathbf{b}$.[12]
- Consider two valuation profiles $\mathbf{v}$ and $\mathbf{v}'$ with corresponding bid profiles $\mathbf{b}$ and $\mathbf{b}'$, respectively. For every $i \in [k]$, if $v_j = v'_j$ for all $j > i$ then $b_j = b'_j$ for all $j > i$.

A byproduct of the proof of Proposition 4.2 is that every linear payment rule in $\mathcal{L}$ satisfies Definition 4.3; it is easy to see that the converse fails and hence the next statement is strictly more general.

Proposition 4.4 For every payment rule $\mathbf{p} \in \mathcal{P}$ that has unique equilibrium bids that are independent of higher valuations, $\chi^{\mathbf{p}}_{i,j} \geq \chi^{GSP}_{i,j}$ for every $i, j \in [k]$. That is, the GSP auction has the minimal set of dependencies over all such rules.

Proof: We need to show that $\chi^{\mathbf{p}}_{i,i+1} = 1$ for every $i \in [k]$. Consider two valuation profiles $\mathbf{v}$ and $\mathbf{v}'$ with corresponding bid profiles $\mathbf{b}$ and $\mathbf{b}'$, respectively. Pick $i \in [k]$ and valuations that are identical after index $i + 1$. Also pick valuations such that $v_{i+1} \neq v'_{i+1}$. This implies that $q_i(\mathbf{v}) \neq q_i(\mathbf{v}')$.

Since $\mathbf{p}$ has unique equilibrium bids that are independent of higher values, $b_j = b'_j$ for all $j > i + 1$. We also know that $p_i(\mathbf{b}) = q_i(\mathbf{v}) \neq q_i(\mathbf{v}') = p_i(\mathbf{b}')$. Since $\mathbf{p}$ is upper-triangular and all bids after index $i + 1$ are identical, yet the $(i + 1)$th price is different for $\mathbf{v}$ and $\mathbf{v}'$, p_i must depend on b_{i+1}. ■

[12] Clearly for an upper-triangular rule b_1 can vary with no affect on the allocation and payments. By "unique" we mean that the vector $(b_2, ..., b_{k+1})$ is unique and b_1 can be any value such that $b_1 \geq b_2$.

5 Future Directions: Right Answers from Wrong Mechanisms

While this paper studies only sponsored search auctions, the results in Section 3 suggest a broader research agenda. Consider an arbitrary welfare maximization problem. In principle, every such problem can be solved by the VCG mechanism. Moreover, the VCG mechanism is essentially the unique mechanism that solves the welfare maximization problem in dominant strategies.

The VCG mechanism is not always used in practice, however. For example, Ausubel and Milgrom [5] state that "practical applications of Vickrey's design are rare at best" and list several reasons why the VCG mechanism is inappropriate for combinatorial auctions.[13] These critiques motivate considering non-truthful mechanisms for welfare maximization problems. One desirable property of such a "wrong" (i.e., non-VCG) mechanism is that the "right" (VCG) outcome arises at a natural equilibrium — this is precisely the property we study in Section 3. An interesting research direction is to study more general allocation problems from this perspective. For example, are there simple and novel payment rules that, when coupled with the allocation rule that maximizes the welfare with respect to the submitted bids, guarantees efficient equilibria in general matching markets (e.g. [12])?

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[13] For the simpler problem of sponsored search, it is unclear if the VCG mechanism would work well in practice. Published accounts of the evolution of the first sponsored search auctions (e.g. [7]) suggest that the auction designers were unaware of the VCG mechanism, and thus did not explicitly reject it.

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A Additional Examples

A.1 The Necessary Conditions in Theorem 3.4 Are Not Always Sufficient

Our necessary conditions for a rule to be efficiency inducing are not sufficient as agents might deviate by bidding up more than one slot.

Example A.1 We show that there exists an anonymous and upper-triangular linear payment rule $\mathbf{p}$ that is onto $Q(\alpha)$ and locally monotone, yet not inducing. We present a rule for $k = 3$. The prices are $p_1(\mathbf{b}) = b_2 - b_3$, $p_2(\mathbf{b}) = b_3 - 2b_4$ and $p_3(\mathbf{b}) = b_4$. This rule is clearly locally monotone. To see that it is onto $Q(\alpha)$ for every α we observe that for every $\mathbf{v}$ any vector $\mathbf{b}$ that corresponds to $\mathbf{v}$ has the same bids (b_2, b_3, b_4). Any corresponding vector $\mathbf{b}$ must satisfy $b_4 = q_3$, $b_3 = q_2 + 2q_3$ and $b_2 = q_1 + b_3 = q_1 + q_2 + 2q_3$. We note that $b_2 \geq b_3 \geq b_4$, so $\mathbf{b} \in S$ and hence the rule in indeed onto $Q(\alpha)$ for every α.

Finally we show that this rule is not efficiency inducing. Consider any decreasing vector of CTRs. For the vector of values $\mathbf{v} = (1, 1, 1, 1)$ the vector of VCG prices is $\mathbf{q} = (1, 1, 1)$ and the utility of all agents is 0. Any corresponding bid vector $\mathbf{b}$ is of the form $(b_1, 4, 3, 1)$. If $b_1 = 4$ then if the agent with the smallest bid raises his bid to bid for the first slot (say by bidding 100), getting positive utility as the price is now $b_1 - 4 = 0$. If on the other hand $b_1 > 4$ the agent with the smallest bid can now deviate to bid between 4 and b_1, getting the second slot and paying $4 - 6 = -2$ (getting paid 2), again ending up with positive utility. We conclude that this rule is not efficiency inducing.

We note that the above example has negative coefficients $\lambda_{i,j}$. This is necessary for a linear rule that is onto Q and not efficiency inducing, as if all coefficients are non-negative the rule satisfies monotonicity which is sufficient for efficiency (see Section 3.2).

A.2 The Sufficient Conditions in Theorem 3.7 Are Not Always Necessary

We next give an example showing that monotonicity is not necessary for $\mathbf{p}$ to be efficiency inducing. The reason is that for the utility not to increase when an agent deviates upwards it is sufficient that the price will not drop but not necessary. It might be that the price drops but not enough to increase the utility with respect to the original slot.

Example A.2 There exists an anonymous linear payment rule $\mathbf{p}$ that is efficiency inducing for every α (thus onto $Q(\alpha)$ and locally monotone) yet is not monotone. That is, the rule has a negative coefficient ($\lambda_{i,j} < 0$ for some i, j), so there is a slot price that is *decreasing* in one of the lower bids. We present a rule with 2 slots ($k = 2$). The prices are $p_1(\mathbf{b}) = (1 + \beta)b_2 - \beta b_3$, $p_2(\mathbf{b}) = b_3$, for any $\beta > 0$. This rule is clearly locally monotone yet not monotone. Assume $1 = \alpha_1 > \alpha_2 > 0$. The VCG prices are $q_1(\mathbf{v}) = (1 - \alpha_2)v_2 + \alpha_2 v_3$ and $q_2(\mathbf{v}) = v_3$. The bid vector $\mathbf{b}$ defined as $b_1 = b_2 = \frac{1-\alpha_2}{1+\beta} v_2 + \frac{\beta+\alpha_2}{1+\beta} v_3$ and $b_3 = v_3$ corresponds to $\mathbf{v}$. Note that the bids are non-increasing ($b_2 \geq b_3$) as b_2 is a convex combination of $b_3 = v_3$ and $v_2 \geq v_3$. We conclude that $\mathbf{p}$ is onto Q.

To complete the proof we show that $\mathbf{b}$ forms an equilibrium. The only deviation that can possibly be beneficial is for the agent with the lowest value v_3 to bid for the first slot. But in this case he will need to pay b_2 per click and as $b_2 \geq v_3$ he will end up with non-positive utility.