Intrinsic Robustness of the Price of Anarchy

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Key Points

• main definition: a “canonical way” to bound the price of anarchy (for pure equilibria)

• theorem 1: every POA bound proved “canonically” is automatically far stronger
  - e.g., even applies “out-of-equilibrium”, assuming no-regret play

• theorem 2: canonical method provably yields optimal bounds in fundamental cases
The Price of Anarchy

Network w/2 players:

\[ s \xrightarrow{2x} 0 \xrightarrow{12} t \]

\[ s \xrightarrow{5} 0 \xrightarrow{5x} t \]
The Price of Anarchy

Nash Equilibrium:

\[\text{cost} = 14 + 14 = 28\]
The Price of Anarchy

Nash Equilibrium:

\[
\begin{align*}
\text{cost} &= 14+14 = 28
\end{align*}
\]

To Minimize Cost:

\[
\begin{align*}
\text{cost} &= 14+10 = 24
\end{align*}
\]

\textit{Price of anarchy} = \frac{28}{24} = \frac{7}{6}.

- if multiple equilibria exist, look at the \textit{worst} one
Congestion Games [Rosenthal 73]

Model: ground set $E$ (e.g. network links)
- players $N$, strategy sets = subsets of $2^E$
- cost function $c_e$ per $e \in E$
  - $c_e(x_e) =$ per-player cost
  - $i$’s cost $C_i(s) : \sum_{e \in A} c_e(x_e)$

- social objective = sum of players’ costs
  - can write $\text{cost}(s) := \sum_i C_i(s) = \sum_e c_e(x_e)x_e$
Abstract Setup

• n players, each picks a strategy $s_i$
• player $i$ incurs a cost $C_i(s)$

Important Assumption: objective function is
$\text{cost}(s) := \sum_i C_i(s)$

Key Definition: A game is $(\lambda, \mu)$-smooth if, for every pair $s, s^*$ outcomes ($\lambda > 0; \mu < 1$):

$$\sum_i C_i(s^*_i, s_{-i}) \leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s) \quad [(*)]$$
Smooth => POA Bound

Next: “canonical” way to upper bound POA (via a smoothness argument).
• notation: \( s \) = a Nash eq; \( s^* \) = optimal

Assuming \((\lambda, \mu)\)-smooth:

\[
\text{cost}(s) = \sum_i C_i(s) \quad \text{[defn of cost]}
\leq \sum_i C_i(s^*_i, s_{-i}) \quad \text{[s a Nash eq]}
\leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s) \quad \text{[(*)]}
\]

Then: POA (of pure Nash eq) \( \leq \lambda/(1-\mu) \).
Why Is Smoothness Stronger?

Key point: to derive POA bound, only needed

\[ \Sigma_i C_i(s^*_i, s_{-i}) \leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s) \quad [(\ast)] \]

to hold in special case where \( s = \) a Nash eq and \( s^* = \) optimal.

Smoothness: requires \( (*) \) for every pair \( s, s^* \) outcomes.

- even if \( s \) is not a pure Nash equilibrium
Example Application

**Definition**: a sequence $s^1, s^2, ..., s^T$ of outcomes is *no-regret* if:

- for each player $i$, each fixed action $q_i$:
  - average cost player $i$ incurs over sequence no worse than playing action $q_i$ every time
  - simple hedging strategies can be used by players to enforce this (for suff large $T$)

**Theorem**: in a $(\lambda, \mu)$-smooth game, average cost of every no-regret sequence at most $(\lambda/(1-\mu) + o(1)) \times$ cost of optimal outcome.
Smooth $\Rightarrow$ POTA Bound

- notation: $s^1, s^2, \ldots, s^T = \text{no regret}; \quad s^* = \text{optimal}$

Assuming $(\Lambda, \mu)$-smooth:

$$\sum_t \text{cost}(s^t) = \sum_t \sum_i C_i(s^t) \quad \text{[defn of cost]}$$
Smooth $\Rightarrow$ POTA Bound

- notation: $s^1, s^2, \ldots, s^T = \text{no regret}; s^* = \text{optimal}$

Assuming $(\lambda, \mu)$-smooth:

$$\sum_t \text{cost}(s^t) = \sum_t \sum_i C_i(s^t) \quad \text{[defn of cost]}$$

$$= \sum_t \sum_i [C_i(s^*_i, s^*_{-i}) + \Delta_{i,t}] \quad \text{[} \Delta_{i,t} = C_i(s^t) - C_i(s^*_i, s^*_{-i}) \text{]}$$
Smooth $\Rightarrow$ POTA Bound

- notation: $s_1, s_2, \ldots, s^T = \text{no regret}; s^* = \text{optimal}

Assuming $(\lambda, \mu)$-smooth:

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\sum_t \text{cost}(s^t) = \sum_t \sum_i C_i(s^t) \quad \text{[defn of cost]}
$$

$$
= \sum_t \sum_i [C_i(s^*_{i}, s^*_{-i}) + \Delta_{i,t}] \quad \text{[} \Delta_{i,t} := C_i(s^t) - C_i(s^*_{i}, s^*_{-i}) \text{]} 
$$

$$
\leq \sum_t [\lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s^t)] + \sum_i \sum_t \Delta_{i,t} \quad \text{[(*)]}
$$
Smooth $\Rightarrow$ POTA Bound

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Assuming $(\lambda, \mu)$-smooth:

$\sum_t \text{cost}(s^t) = \sum_t \sum_i C_i(s^t)$ \hspace{1cm} \text{[defn of cost]}

$= \sum_t \sum_i [C_i(s^*_i, s^t_{-i}) + \Delta_{i,t}]$ \hspace{1cm} \text{[$\Delta_{i,t} := C_i(s^t) - C_i(s^*_i, s^t_{-i})$]}

$\leq \sum_t [\lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s^t)] + \sum_i \sum_t \Delta_{i,t}$ \hspace{1cm} \text{[(*)]}

No regret: $(\sum_t \Delta_{i,t})/T = o(1)$ for each $i$.

To finish proof: divide through by $T$. 
Why Important?

• bound on “price of total anarchy” implies bound of inefficiency of mixed + correlated equilibria

• bound applies even to sequences that don’t converge in any sense
  • no regret much weaker than reaching equilibrium
  • [Blum/Even-Dar/Ligett PODC 06], [Blum/Hajiaghayi/Ligett/Roth STOC 08]
Further Applications

Theorem: in a $(\lambda, \mu)$-smooth game, everything in these sets costs (essentially) $\lambda / (1-\mu) \times \text{OPT}$.
Some Smoothness Bounds

Examples: selfish routing, linear cost fns.

• every nonatomic game is (1,1/4)-smooth
  - implicit in [Roughgarden/Tardos 00]
  - less implicit in [Correa/Schulz/Stier Moses 05]
  - implies bound of 4/3 (tight even for pure eq)

• every atomic game is (5/3,1/3)-smooth
  - follows directly from analysis in
    [Awerbuch/Azar/Epstein 05],
    [Christodoulou/Koutsoupias 05]
  - implies bound of 5/2 (tight even for pure eq)
Some Smoothness Bounds

Claim: $(5/3,1/3)$-smoothness in atomic, affine case

- [Christodoulou/Koutsoupias 05]: for all integers $y,z$:
  \[ y(z+1) \leq (5/3)y^2 + (1/3)z^2 \]
Some Smoothness Bounds

Claim: \((5/3,1/3)\)-smoothness in atomic, affine case

- [Christodoulou/Koutsoupias 05]: for all integers \(y,z\):
  \[y(z+1) \leq (5/3)y^2 + (1/3)z^2\]

- so: \(ay(z+1) + by \leq (5/3)[ay^2 + by] + (1/3)[az^2 + bz]\)
  - for all integers \(y,z\) and \(a,b \geq 0\)
Some Smoothness Bounds

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- so: \(\sum_e [a_e(x_e+1) + b_e)x_e^*] \leq (5/3) \sum_e [(a_e x_e^* + b_e)x_e^*] + (1/3) \sum_e [(a_e x_e + b_e)x_e]\)
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- so: \(\sum_i C_i(s^*_i,s_{-i}) \leq (5/3) \cdot \text{cost}(s^*) + (1/3) \cdot \text{cost}(s)\)
So: in every \((\lambda, \mu)\)-smooth game with a sum objective, inefficiency of outcomes in the various sets looks like:
Tight Game Classes

For the tightest choice of $\lambda, \mu$, the optimal outcome is given by $\lambda/(1-\mu)$.

- Pure correlated Nash equilibrium
- Mixed Nash
- No regret sequence

- Optimal outcome
- $1$
Tight Game Classes

**Theorem:** for every set $C$, congestion games with cost functions restricted to $C$ are **tight**.

$$\text{maximum [pure POA]} = \text{minimum } [\lambda/(1-\mu)]$$

**congestion games w/cost functions in } C \quad (\lambda, \mu): \text{ all such games are (}\lambda, \mu\text{-smooth}$$

- optimal outcome
- pure correlated Nash equilibrium
- mixed Nash
- no regret sequence

1 \rightarrow \lambda/(1-\mu) \quad \text{for tightest choice of } \lambda, \mu
First Main Proof Step

Step 1: characterize optimal smoothness parameters $\lambda, \mu$ as vertex of a 2-D polyhedron

- intersection of halfplanes of the form
  \[ yc(z+1) \leq \lambda \cdot c(y)y + \mu \cdot c(z)z \]
  for all integers $y, z$ and cost fns $c \in C$
Second Main Proof Step

Step 2: exhibit example with POA = \lambda/(1-\mu)

- use two “parallel cycles” (one per tight halfplane)
- each player has “short” and “long” strategy
  - each strategy uses resources of both cycles

• OPT = all use short strategies;
• worst Nash = all use long strategies
Corollaries

Corollary 1: first characterization of “universal worst-case congestion games” in the atomic case.
• analog of “Pigou-like (2-node, 2-link) networks are the worst” in nonatomic case [Roughgarden 03]
• here: “2 parallel cycles always suffice”
  – and are generally necessary for minimal worst-case examples

Corollary 2: first (tight) POA bounds for (atomic) congestion games with general cost functions.
• previous exact bounds for polynomials +w/nonnegative coefficients: [Aland et al 06], [Olver 06]
Take-Home Points

• the most common way of proving POA bounds automatically yields a much more robust guarantee

• and this technique often gives tight bounds

• future work: characterize tight game classes (where smoothness gives optimal POA bounds, even for pure NE)