Near-Optimal Equilibria

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Example Theorem: [Syrgkanis/Tardos 13] (improving [Hassidim/Kaplan/Nisan/Mansour 11]) Suppose $m$ items are sold simultaneously via first-price single-item auctions:

- for every product distribution over submodular bidder valuations (independent, not necessarily identical), and
- for every (mixed) Bayes-Nash equilibrium, expected welfare of the equilibrium is within 63% of the maximum possible.

(matches best-possible algorithms!)
Outline

1. *Smooth Games, Extension Theorems, and Robust POA Bounds*
2. Smooth Mechanisms and Bayes-Nash POA Bounds
3. Reducing Complex Mechanisms to Simple Mechanisms Using Composition Theorems
4. Complexity-Based POA Lower Bounds
THE PRICE OF ANARCHY

IS IT OKAY TO BE SELFISH?
The Price of Anarchy

Network with 2 players:
The Price of Anarchy

Nash Equilibrium:

\[ \text{cost} = 14 + 14 = 28 \]
The Price of Anarchy

Nash Equilibrium:

To Minimize Cost:

\[
\begin{align*}
\text{cost} &= 14 + 14 = 28 \\
\text{cost} &= 14 + 10 = 24
\end{align*}
\]

\textit{Price of anarchy (POA)} = \frac{28}{24} = \frac{7}{6}.

- if multiple equilibria exist, look at the \textit{worst} one
  - [Koutsoupias/Papadimitriou 99]
What Do POA Bounds Look Like?

- n players, each picks a strategy $s_i$
- player $i$ incurs a cost $C_i(s)$

Objective function: $\text{cost}(s) := \sum_i C_i(s)$
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To Bound POA: (let $s =$a Nash eq; $s^*$ =optimal)

$$\text{cost}(s) = \sum_i C_i(s) \quad [\text{defn of cost}]$$
What Do POA Bounds Look Like?

- n players, each picks a strategy $s_i$
- player $i$ incurs a cost $C_i(s)$

Objective function: $\text{cost}(s) := \Sigma_i C_i(s)$

To Bound POA: (let $s$ =a Nash eq; $s^*$ =optimal)

$$\text{cost}(s) = \Sigma_i C_i(s) \quad [\text{defn of cost}]$$
$$\leq \Sigma_i C_i(s^*_i, s_{-i}) \quad [s \text{ a Nash eq}]$$

“baseline” strategies
What Do POA Bounds Look Like?

Suppose: we prove that (for \( \lambda > 0; \mu < 1 \))

\[
\sum_i C_i(s_i^*, s_{-i}) \leq \lambda \cdot cost(s^*) + \mu \cdot cost(s) \quad [\text{(*)}]
\]
What Do POA Bounds Look Like?

Suppose: we prove that (for $\lambda > 0; \mu < 1$)

$$\sum_i C_i(s^*_i, s_{-i}) \leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s) \quad [(*')]$$

Implies: $\text{cost}(s) \leq \sum_i C_i(s^*_i, s_{-i}) \quad [s \text{ a Nash eq}]$

$$\leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s) \quad [\text{by (*)}]$$

So: POA (of pure Nash equilibria) $\leq \lambda/(1-\mu)$. 
Canonical Example

Claim [Christodoulou/Koutsoupias 05] (see also [Awerbuch/Azar Epstein 05]) worst-case POA in routing games with affine cost functions is $5/2$.

- for all integers $y,z$: $y(z+1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2$
- so: $ay(z+1) + by \leq \frac{5}{3}[ay^2 + by] + \frac{1}{3}[az^2 + bz]$
  - for all integers $y,z$ and $a,b \geq 0$
- so: $\sum_e [a_e(x_e+1) + b_e)x_e^*] \leq \frac{5}{3} \sum_e [(a_e x_e^* + b_e)x_e^*]$
  + $\frac{1}{3} \sum_e [(a_e x_e + b_e)x_e]$
- so: $\sum_i C_i(s_i^*,s_{-i}) \leq \frac{5}{3} \cdot \text{cost}(s^*) + \frac{1}{3} \cdot \text{cost}(s)$
Smooth Games

Definition: [Roughgarden 09] A game is \((\lambda, \mu)\)-smooth w.r.t. baselines \(s^*\) if, for every outcome \(s\) (\(\lambda > 0; \mu < 1\)):

\[
\sum_i C_i(s^*, s_{-i}) \leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s) \quad [(*)]
\]
Smooth Games

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\]

Implies: \(\text{cost}(s) \leq \sum_i C_i(s^*_i, s_{-i}) \quad [s \text{ a Nash eq}]
\]

\[
\leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s) \quad \text{[by (*)]}
\]

So: if \((\lambda, \mu)\)-smooth w.r.t. optimal outcome, then POA (of pure Nash equilibria) is at most \(\lambda/(1-\mu)\).

(using (*) only in the special case where \(s = \text{equilibrium}\))
POA Bounds Without Convergence

Meaning of a POA bound: *if* the game is at an equilibrium, *then* outcome is near-optimal.

**Problem:** what if can’t reach an equilibrium?
- non-existence (pure Nash equilibria)
- intractability (mixed Nash equilibria)

[Daskalakis/Goldberg/Papadimitriou 06], [Chen/Deng/Teng 06], [Etessami/Yannakakis 07]

**Worry:** fail to converge, POA bound won’t apply.
Learnable Equilibria

Fact: simple strategies converge quickly to more permissive equilibrium sets.

- correlated equilibria: [Foster/Vohra 97], [Fudenberg/Levine 99], [Hart/Mas-Colell 00], ...
- coarse/weak correlated equilibria (of [Moulin/Vial 78]): [Hannan 57], [Littlestone/Warmuth 94], ...

Question: are there good “robust” POA bounds, which hold more generally for such “easily learned” equilibria?

[Mirrokni/Vetta 04], [Goemans/Mirrokni/Vetta 05], [Awerbuch/Azar/Epstein/Mirrokni/Skopalk 08], [Christodoulou/Koutsoupias 05], [Blum/Even-Dar/Ligett 06], [Blum/Hajiaghayi/Ligett/Roth 08]
A Hierarchy of Equilibria

Recall: POA determined by \textit{worst} equilibrium (only increases with the equilibrium set).
An Out-of-Equilibrium Bound

**Theorem:** [Roughgarden 09] if game is $(\lambda, \mu)$-smooth w.r.t. an optimal outcome, then the average cost of every no-regret sequence is at most

$$\frac{\lambda}{1-\mu} \cdot \text{cost of optimal outcome}.$$  

(the same bound as for pure Nash equilibria!)
No-Regret Sequences

Definition: a sequence $s_1, s_2, ..., s_T$ of outcomes of a game is no-regret if:

• for each $i$, each (time-invariant) deviation $q_i$:

$$\frac{1}{T} \sum_t C_i(s^t) \leq \frac{1}{T} \sum_t C_i(q_i, s_{-i}^t) [\pm o(1)]$$

(will ignore the “$o(1)$” term)
Smooth => No-Regret Bound

- notation: $s^1,s^2,...,s^T =$ no regret; $s^*$ = optimal

Assuming $(\lambda, \mu)$-smooth:

$$\sum_t \text{cost}(s^t) = \sum_t \sum_i C_i(s^t) \quad \text{[defn of cost]}$$
Smooth $\Rightarrow$ No-Regret Bound

- notation: $s^1, s^2, \ldots, s^T = \text{no regret}; s^* = \text{optimal}$

Assuming $(\lambda, \mu)$-smooth:

$$\Sigma_t \text{cost}(s^t) = \Sigma_t \Sigma_i C_i(s^t)$$

[defn of cost]

$$= \Sigma_t \Sigma_i [C_i(s^*_i, s^t_{-i}) + \Delta_{i,t}]$$

[\(\Delta_{i,t} := C_i(s^t) - C_i(s^*_i, s^t_{-i})\)]
Smooth => No-Regret Bound

- notation: \( s^1, s^2, \ldots, s^T \) = no regret; \( s^* \) = optimal

**Assuming \((\lambda, \mu)\)-smooth:**

\[
\sum_t \text{cost}(s^t) = \sum_t \sum_i C_i(s^t) \quad \text{[defn of cost]}
\]

\[
= \sum_t \sum_i [C_i(s^*_i, s^t_{-i}) + \Delta_{i,t}] \quad [\Delta_{i,t} := C_i(s^t) - C_i(s^*_i, s^t_{-i})]
\]

\[
\leq \sum_t [\lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s^t)] + \sum_i \sum_t \Delta_{i,t} \quad \text{[smooth]}
\]
Smooth => No-Regret Bound

- notation: $s^1, s^2, ..., s^T = \text{no regret}; \ s^* = \text{optimal}$

Assuming $(\lambda, \mu)$-smooth:

$$\Sigma_t \ cost(s^t) = \Sigma_t \Sigma_i C_i(s^t)$$  \hspace{1cm} \text{[defn of cost]}

$$= \Sigma_t \Sigma_i \left[ C_i(s^*, s^t_{-i}) + \Delta_{i,t} \right]$$  \hspace{1cm} \text{[$\Delta_{i,t} := C_i(s^t) - C_i(s^*, s^t_{-i})$]}

$$\leq \Sigma_t \left[ \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s^t) \right] + \Sigma_i \Sigma_t \Delta_{i,t}$$  \hspace{1cm} \text{[smooth]}

No regret: $\Sigma_t \Delta_{i,t} \leq 0$ for each $i$.

To finish proof: divide through by $T$. 

Extension Theorems

permissive equilibrium concept (e.g., no-regret outcomes)

what we care about
Extension Theorems

pure Nash equilibria
what’s easy to analyze

permissive equilibrium concept (e.g., no-regret outcomes)
what we care about

easier
Extension Theorems

- Pure Nash equilibria
- What's easy to analyze
- Easier
- POA extension theorem
- Permissive equilibrium concept (e.g., no-regret outcomes)
- What we care about
Bells and Whistles

- can allow baseline \( s^*_i \) to depend on \( s_i \), but not \( s_{-i} \)
- POA bound extends to correlated equilibria
- but *not* to no-regret sequences
- applications include:
  - splittable routing games [Roughgarden/Schoppman 11]
  - opinion formation games [Bhawalkar/Gollapudi/Munagala 13]
  - sequential composition of auctions [Syrgkanis/Tardos 13]
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Incomplete-Information Games

Game of incomplete information: [Harsanyi 67,68] specified by players, types, actions, payoffs.

- e.g., type = private valuation for a good
- player payoff depends on outcome and type
- strategy: function from types to actions
  - semantics: “if my type is t, then I will play action a”

Common Prior Assumption: types drawn from a distribution known to all players (independent, or not)

- realization of type i known only to player i
Example: First-Price Auction

Bayes-Nash Equilibrium: every player picks expected utility-maximizing action, given its knowledge.

Exercise: with n bidders, valuations drawn i.i.d. from $U[0,1]$, the following is a Bayes-Nash equilibrium: all bidders use the strategy $v_i \rightarrow [(n-1)/n] \cdot v_i$.

- highest-valuation player wins (maximizes welfare)
Bayes-Nash Equilibrium: every player picks expected utility-maximizing action, given its knowledge.

Exercise: with n bidders, valuations drawn i.i.d. from $U[0,1]$, the following is a Bayes-Nash equilibrium: all bidders use the strategy $v_i \rightarrow [(n-1)/n] \cdot v_i$.

- highest-valuation player wins (maximizes welfare)

Exercise: with 2 bidders, valuations from $U[0,1]$ and $U[0,2]$, no Bayes-Nash equilibrium maximizes expected welfare. (Second bidder shades bid more.)
POA with Incomplete Information: The Best-Case Scenario

**Ideal:** POA bounds w.r.t an *arbitrary* prior distribution.  
(or maybe assuming only independence)

**Observation:** point mass prior distribution $\Leftrightarrow$ game of full-information (Bayes-Nash equilibria $\Leftrightarrow$ Nash eq).
POA with Incomplete Information: The Best-Case Scenario

**Ideal:** POA bounds w.r.t an *arbitrary* prior distribution.
   (or maybe assuming only independence)

**Observation:** point mass prior distribution $\Leftrightarrow$ game of full-information (Bayes-Nash equilibria $\Leftrightarrow$ Nash eq).

**Coolest Statement That Could Be True:** POA of Bayes-Nash equilibria (for worst-case prior distribution) same as that of Nash equilibria in worst induced full-info game. (Observation above $\Rightarrow$ can only be worse)
Ideal Extension Theorem

**Hypothesis:** in every induced full-information game, a smoothness-type proof shows that the POA of (pure) Nash equilibria is $\alpha$ or better.

- induced full-info game $\iff$ specific type profile
- ex: first-price auction with known valuations

**Conclusion:** for every common prior distribution, the POA of (mixed) Bayes-Nash equilibria is $\alpha$ or better.
Extension Theorem (Informal)

- incomplete-info games
  - i.e., uncertain payoffs
- mixed Bayes-Nash equilibria

what we care about
(e.g., for auctions)
Extension Theorem (Informal)

- Full-information games
  - i.e., certain payoffs
  - Pure Nash equilibria
  - What’s easy to analyze

- Incomplete-info games
  - i.e., uncertain payoffs
  - Mixed Bayes-Nash equilibria
  - What we care about (e.g., for auctions)

Easier
Extension Theorem (Informal)

- Full-information games • i.e., certain payoffs
  - Pure Nash equilibria

- Incomplete-info games • i.e., uncertain payoffs
  - Mixed Bayes-Nash equilibria

POA extension theorem

What we care about (e.g., for auctions)

What’s easy to analyze

easier
1. Fix a game.
   (fixes optimal outcomes)

2. Choose baseline \( s^* = \text{some optimal outcome} \).
   (in many games, only one option)

3. Fix outcome \( s \).

4. Prove \( \Sigma_i C_i(s^*_i, s_{-i}) \leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s) \).

5. Conclude that POA of no-regret sequences \( \leq \lambda/(1-\mu) \).
Smoothness Paradigm
(Full => Incomplete)

1. Fix a setting *and the private valuations*. (fixes optimal outcomes)

2. Choose baseline \( s^* = \) some optimal outcome. (in many games, only one option)

3. Fix outcome \( s \).

4. Prove \( \sum_i C_i(s^*_i, s_{-i}) \leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s) \).

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Smoothness Paradigm
(Full ⇒ Incomplete)

1. Fix a setting and the private valuations. (fixes optimal outcomes)

2. Choose baseline $b^* = \text{some optimal outcome.}$ (note the large number of possible options)

3. Fix outcome $s.$

4. Prove $\sum_i C_i(s^*_i, s_{-i}) \leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s).$

5. Conclude that POA of no-regret sequences $\leq \lambda/(1-\mu).$
1. Fix a setting \textit{and the private valuations}.  
\hspace{1cm} (fixes optimal outcomes)

2. Choose baseline $b^* = \text{some optimal outcome}$.  
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3. Fix outcome $b$.

4. Prove $\sum_i C_i(s_i^*, s_{-i}) \leq \lambda \cdot \text{cost}(s^*) + \mu \cdot \text{cost}(s)$.

5. Conclude that POA of no-regret sequences $\leq \lambda/(1-\mu)$. 
Smoothness Paradigm
(Full => Incomplete)

1. Fix a setting *and the private valuations*.  
   (fixes optimal outcomes)

2. Choose baseline $b^* = \text{some optimal outcome}$.  
   (note the large number of possible options) \cite{SyrgkanisTardos13}

3. Fix outcome $b$.

4. Prove $\Sigma_i u_i(b^*_i, b_{-i}) \geq \lambda \cdot [\text{OPT Welfare}] - \text{Revenue}(b)$.

5. Conclude that POA of no-regret sequences $\leq \lambda/(1-\mu)$.
Smoothness Paradigm (Incomplete Information)

1. Fix a setting *and the private valuations.*
   (Fixes optimal outcomes)

2. Choose baseline \( b^* \) = some optimal outcome.
   (Note the large number of possible options)

3. Fix outcome \( b \).

4. Prove \( \sum_i u_i(b^*_i, b_{-i}) \geq \lambda \cdot [\text{OPT Welfare}] - \text{Revenue}(b) \).

5. Conclude that POA of Bayes-Nash equilibria is \( \geq \lambda \).

[Syrgkanis/Tardos 13]
Smoothness Paradigm (Incomplete Information)

1. Fix a setting and the private valuations.  
   (fixes optimal outcomes)

2. Choose baseline $b^* = \text{some optimal outcome}$.  
   (note the large number of possible options)

3. Fix outcome $b$.

4. Prove $\sum_i u_i(b^*, b_{-i}) \geq \lambda \cdot \text{[OPT Welfare]} - \text{Revenue}(b)$.  

5. Conclude that POA of Bayes-Nash equilibria is $\geq \lambda$.  

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Claim: for suitable choice of \( b^* \), for every \( b \),
\[
\sum_i u_i(b^*_i, b_{-i}) \geq \frac{1}{2} \cdot [\text{OPT Welfare}] - \text{Revenue}(b).
\]

Proof: Set \( b^*_i = \frac{v_i}{2} \) for every \( i \). (a la [Lucier/Paes Leme 11])
- since LHS \( \geq 0 \), can assume \( \frac{1}{2} \cdot [\max_i v_i] > \max_i b_i \)
- suppose bidder 1 has highest valuation. Then:
  \[
u_1(b^*_1, b_{-1}) = v_1 - (v_1/2) = v_1/2 \geq \frac{1}{2} \cdot [\text{OPT Welfare}]\]

Optimization: [Syrgkanis 12] 50% \( \Rightarrow \) 63% (different \( b^* \))
1. Fix a setting *and the private valuations.*
   (fixes optimal outcomes)

2. Choose baseline $b^* = $ some optimal outcome.
   (note the large number of possible options)

3. Fix outcome $b$.

4. Prove $\sum_i u_i(b^*_i, b_{-i}) \geq \lambda \cdot [\text{OPT Welfare}] - \text{Revenue}(b)$.

5. **Conclude that POA of Bayes-Nash equilibria is $\geq \lambda$.**
Extension Theorem (PNE)

Assume: for suitable choice of $b^*$, for every $b$,
$$\Sigma_i u_i(b^*_i, b_{-i}) \geq \lambda \cdot [\text{OPT Welfare}] - \text{Rev}(b).$$

Claim: POA of pure Nash equilibria is $\geq \lambda$. 
Assume: for suitable choice of $b^*$, for every $b$,
\[ \sum_i u_i(b^*_i, b_{-i}) \geq \lambda \cdot [\text{OPT Welfare}] - \text{Rev}(b). \]

Claim: POA of pure Nash equilibria is $\geq \lambda$.

Proof: Let $b$ = a pure Nash equilibrium. Then:
\[
\text{welfare}(b) = \text{Rev}(b) + \sum_i u_i(b) \quad \text{[defn of utility]}
\geq \text{Rev}(b) + \sum_i u_i(b^*_i, b_{-i}) \quad \text{[b a Nash eq]}
\geq \text{Rev}(b) + [\lambda \cdot [\text{OPT Welfare}] - \text{Rev}(b)]
= \lambda \cdot [\text{OPT Welfare}]
\]
Assume: for suitable choice of \( b^* \), for every \( b \),
\[
\sum_i u_i(b^*_i, b_{-i}) \geq \lambda \cdot [\text{OPT Welfare}] - \text{Rev}(b).
\]

Claim: \((\approx [\text{Lucier/Paes Leme 11}])\) for all (possibly correlated) valuation distributions, POA of Bayes-Nash eq is \( \geq \lambda \).

Proof: Let \( b() \) = a Bayes-Nash equilibrium. Then:
\[
E_v[\text{welfare}(b(v))] = E_v[\text{Rev}(b(v))] + \sum_i E_v[u_i(b(v))] \quad [\text{defn of utility}]
\geq E_v[\text{Rev}(b(v))] + \sum_i E_v[u_i(b^*_i(v_i), b_{-i}(v_{-i}))] \quad [b \text{ a BNE}]
\geq E_v[\text{Rev}(b(v))] + [\lambda \cdot E_v[\text{OPT Welfare}] - E_v[\text{Rev}(b(v))]]
= \lambda \cdot E_v[\text{OPT Welfare}]
\]
First-Price Auctions

Summary: for all (possibly correlated) valuation distributions, every Bayes-Nash equilibrium of a first-price auction has welfare at least 50% (or even 63%) of the maximum possible.

• 63% is tight for correlated valuations [Syrgkanis 14]
• independent valuations = worst-case POA unknown
  • worst known example = 87% [Hartline/Hoy/Taggart 14]
• 63% extends to simultaneous single-item auctions (covered tomorrow)
Further Applications

- first-price sponsored search auctions
  [Caragiannis/Kaklamanis/Kanellopoulos/Kyropoulou/Lucier/Paes Leme/Tardos 12]

- greedy pay-as-bid combinatorial auctions
  [Lucier/Borodin 10]

- pay-as-bid mechanisms based on LP rounding
  [Duetting/Kesselheim/Tardos 15]
Second-Price Rules

- simultaneous second-price auctions [Christodoulou/Kovacs/Schapira 08]
  - worst-case POA = 50%, and this is tight (even for PNE)
- truthful greedy combinatorial auctions [Borodin/Lucier 10]
  - worst-case POA close to greedy approximation ratio
- can be reinterpreted via modified smoothness condition [Roughgarden 12, Syrgkanis 12]
- “bluffing equilibria” => need a no overbidding condition for non-trivial POA bounds
Revenue Covering

- [Hartline/Hoy/Taggart 14] define “revenue covering”
- for every \( b \), \( \text{Rev}(b) \geq \) critical bids of winners in OPT
- implies smoothness condition
  - near-equivalent in some cases [Duetting/Kesselheim 15]
- application #1: POA bounds w.r.t. revenue objective
  - e.g., simultaneous first-price auctions with monopoly reserves
- application #2: [Hoy/Nekipelov/Syrgkanis 15] bound the “empirical POA” from data
  - do not need to explicitly estimate valuations!
  - can prove instance-by-instance bounds that beat the worst-case bound
first POA guarantees when bidder population changing (p fraction drops out each time step, replaced by new bidders).

- convergence to (Nash) equilibrium hopeless
- positive results for “adaptive learners” (assume agents use sufficiently good learning algorithm)
- need baseline near-optimal strategy profiles (one per time step) s.t. no player changes frequently
- novel use of differential privacy! (in the analysis)
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Multi-Item Auctions

- suppose $m$ different items
- for now: *unit-demand* valuations
- each bidder $i$ has private valuation $v_{ij}$ for each item $j$
- $v_i(S) := \max_{j \in S} v_{ij}$
Simultaneous Composition

- suppose have mechanisms $M_1,\ldots,M_m$
- in their *simultaneous composition*:
  - new action space = product of the $m$ action spaces
  - new allocation rule = union of the $m$ allocation rules
  - new payment rule = sum of the $m$ payment rules

- example: each $M_j$ a single-item first-price auction

**Question**: as a unit-demand bidder, how should you bid?  
(not so easy)
Hypothesis: every single-item auction $M_j$ is $\lambda$-smooth: for every $v$, there exists $b^*$ such that, for every $b$,

$$\sum_i u_i(b^*_i, b_{-i}) \geq \lambda \cdot [\text{OPT Welfare}(v)] - \text{Rev}(b).$$

Theorem: [Syrgkanis/Tardos 13] if bidders are unit-demand, then composed mechanism is also $\lambda$-smooth.

- holds more generally from arbitrary smooth $M_j$’s and “XOS” valuations (generalization of submodular)
Composition Preserves Smoothness

Hypothesis: every single-item auction $M_j$ is $\lambda$-smooth: for every $\mathbf{v}$, there exists $\mathbf{b}^*$ such that, for every $\mathbf{b}$,

$$\sum_i u_i(b^*_i, b_{-i}) \geq \lambda \cdot \text{OPT Welfare}(\mathbf{v}) - \text{Rev}(\mathbf{b}).$$

Theorem: [Syrgkanis/Tardos 13] if bidders are unit-demand, then composed mechanism is also $\lambda$-smooth.

Proof idea: Fix unit-demand valuations $\mathbf{v}$, fixes OPT.

• baseline strategy for a bidder $i$ that gets item $j$ in OPT
  • bid 0 in mechanisms other $M_j$
  • in $M_j$, use assumed baseline strategy for $M_j$
Simultaneous First-Price Auctions (First Try)

**Consequence:** for all (possibly correlated) unit-demand valuation distributions, every Bayes-Nash equilibrium of simultaneous first-price auctions has welfare at least 50% (or even 63%) of the maximum possible.

- prove smoothness inequality for first-price auction
- use composition theorem to extend smoothness to simultaneous first-price auctions
- use extension theorem to conclude Bayes-Nash POA bound for simultaneous first-price auctions
Fact: [Feldman/Fu/Gravin/Lucier 13], following [Bhawalkar/Roughgarden 11] there are (highly correlated) valuation distributions over unit-demand valuations such that every Bayes-Nash equilibrium has expected welfare arbitrary smaller than the maximum possible.

- idea: plant a random matching plus some additional highly demanded items; by symmetry, a bidder can’t detect the item “reserved” for it
Revised Statement

**Consequence:** for all *product* unit-demand valuation distributions, every Bayes-Nash equilibrium of simultaneous first-price auction has welfare at least 50% (or even 63%) of the maximum possible.

- prove smoothness inequality for first-price auction
- use composition theorem to extend smoothness to simultaneous first-price auctions
- use *modified* extension theorem to conclude Bayes-Nash POA bound for simultaneous first-price auctions
Private Baseline Strategies

First-price auction: set $b^*_i = v_i/2$ for every $i$.
- independent of $v_{-i}$ ("private" baseline strategies)

Simultaneous first-price auctions: $b^*_i$ is "bid half your value only on the item $j$ you get in $OPT(v)$".
- "public" baseline strategies
- not well defined unless $v_{-i}$ known
Assume: for suitable choice of $b^*$, for every $b$,
\[ \sum_i u_i(b^*_i, \mathbf{b}_{-i}) \geq \lambda \cdot [\text{OPT Welfare}] - \text{Rev}(b). \]

Claim: ($\approx$ [Lucier/Paes Leme 11]) for all (possibly correlated) valuation distributions, POA of Bayes-Nash eq is $\geq \lambda$.

Proof: Let $b() = a$ Bayes-Nash equilibrium. Then:
\[
E_v[\text{welfare}(b(v))] = E_v[\text{Rev}(b(v))] + \sum_i E_v[u_i(b(v))] \quad \text{[defn of utility]}
\]
\[
\geq E_v[\text{Rev}(b(v))] + \sum_i E_v[u_i(b^*_i(v_i), \mathbf{b}_{-i}(v_{-i}))] \quad \text{[b a BNE]}
\]
\[
\geq E_v[\text{Rev}(b(v))] + [\lambda \cdot E_v[\text{OPT Welfare}] - E_v[\text{Rev}(b(v))]]
\]
\[
= \lambda \cdot E_v[\text{OPT Welfare}]
\]

development can depend on $v_i$ but not $v_{-i}$
Extension Theorem (BNE)

Assume: for suitable choice of private $b^*$, for every $b$, 
\[
\Sigma_i u_i(b^*_i, b_{-i}) \geq \lambda \cdot [\text{OPT Welfare}] - \text{Rev}(b).
\]

Claim: ($\approx$[Lucier/Paes Leme 11]) for all (possibly correlated) valuation distributions, POA of Bayes-Nash eq is $\geq \lambda$.

Proof: Let $b() = a$ Bayes-Nash equilibrium. Then:
\[
E_v[\text{welfare}(b(v))] = E_v[\text{Rev}(b(v))] + \Sigma_i E_v[u_i(b(v))] \quad \text{[defn of utility]}
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\geq E_v[\text{Rev}(b(v))] + \Sigma_i E_v[u_i(b^*_i(v_i), b_{-i}(v_{-i}))] \quad \text{[b a BNE]}
\]
\[
\geq E_v[\text{Rev}(b(v))] + [\lambda \cdot E_v[\text{OPT Welfare}] - E_v[\text{Rev}(b(v))]]
\]
\[
= \lambda \cdot E_v[\text{OPT Welfare}]
\]

deviation can depend on $v_i$ but not $v_{-i}$
Modified Extension Theorem

Assume: for suitable choice of public $b^*$, for every $b$,
\[ \sum_i u_i(b^*_i,b_{-i}) \geq \lambda \cdot [\text{OPT Welfare}] - \text{Rev}(b). \]

Theorem: [Syrgkanis/Tardos 13], following [Christodoulou/Kovacs/Schapira 08] for all product valuation distributions, POA of Bayes-Nash eq is $\geq \lambda$.

Proof idea: to transform public $b^*_i$ to a deviation:
- sample $w_{-i}$ from prior distribution
- play baseline strategy for valuation profile $(v_i,w_{-i})$
Outline

1. Smooth Games, Extension Theorems, and Robust POA Bounds
2. Smooth Mechanisms and Bayes-Nash POA Bounds
3. Reducing Complex Mechanisms to Simple Mechanisms Using Composition Theorems
4. Complexity-Based POA Lower Bounds
Tight POA Bounds

Theorem: [Feldman/Fu/Gravin/Lucier 13], [Christodoulou/Kovacs/Sgouritsa/Tang 14]
the worst-case POA of S1A’s with subadditive bidder valuations is precisely 2.

monotone \textit{subadditive} valuations:
• \( v_i(A \cup B) \leq v_i(A) + v_i(B) \) for all disjoint \( A, B \)
Tight POA Bounds

Theorem: [Feldman/Fu/Gravin/Lucier 13], [Christodoulou/Kovacs/Sgouritsa/Tang 14]

the worst-case POA of S1A’s with subadditive bidder valuations is precisely 2.

Explicit Lower Bound:
Tight POA Bounds

Theorem: [Feldman/Fu/Gravin/Lucier 13], [Christodoulou/Kovacs/Sgouritsa/Tang 14]

The worst-case POA of S1A’s with subadditive bidder valuations is precisely 2.

Question: Can we do better?

(without resorting to the VCG mechanism)
The Upshot

**Meta-theorem:** equilibria are generally bound by the same limitations as algorithms with polynomial computation or communication.

- lower bounds without explicit constructions!

**Caveats:** requires that equilibria are

- guaranteed to exist (e.g., mixed Nash equilibria)
- can be efficiently verified

**Example consequence:** no “simple” auction has POA < 2 for bidders with subadditive valuations.
Theorem: [Roughgarden 14] Suppose:

- no nondeterministic subexponential-communication protocol approximates the welfare-maximization problem (with valuations V) to within factor of $\alpha$.
  - i.e., impossible to decide $\text{OPT} \geq W^*$ vs. $\text{OPT} \leq W^*/\alpha$

Then worst-case POA of $\epsilon$-approximate mixed Nash equilibria of every “simple” mechanism is at least $\alpha$.

- simple = number of strategies sub-doubly-exponential in m
- $\epsilon$ can be as small as inverse polynomial in n and m

Point: reduces lower bounds for equilibria to lower bounds for communication protocols.
Consequences

Corollary: (via [Nisan/Segal 06], [Dobsinski/Nisan/Schapira 05])

- With subadditive bidder valuations, no simple auction guarantees equilibrium welfare better than 50% OPT.
  - “simple”: bid space dimension \( \leq \) subexponential in \# of goods
- With general valuations, no simple auction guarantees non-trivial equilibrium welfare.

Take-aways:

1. In these cases, S1A’s optimal among simple auctions.
2. With complements, complex bid spaces (e.g., package bidding) necessary for welfare guarantees.
Why Approximate MNE?

**Issue:** in an S1A, number of strategies = \((V_{\text{max}} + 1)^m\)
- valuations, bids assumed integral and poly-bounded

**Consequence:** can’t efficiently guess/verify a MNE.

**Theorem:** [Lipton/Markakis/Mehta 03] a game with \(n\) players and \(N\) strategies per player has an \(\varepsilon\)-approximate mixed Nash equilibrium with support size polynomial in \(n\), \(\log N\), and \(\varepsilon^{-1}\).
- proof idea based on sampling from an exact MNE
Nondeterministic Protocols

- each of n players has a private valuation $v_i$
- a “referee” wants to convince the players that the value of some function $f(v_1,\ldots,v_n)$ has the value $z$
- referees knows all $v_i$’s and writes, in public view, an alleged proof $P$ that $f(v_1,\ldots,v_n) = z$
- protocol accepts if and only if every player $i$ accepts the proof $P$ (knowing only $v_i$)
- communication used = length (in bits) of proof $P$
- example: Non-Equality vs. Equality
Theorem: [Roughgarden 14] Suppose:

• no nondeterministic subexponential-communication protocol approximates the welfare-maximization problem (with valuations V) to within factor of $\alpha$.
  • i.e., impossible to decide $\text{OPT} \geq W^* \text{ vs. } \text{OPT} \leq W^*/\alpha$

Then worst-case POA of $\varepsilon$-approximate mixed Nash equilibria of every “simple” mechanism is at least $\alpha$.

• simple = number of strategies sub-doubly-exponential in m
• $\varepsilon$ can be as small as inverse polynomial in n and m

Point: reduces lower bounds for equilibria to lower bounds for communication protocols.
Proof of Theorem

Suppose worst-case POA of $\varepsilon$-MNE is $\rho < \alpha$:

**Input:** game $G$ s.t. either

(i) $\text{OPT} \geq W^*$
or (ii) $\text{OPT} \leq \frac{W^*}{\alpha}$
Proof of Theorem

Suppose worst-case POA of $\varepsilon$-MNE is $\rho < \alpha$:

**Input:** game $G$ s.t. either
(i) $\text{OPT} \geq W^*$
or (ii) $\text{OPT} \leq W^*/\alpha$

**Protocol:**
“proof” = $\varepsilon$ -MNE $x$ with small support (exists by LMM); players verify it privately
Proof of Theorem

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Input: game $G$ s.t. either (i) $\text{OPT} \geq W^*$ or (ii) $\text{OPT} \leq W^*/\alpha$

Protocol: "proof" = $\varepsilon$-MNE $x$ with small support (exists by LMM); players verify it privately

if $E[\text{wel}(x)] > W^*/\alpha$ then $\text{OPT} > W^*/\alpha$ so in case (i)
Proof of Theorem

Suppose worst-case POA of $\varepsilon$-MNE is $\rho < \alpha$:

**Input:** game $G$ s.t. either (i) $\text{OPT} \geq W^*$ or (ii) $\text{OPT} \leq W^*/\alpha$

**Protocol:**
- "proof" = $\varepsilon$ -MNE $x$ with small support (exists by LMM); players verify it privately

If $E[wel(x)] > W^*/\alpha$ then $\text{OPT} > W^*/\alpha$ so in case (i)

If $E[wel(x)] \leq W^*/\alpha$ then $\text{OPT} \leq (\rho/\alpha)W^* < W^*$ so in case (ii)

**Key point:** every $\varepsilon$-MNE is a short, efficiently verifiable certificate for membership in case (ii).
Claim: POA lower bounds for $\varepsilon$-MNE with small enough $\varepsilon$ essentially as good as for exact MNE. Reasons:

1. All known upper bound techniques apply automatically to approximate equilibria.
   1. e.g., “smoothness proofs” [Roughgarden 09]
   2. so our lower bounds limit all known proof techniques

2. Lower bounds for approximate equilibria can sometimes be translated into bounds for exact equilibria.

3. If POA of exact equilibria $\ll$ POA of approximate equilibria, the latter is likely more relevant (and robust).
More Applications

- optimality results for “simple” auctions with other valuation classes (general, XOS)
- analogous results for combinatorial auctions with succinct valuations (if coNP not in MA)
- impossibility results for low-dimensional price equilibria (assuming NP ≠ coNP) [Roughgarden/Talgam-Cohen 15]
- unlikely to reduce planted clique to ε-Nash hardness
Open Questions

1. **Tight POA bounds for important auction formats**
   1. e.g. first-price auctions with independent valuations

2. **Best “simple” auction for submodular valuations?**
   1. S1A’s give 63% [Syrgkanis/Tardos 13], [Christodoulou et al 14]
   2. > 77% impossible [Dobzinski/Vondrak 13] + [R14]
   3. > 63% is possible with poly communication [Feige/Vondrak 06]

3. **Design “natural” games with POA matching hardness lower bound for the underlying optimization problem.**
   1. e.g., many auction and scheduling problems
FIN