How To Think About Algorithmic Mechanism Design

[Tutorial at FOCS 2010]

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An eBay Single-Good Auction

- winner = highest bidder above reserve price
- price = max \{second-highest bid, reserve\}
Truthful Auctions

**Utility Model:** bidder $i$ has *valuation* $v_i$
- maximum willingness to pay
- known to bidder, unknown to seller

- *utility* $= v_i - \text{price paid}; \text{or } 0 \text{ if loses auction}$
- submits *bid* $b_i$ to maximize its utility

**Claim:** an eBay auction is *truthful*
- truthful bidding ($b_i = v_i$) is “foolproof”
- i.e., a false bid never outperforms a true bid
Fix player $i$, reserve $r$, other bids $b_{-i}$

Observation #1: bidder $i$ effectively faces a “take-it-or-leave it" offer at a fixed price $p = \max \{\text{reserve, highest other bid}\}$.

Observation #2: truthful bidding guaranteed to maximize utility (a "dominant strategy")
- case 1: $(v \leq p)$ max utility $= 0$, achieved when $b = v$
- case 2: $(v \geq p)$ max utility $= v - p$, achieved when $b = v$
Overarching Goals

- want to design "optimal" truthful mechanisms and auctions
  - for a wide range of problems
    - combinatorial auctions, scheduling, etc.
  - for different objectives (welfare, revenue)
  - often require polynomial running time as well

- general design techniques, analysis frameworks

- prove limits on what is possible
Why Truthful?

- many mechanisms "in the wild" not truthful
  - sponsored search, combinatorial auctions
  - important for practical implementations

- not clear when other mechanisms (with no dominant strategies) are fundamentally more powerful than truthful ones; sometimes have equivalence
  - e.g., "Revenue Equivalence" theorems

- truthful mechanisms definitely a good "first-cut abstraction" for foundations of mechanism design
How Theory CS Can Contribute

Unsurprising fact: very rich tradition and literature on mechanism design in economics.
- largely "Bayesian" (i.e., average-case) settings
- emphasizes exact solutions/characterizations
- usually ignores communication/computation

What we have to offer:
1. worst-case guarantees
2. approximation bounds
3. computational complexity
How To Think About Algorithmic Mechanism Design

Philosophy: designing truthful mechanisms boils down to designing algorithms in a certain "restricted computational model".

Next: focus on simple class of problems where this point is particularly clear and well understood.
Single-Parameter Problems

Outcome space: a set of vectors of the form
\((x_1, x_2, \ldots, x_n)\) [amount of "stuff" per player]

Utility Model: bidder \(i\) has private valuation \(v_i\)
(per unit of "stuff")

- utility = \(v_i \times i - \text{payment}\)
- submits bid \(b_i\) to maximize its utility

Examples: \(k\)-unit auction, "unit-demand" bidders; job scheduling on related machines
Mechanism Design Space

The essence of any truthful mechanism (formalized via the "Revelation Principle"): 

- collect bid \( b_i \) from each player \( i \)
- invoke (randomized) allocation rule: \( b_i \)'s \( \rightarrow \) \( x_i \)'s
  - who gets how much (expected) stuff
- invoke (randomized) payment rule: \( b_i \)'s \( \rightarrow \) \( p_i \)'s
  - and who pays what
- truthfulness: for every \( i \), \( v_i \), other bids, setting \( v_i = b_i \) maximizes expected utility \( v_i x_i(b) - p_i(b) \)
Two Definitions

Implementable Allocation Rule: is a function $x$ (from bids to expected allocations) that admits a payment rule $p$ such that $(x, p)$ is truthful.

- i.e., truthful bidding $[b_i := v_i]$ always maximizes a bidder's (expected) utility
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**Monotone Allocation Rule:** for every fixed bidder $i$, fixed other bids $b_{-i}$, expected allocation only increases in the bid $b_i$.

- example: highest bidder wins
- non-example: 2nd-highest bidder wins
Myerson's Lemma

Myerson's Lemma: [1981; also Archer-Tardos FOCS 01] an allocation rule $x$ is implementable if and only if it is monotone.
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Explicit formula for $p_i(b)$:
- keep $b_{-i}$ fixed, increase $z$ from 0 to $b_i$
- consider breakpoints $y_1, \ldots, y_q$ at which $x_i$ jumps
- set $p_i(b) := \Sigma_j y_j \bullet [\text{jump in } x_i \text{ at } y_j]$
Myerson's Lemma (Proof Idea)

Proof idea: let \( x \) be an allocation rule, fix \( i \) and \( b_{-i} \).
Write \( x(z) \), \( p(z) \) for \( x_i(z, b_{-i}) \), \( p_i(z, b_{-i}) \).

- apply purported truthfulness of \((x, p)\) to two scenarios: true value = \( z \), false bid = \( z + \varepsilon \) and true value = \( z + \varepsilon \), false bid = \( z \)

- take \( \varepsilon \) to zero get
  
  \[ p'(z) = z \circ x'(z) \quad \text{[if } x \text{ differentiable at } z \text{]} \text{ or} \]
  
  jump in \( p \) at \( z = z \circ \) [jump in \( x \) at \( z \)]

Integrating from 0 to \( b_i \), get sole candidate:

\[ p_i(b) := \sum_j y_j \bullet [\text{jump in } x_i \text{ at } y_j] \]
Example: Profit Extractor

[Fiat/Goldberg/Hartline/Karlin STOC 02]
Allocation Rule: bids $b +$ revenue target $R$:

- initialize $S = \text{all bidders}$
- while there is an $i$ in $S$ such that $b_i < R/|S|$:  
  - remove such a bidder from $S$
- winners = final set $S$
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- winners = final set $S$

Note: allocation rule is monotone.

By Myerson's Lemma: forms a truthful auction if and only if every winner charged price $p = \frac{R}{|S|}$
- if halts with non-empty set, raises revenue $R$
Revenue Maximization

**Setting:** k-item auction, n unit-demand bidders.

**Goal:** truthful auction with "optimal" revenue.
- but different auctions do better on different inputs
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Approach #1: Bayesian/average-case analysis.
   - "optimal" auction maximizes expected revenue

Approach #2: worst-case guarantee.
   - "optimal" auction tricky to define, standard competitive analysis is useless
   - use "Bayesian thought experiment" instead
Bayesian Profit Maximization

Example: 1 bidder, 1 item, $v \sim$ known distribution $F$

- truthful auctions = posted prices $p$
- expected revenue of $p$: $p(1-F(p))$
  - given $F$, can solve for optimal $p^*$
  - e.g., $p^* = \frac{1}{2}$ for $v \sim$ uniform[0,1]
- but: what about $k,n > 1$ (with i.i.d. $v_i$'s)?
Bayesian Profit Maximization

**Example:** 1 bidder, 1 item, \( v \sim \) known distribution \( F \)

- **truthful auctions = posted prices** \( p \)
- **expected revenue of** \( p \): \( p(1-F(p)) \)
  - given \( F \), can solve for optimal \( p^* \)
  - e.g., \( p^* = \frac{1}{2} \) for \( v \sim \) uniform\([0,1]\)
- **but:** what about \( k, n > 1 \) (with i.i.d. \( v_i \)'s)?

**Theorem:** [Myerson 81] auction with max expected revenue is Vickrey with above reserve \( p^* \).
  - note \( p^* \) is *independent of* \( k \) and \( n \)
Toward Worst-Case Analysis

Goal: prove approximation results of the form:

"Theorem: for every valuation profile $v$: auction $A$'s revenue on $v$ is at least $\frac{\text{OPT}(v)}{\alpha}$." (for a hopefully small constant $\alpha$)
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(for a hopefully small constant \( \alpha \))

Idea for \( \text{OPT}(v) \): sum of \( k \) largest \( v_i \)'s.

Problem: too strong, not useful.

- makes all auctions \( A \) look equally bad.
- every auction \( A \) has a bad \( v \) [no finite \( \alpha \) possible]
Bayesian Thought Experiment

Question: what would an i.i.d. Bayesian do?

- formulate prior F, run the optimal auction for F
  [by Myerson => Vickrey with suitable reserve]

Ambition: design auction $A$ that is simultaneously competitive with all Bayesian optimal auctions!

I.e.: For every F, corresponding opt auction $A_F$:

$$A's\ expected\ revenue \geq (A_F's\ expected\ revenue)/\alpha$$

- [Bulow/Klemperer AER 96], [Hartline/Roughgarden EC 09], [Dhangwotnotai/Roughgarden/Yan EC 10]
Distribution-Free Benchmarks

Myerson: for all $F$, Vickrey + a reserve is optimal.

Corollary: for all $F$ and all $v$, behavior of optimal auction for $F$ equivalent to offering every bidder a common take-it-or-leave-it offer.

- namely: $\max \{\text{reserve price, } (k+1)\text{th highest bid of } v\}$

Upper Bound: $RB(v) := \max_{i \leq k} iv_i$ [assume sorted $v_i$'s]

By Design: if auction $A$ achieves revenue $RB(v)/\alpha$ for every $v$, then it also has "simultaneous Bayesian" guarantee.

- [Goldberg/Hartline/Karlin/Saks/Wright GEB 06]
- [Hartline/Roughgarden STOC 08], [Devanur/Hartline EC 09]
Intermission

GO GIANTS!
Combinatorial Auctions (CA)

Setting: n bidders, m goods. Player i has private valuation $v_i(S)$ for each subset S of goods.

Assume: $v_i(\emptyset) = 0$ and $v_i$ is
- monotone: $S$ subset of $T \Rightarrow v_i(S) \leq v_i(T)$
- subadditive: $v_i(S \cup T) \leq v_i(S) + v_i(T)$
- ignore representation issues
  [want running time polynomial in $n$ and $m$]

Facts: there is a poly-time 2-approximation for welfare $\Sigma_i v_i(S_i)$ [Feige STOC 06]. No good truthful approximation known.
Multi-Parameter Problems

Outcome space: an abstract set $\Omega$

Utility Model: bidder $i$ has private valuation $v_i(\omega)$ for each outcome $\omega$

- utility = $v_i(\omega)$ - payment

Example: in a combinatorial auction, $\Omega$ = all possible allocations of goods to players
How To Think About Algorithmic Mechanism Design

Philosophy: designing truthful mechanisms boils down to designing algorithms in a certain "restricted computational model".

Single-Parameter Special Case:

implementable rules
= monotone rules
(Myerson's Lemma)
The Multi-Parameter World

Implementable rules
= "cyclic monotone" rules
(still have uniqueness of truthful payment rule) [Rochet]
The Multi-Parameter World

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mechanisms that we understand

VCG
The VCG Mechanism

Utility Model: bidder i's utility: \( v_i(\omega) \) - payment

Vickrey-Clarke-Groves: (1961/71/73)

- collect bid \( b_i(\omega) \) for all \( i \), all outcomes \( \omega \) in \( \Omega \)
- select \( \omega^* \) in argmax \( \{ \Sigma_i b_i(\omega) \} \)
- charge \( p_i = [-\Sigma_{j \neq i} b_i(\omega)] + \text{suitable constant} \)
  - align private objectives with global one

Facts: truthful, maximizes welfare \( \Sigma_i v_i(\omega) \) over \( \Omega \)
(assuming truthful bids).
Approximation Mechanisms

**Assume:** want to maximize welfare $\Sigma_i v_i(\omega)$
- revenue also interesting, wide open

**Why Not VCG?:** communication/computation lower bounds for many important problems.
- e.g., players = nodes of graph G;
- $\Omega =$ independent sets of G;
- $v_i(\omega) = 1$ if $i$ in $\omega$, 0 otherwise

**Goal:** mechanisms that are (1) truthful; (2) run in time polynomial in natural parameters; and (3) guarantee near-optimal welfare
Approximation Mechanisms

Goals: [Nisan/Ronen 99] (1) truthful; (2) run in time polynomial in natural parameters; and (3) guarantee near-optimal welfare

Best-case scenario: match approximation factor of best polynomial-time approximation algorithm (with valuations given freely as input).

Holy Grail: "black-box reduction" that turns an approximation algorithm into a truthful approximation mechanism.
Approximation Mechanisms

Idea: [Nisan/Ronen 00] use VCG mechanism but substitute approximation algorithm for the previous step "select $\omega^*$ in $\text{argmax} \{\Sigma_i b_i(\omega)\}". 

implementable = "cyclic monotone"

mechanisms we understand VCG
Approximation Mechanisms

Idea: [Nisan/Ronen 00] use VCG mechanism but substitute approximation algorithm for the previous step "select $\omega^*$ in $\arg\max \{\Sigma_i b_i(\omega)\}".

Issue: only truthful for a very special type of approximation algorithm (discussed next).

implementable = "cyclic monotone"

mechanisms we understand

more on this next
VCG-Based Mechanisms

Outcome space: an abstract set $\Omega$

Utility Model: bidder i's utility: $v_i(\omega)$ - payment

Step 1: pre-commit to a subset $\Omega'$ of $\Omega$
Step 2: run VCG with respect to $\Omega'$

Facts: truthful, maximizes welfare $\Sigma_i v_i(\omega)$ over $\Omega'$

Hope: can choose $\Omega'$ to recover tractability while controlling approximation factor.
Combinatorial Auctions (CA)

Setting: n bidders, m goods. Player i has private valuation $v_i(S)$ for each subset S of goods.

Assume: $v_i(\emptyset) = 0$ and $v_i$ is

- **monotone**: S subset of T => $v_i(S) \leq v_i(T)$
- **subadditive**: $v_i(S \cup T) \leq v_i(S) + v_i(T)$
- ignore representation issues
  [want running time polynomial in n and m]

Fact: there is a 2-approximation for welfare $\Sigma_i v_i(S_i)$
[Feige STOC 06], but this allocation rule is not implementable.
VCG-Based Solution

Key Claim: for every instance, there is a \( \frac{1}{2\sqrt{m}} \)-approximate allocation that either:
- assigns all goods to a single player; OR
- assigns at most one good to each player
VCG-Based Solution

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Corollary: [Dobzinski/Nisan/Schapira STOC 05] there is a truthful \((1/2\sqrt{m})\)-approximate mechanism for CAs with subadditive bidder valuations.

Proof: define \(\Omega'\) as above; can optimize in poly-time via max-weight matching + case analysis.
VCG-Based Solution

Proof of Key Claim: Fix $v_i$'s. Call a player big if it gets $> \sqrt{m}$ goods in the optimal allocation. (So there are at most $\sqrt{m}$ of them.)

Case 1: big players account for more than half of optimal welfare, so one big player accounts for a $1/2\sqrt{m}$ fraction. Give all goods to this player.

Case 2: otherwise, small players account for half. Give each its favorite good; by subadditivity, still have a $1/2\sqrt{m}$ fraction of optimal welfare.
Can We Do Better?

[Dobzinski/Nisan STOC 07]: Can't do much better using a deterministic VCG-based mechanism.

- results and techniques launched very active research agenda on lower bounds
  - [Papadimitriou/Schapira/Singer FOCS 08], ...

The good news: randomized mechanisms seem to hold much promise, for specific problems and for black-box reductions.

- some rigorous randomized vs. deterministic separations already known
Randomized VCG-Based Mechanisms

Step 1: precommit to subset \( \Delta' \) of \( \Delta(\Omega) \)
- "lotteries" over outcomes

Step 2: run VCG with respect to \( \Delta' \)

Facts: truthful (in expectation), maximizes expected welfare \( E[\sum_i v_i(\omega)] \) over \( \Delta' \)

Hope: can choose \( \Delta' \) to recover tractability while controlling approximation factor.
- [Lavi/Swamy FOCS 05], [Dobzinski/Dughmi FOCS 09]
A Black-Box Reduction

**Theorem:** [Dughmi/Roughgarden FOCS 10] If a welfare-maximization problem admits an FPTAS, then it admits a truthful FPTAS.

**Proof idea:** Choosing $\Delta'$ suitably and "dualizing", the relevant optimization problem is a slightly perturbed version of the original one. Can use techniques from smoothed analysis [Roglin/Teng FOCS 09] to get expected polynomial running time.
Black-Box Reduction for Bayes-Nash Implementations

Theorem: [Hartline/Lucier STOC 10], [Bei/Hartline/Huang/Kleinberg/Malekian SODA 11] In many Bayesian settings (where valuations are drawn from known distributions), every approximation algorithm for welfare maximization can be transmuted into an equally good truthful (in Bayes-Nash equilibrium) approximation mechanism.

Suggestive: Bayes-Nash implementations might elude lower bounds for dominant-strategy truthful mechanisms (should such lower bounds exist).
Recap: Mechanism Design as Constrained Algorithm Design

**Philosophy:** designing truthful mechanisms boils down to designing algorithms in a certain "restricted computational model".

- single-parameter $\iff$ monotone algorithms
- multi-parameter: includes all the obvious VCG variants, but what else?

**Research Challenge:** usefully characterize the implementable allocation rules for as many multi-parameter problems as possible.
Recap: Revenue Maximization

- Bayesian single-parameter case well solved
- worst-case guarantees for single-parameter problems: need novel analysis frameworks ("Bayesian thought experiment") but lots of recent progress

Research Challenges:
- non-i.i.d. version of Bayesian thought experiment
- (approximate) analog of Myerson's theory for multi-parameter problems (even relatively simple ones) [Bhattacharya et al STOC 10], [Chawla et al STOC 10]
- worst-case guarantees for multi-parameter problems
Recap: Welfare Maximization

- ignoring tractability, VCG works even for arbitrary multi-parameter problems
- truthful approximation mechanisms so far mostly restricted to randomized variants of VCG
- but this already enough for some interesting results

Research Challenges:
- better (randomized) approximation mechanisms for combinatorial auctions
- more general black-box reductions
- better lower bounds, especially for randomized mechanisms