Approximation in Algorithmic Game Theory

Robust Approximation Bounds for Equilibria and Auctions

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Motivation

Clearly: many modern applications in CS involve autonomous, self-interested agents

- motivates noncooperative games as modeling tool

Unsurprising fact: this often makes full optimality hard/impossible.

- equilibria (e.g., Nash) of noncooperative games are typically suboptimal
- auctions lose revenue from strategic behavior
- incentive constraints can make poly-time approximation of NP-hard problems even harder

Approximation in AGT

- The Price of Anarchy (etc.)
  - worst-case approximation guarantees for equilibria
- Revenue Maximization
  - guarantees for auctions in non-Bayesian settings (information-theoretic)

this talk

- Algorithm Mechanism Design
  - approximation algorithms robust to selfish behavior (computational)

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- Computing Approximate Equilibria
  - e.g., is there a PTAS for computing an approximate Nash equilibrium?


THE PRICE OF ANARCHY
IS IT OKAY TO BE SELFISH?
FIRE DEPT.
39

Price of Anarchy

Price of anarchy: [Koutsoupias/Papadimitriou 99]

quantify inefficiency w.r.t some objective function.

- e.g., *Nash equilibrium*: an outcome such that no player better off by switching strategies

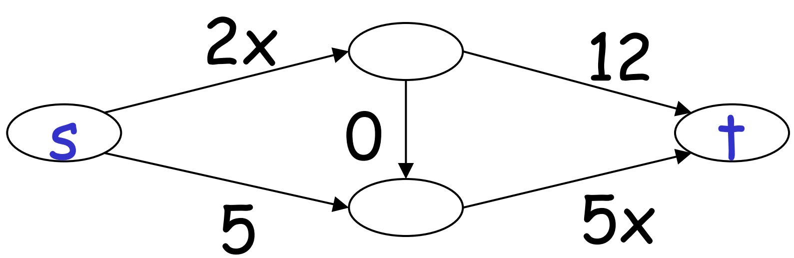
Definition: *price of anarchy (POA)* of a game (w.r.t. some objective function):

$$\frac{\text{equilibrium objective fn value}}{\text{optimal obj fn value}}$$

the closer to 1 the better

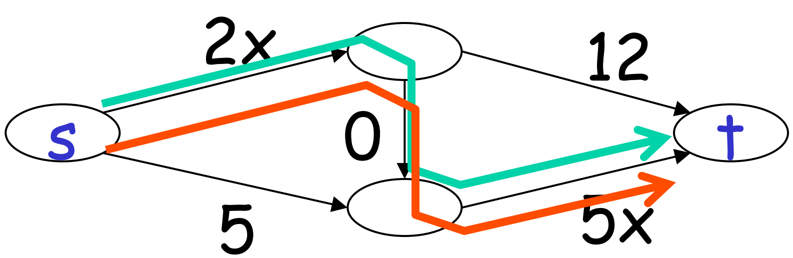
The Price of Anarchy

Network w/2 players:



The Price of Anarchy

Nash Equilibrium:



cost = 14+14 = 28

The Price of Anarchy

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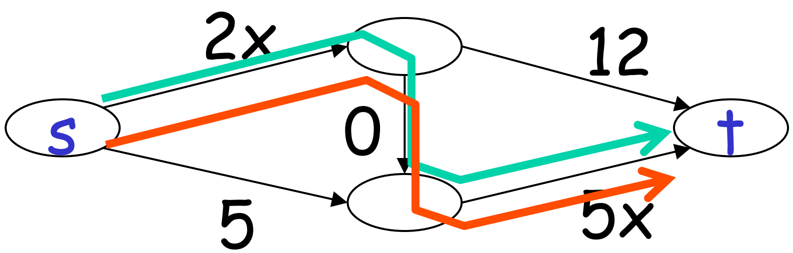

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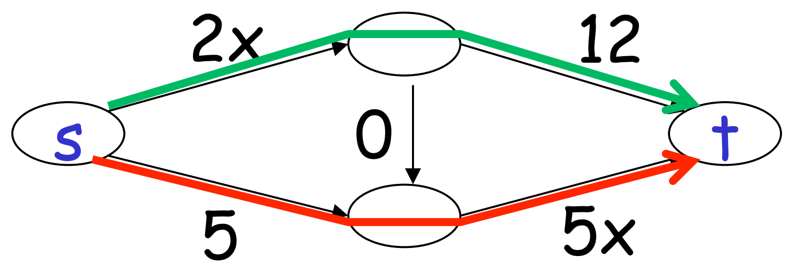
To Minimize Cost:



cost = 14+10 = 24

Price of anarchy = 28/24 = 7/6.

- if multiple equilibria exist, look at the *worst* one

The Need for Robustness

Meaning of a POA bound: *if* the game is at an equilibrium, *then* outcome is near-optimal.

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Meaning of a POA bound: *if* the game is at an equilibrium, *then* outcome is near-optimal.

Problem: what if can't reach equilibrium?

- (pure) equilibrium might not exist
- might be hard to compute, even centrally
  - [Fabrikant/Papadimitriou/Talwar], [Daskalakis/ Goldbeg/ Papadimitriou], [Chen/Deng/Teng], etc.
- might be hard to learn in a distributed way

Worry: are our POA bounds "meaningless"?

Robust POA Bounds

High-Level Goal: worst-case bounds that apply *even to non-equilibrium outcomes!*

- best-response dynamics, pre-convergence
  - [Mirrokni/Vetta 04], [Goemans/Mirrokni/Vetta 05], [Awerbuch/Azar/Epstein/Mirrokni/Skopalik 08]
- correlated equilibria
  - [Christodoulou/Koutsoupias 05]
- coarse correlated equilibria aka "price of total anarchy" aka "no-regret players"
  - [Blum/Even-Dar/Ligett 06], [Blum/Hajiaghayi/Ligett/Roth 08]

Abstract Setup

- n players, each picks a strategy s_i
- player i incurs a cost $C_i(\mathbf{s})$

Important Assumption: objective function is
$\text{cost}(\mathbf{s}) := \Sigma_i C_i(\mathbf{s})$

Key Definition: A game is *(λ, μ)-smooth* if, for every pair $\mathbf{s}, \mathbf{s}^*$ outcomes ($\lambda > 0$; $\mu < 1$):

$$\Sigma_i C_i(s^*_i, s_{-i}) \leq \lambda \bullet \text{cost}(\mathbf{s}^*) + \mu \bullet \text{cost}(\mathbf{s}) \quad [(*)]$$

Smooth => POA Bound

Next: "canonical" way to upper bound POA (via a smoothness argument).

- notation: **s** = a Nash eq; **s*** = optimal

Assuming (λ,μ)-smooth:

$$\text{cost}(\mathbf{s}) = \Sigma_i C_i(\mathbf{s}) \qquad \text{[defn of cost]}$$

$$\leq \Sigma_i C_i(s^*_i, s_{-i}) \qquad [\mathbf{s} \text{ a Nash eq}]$$

$$\leq \lambda \cdot \text{cost}(\mathbf{s}^*) + \mu \cdot \text{cost}(\mathbf{s}) \qquad [(*)]$$

Then: POA (of pure Nash eq) ≤ λ/(1-μ).

Why Is Smoothness Stronger?

Key point: to derive POA bound, only needed

$\Sigma_i C_i(s^*_i, s_{-i}) \leq \lambda \bullet \text{cost}(\mathbf{s}^*) + \mu \bullet \text{cost}(\mathbf{s})$ [(*)]

to hold in special case where $\mathbf{s}$ = a Nash eq and $\mathbf{s}^*$ = optimal.

Smoothness: requires (*) for *every* pair $\mathbf{s},\mathbf{s}^*$ outcomes.

- even if $\mathbf{s}$ is *not* a pure Nash equilibrium

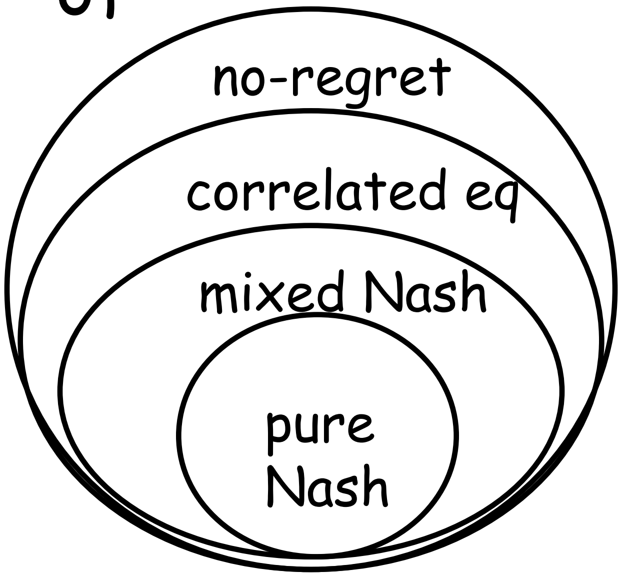
Some Smoothness Bounds

- atomic (unweighted) selfish routing [Awerbuch/Azar/Epstein 05], [Christodoulou/Koutsoupias 05], [Aland/Dumrauf/Gairing/Monien/Schoppmann 06], [Roughgarden 09]
- nonatomic selfish routing [Roughgarden/Tardos 00],[Perakis 04] [Correa/Schulz/Stier Moses 05]
- weighted congestion games [Aland/Dumrauf/Gairing/Monien/Schoppmann 06], [Bhawalkar/Gairing/Roughgarden 10]
- submodular maximization games [Vetta 02], [Marden/Roughgarden 10]
- coordination mechanisms [Cole/Gkatzelis/Mirrokni 10]

Beyond Nash Equilibria

Definition: a sequence $s^1, s^2, \ldots, s^T$ of outcomes is *no-regret* if:

- for each player i, each fixed action q_i:
  - average cost player i incurs over sequence no worse than playing action q_i every time
  - if every player uses e.g. "multiplicative weights" then get o(1) regret in poly-time
  - empirical distribution = "*coarse correlated eq*"



An Out-of-Equilibrium Bound

Theorem: [Roughgarden STOC 09]
in a (λ,μ)-smooth game, average cost of every no-regret sequence at most

$$[\lambda/(1-\mu)] \times \text{cost of optimal outcome.}$$

(the same bound we proved for pure Nash equilibria)

Smooth => No-Regret Bound

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$= \Sigma_t \Sigma_i \, [C_i(s^*_i, s^t_{-i}) + \Delta_{i,t}]$ $[\Delta_{i,t} := C_i(\mathbf{s}^t) - C_i(s^*_i, s^t_{-i})]$

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No regret: $\Sigma_t \, \Delta_{i,t} \leq 0$ for each i.

To finish proof: divide through by T.

Intrinsic Robustness

Theorem: [Roughgarden STOC 09] for every set C, unweighted congestion games with cost functions restricted to C are *tight*:

maximum [pure POA] = minimum [λ/(1-μ)]

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- *weighted* congestion games [Bhawalkar/ Gairing/ Roughgarden ESA 10] and submodular maximization games [Marden/Roughgarden CDC 10] are also tight in this sense

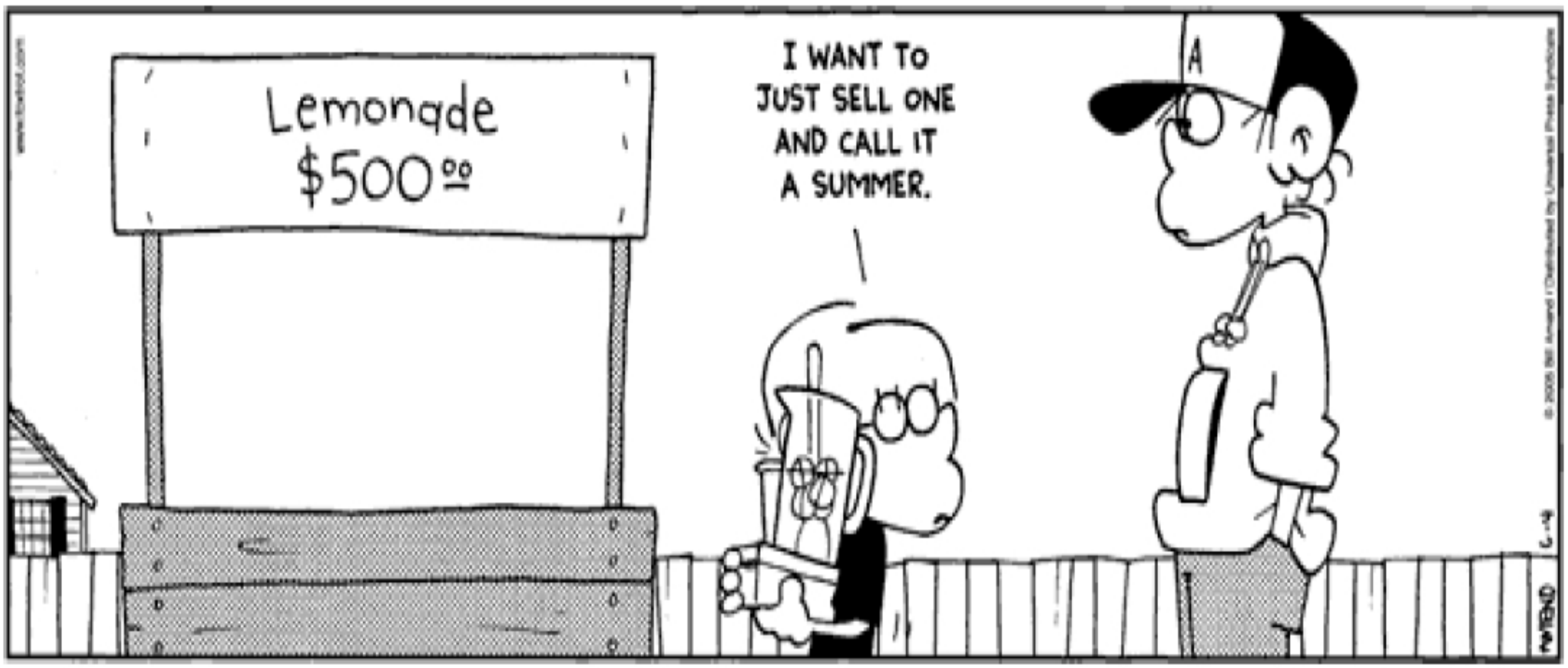
What's Next?

- beating worst-case POA bounds: want to reach a non-worst-case equilibrium
  - because of learning dynamics [Charikar/Karloff/Mathieu/Naor/Saks 08], [Kleinberg/Pilouras/Tardos 09], etc.
  - from modest intervention [Balcan/Blum/Mansour], etc.
- POA bounds for auctions
  - practical auctions often lack "dominant strategies" (sponsored search, combinatorial auctions, etc.)
  - want bounds on their (Bayes-Nash) equilibria [Christodoulou et al 08], [Paes Leme/Tardos 10], [Bhawalkar/Roughgarden 11], [Hassadim et al 11]

Key Points

- smoothness: a "canonical way" to bound the price of anarchy (for pure equilibria)
- robust POA bounds: smoothness bounds extend automatically beyond Nash equilibria
- tightness: smoothness bounds provably give optimal POA bounds in fundamental cases
- extensions: approximate equilibria; best-response dynamics; local smoothness for correlated equilibria; also Bayes-Nash eq

Reasoning About Auctions



Competitive Analysis Fails

Observation: which auction (e.g., opening bid) is best depends on the (unknown) input.

- e.g., opening bid of $0.01 or $10 better?

Competitive analysis: compare your revenue to that obtained by an omniscient opponent.

Problem: fails miserably in this context.

- predicts that all auctions are equally terrible
- novel analysis framework needed

A New Analysis Framework

Prior-independent analysis framework: [Hartline/ Roughgarden STOC 08, EC 09] compare revenue to that of opponent with *statistical information* about input.

Goal: design a distribution-independent auction that is always near-optimal for the underlying distribution (no matter what the distribution is).

- distribution over inputs not used in the *design* of the auction, only in its *analysis*

Bulow-Klemperer ('96)

Setup: single-item auction. Let F be a *known* valuation distribution. [Needs to be "regular".]

Theorem: [Bulow-Klemperer 96]: for every n:

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Interpretation: small increase in competition more important than running optimal auction.

- a "bicriteria bound"!

Bayesian Profit Maximization

Example: 1 bidder, 1 item, $v \sim$ known distribution F

- want to choose optimal reserve price p
- expected revenue of p: $p(1-F(p))$
  - given F, can solve for optimal p^*
  - e.g., $p^* = ½$ for $v \sim$ uniform[0,1]
- but: what about $n > 1$ (with i.i.d. v_i's)?

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- but: what about n >1 (with i.i.d. v_i's)?

need minor technical conditions on F

Theorem: [Myerson 81] auction with max expected revenue is second-price with above reserve p^*.

- note p^* is *independent of n*

Reformulation of BK Theorem

Theorem: [Bulow-Klemperer 96]: for every n:

Vickrey's revenue $\geq$ OPT's revenue

[with (n+1) i.i.d. bidders] [with n i.i.d. bidders]

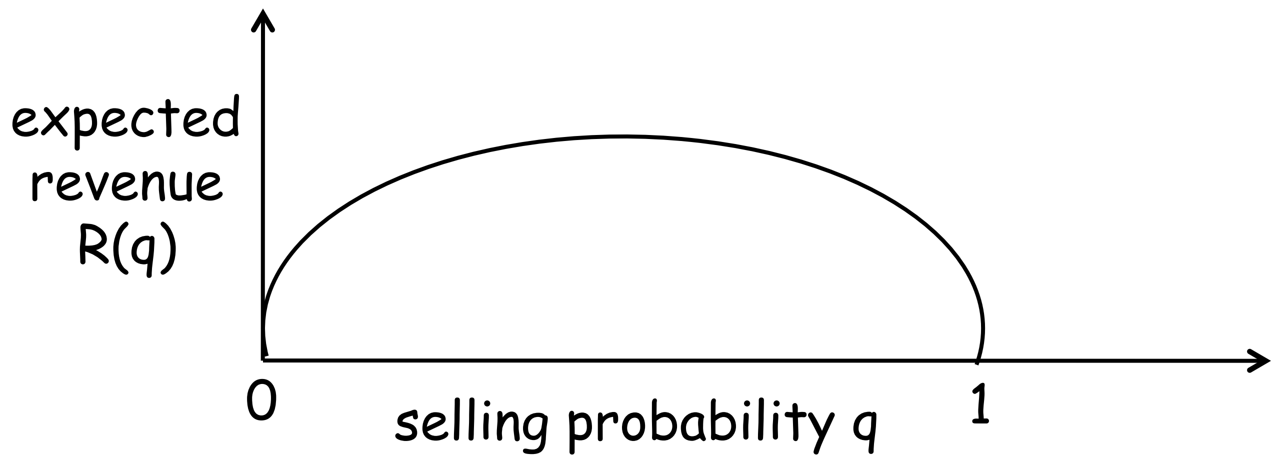
Lemma: if true for n=1, then true for all n.

- relevance of OPT reserve price decreases with n

Reformulation for n=1 case:

2 x Vickrey's revenue with n=1 and random reserve [drawn from F] $\geq$ Vickrey's revenue with n=1 and opt reserve r^*

Proof of BK Theorem



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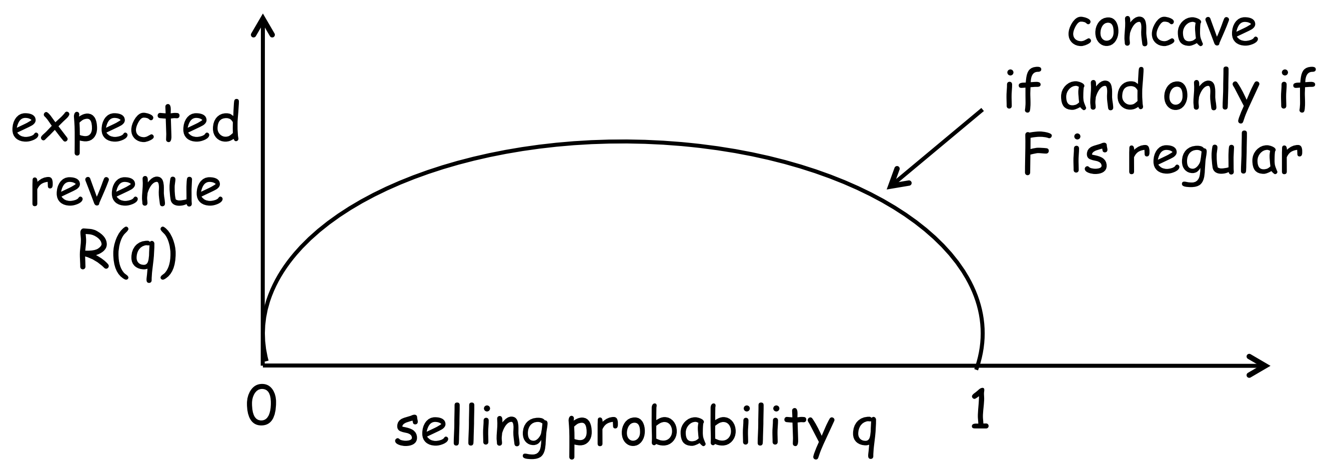

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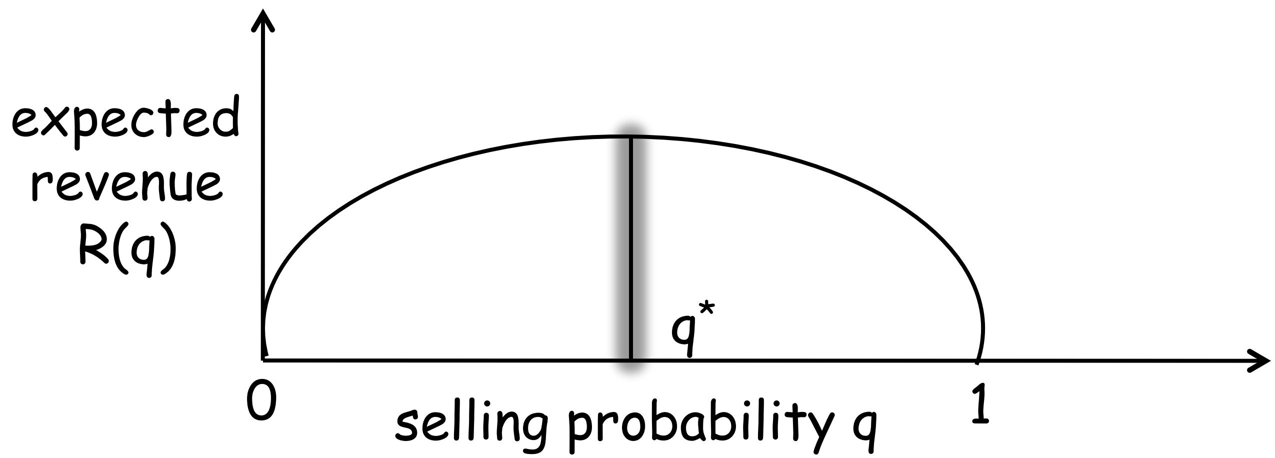

- opt revenue = $R(q^*)$

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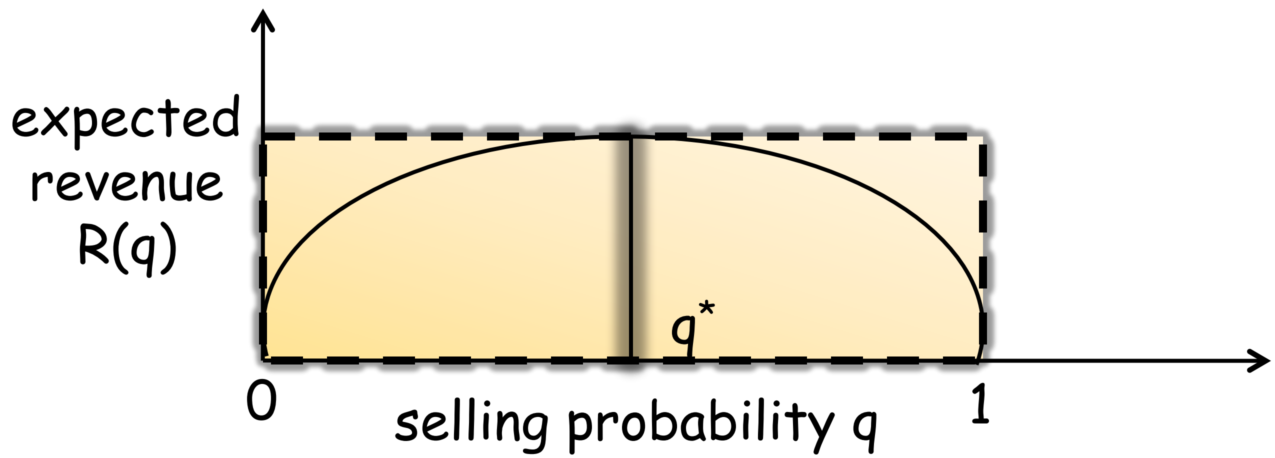

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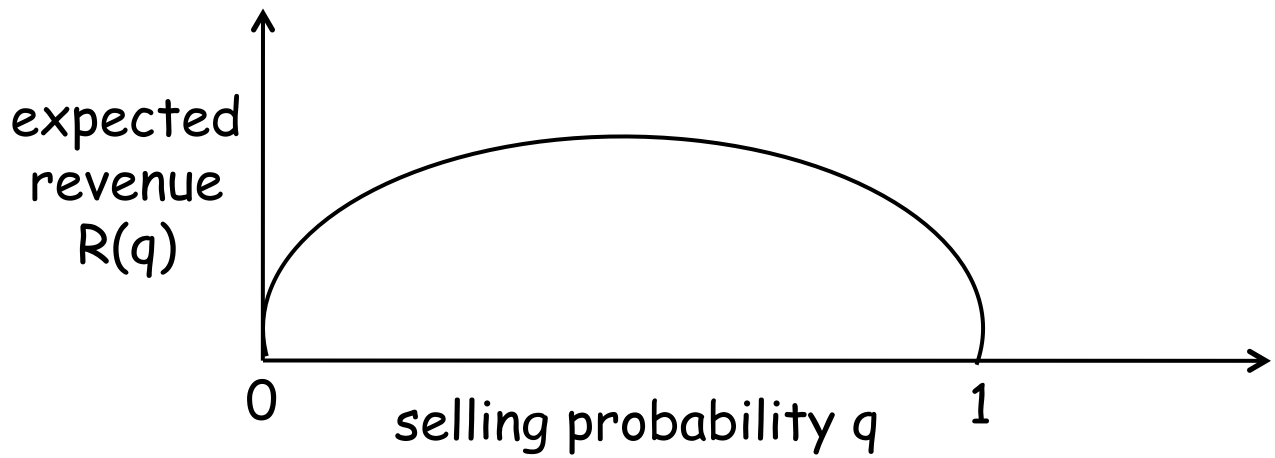

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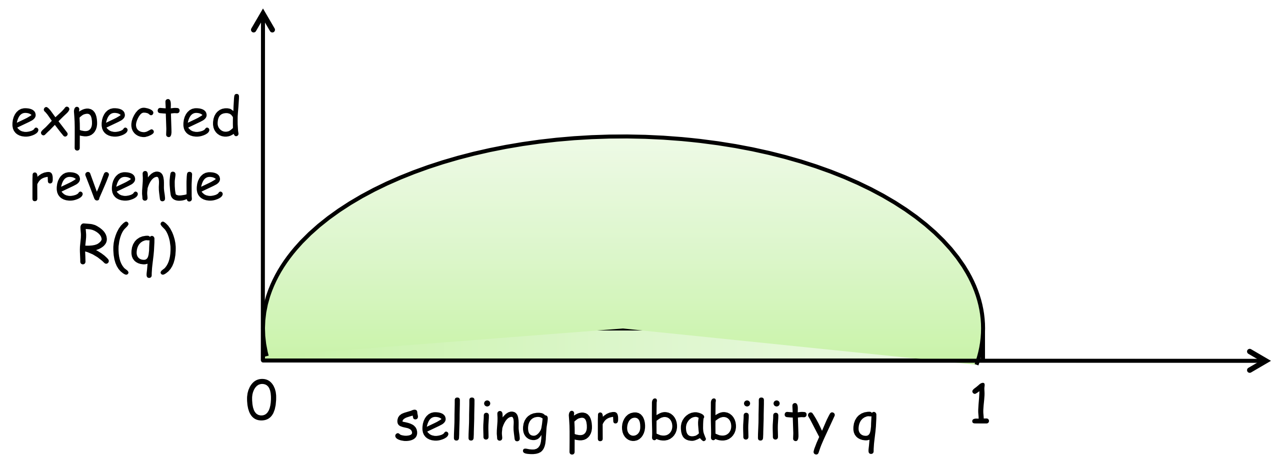

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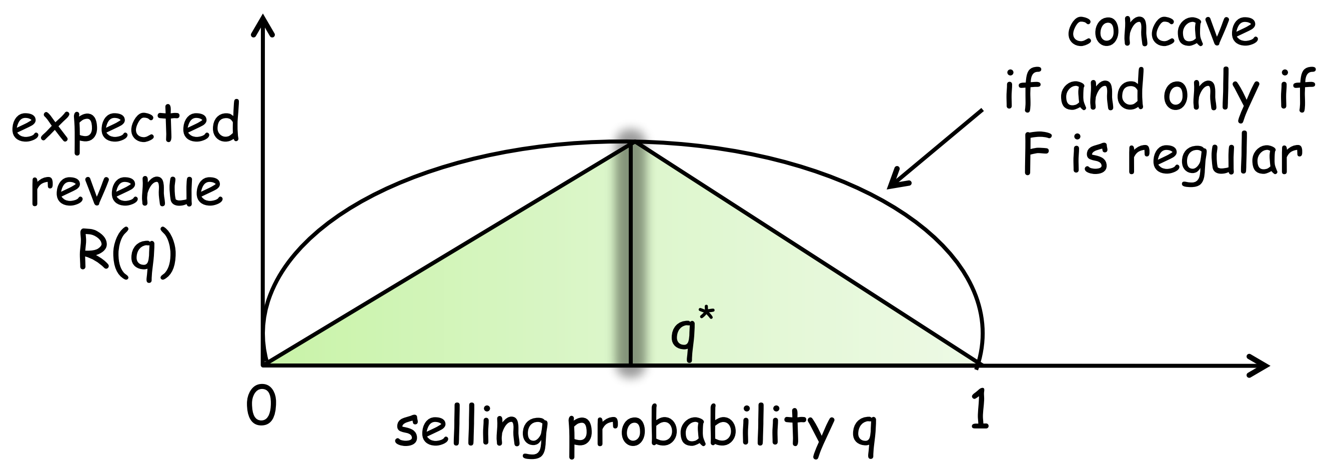

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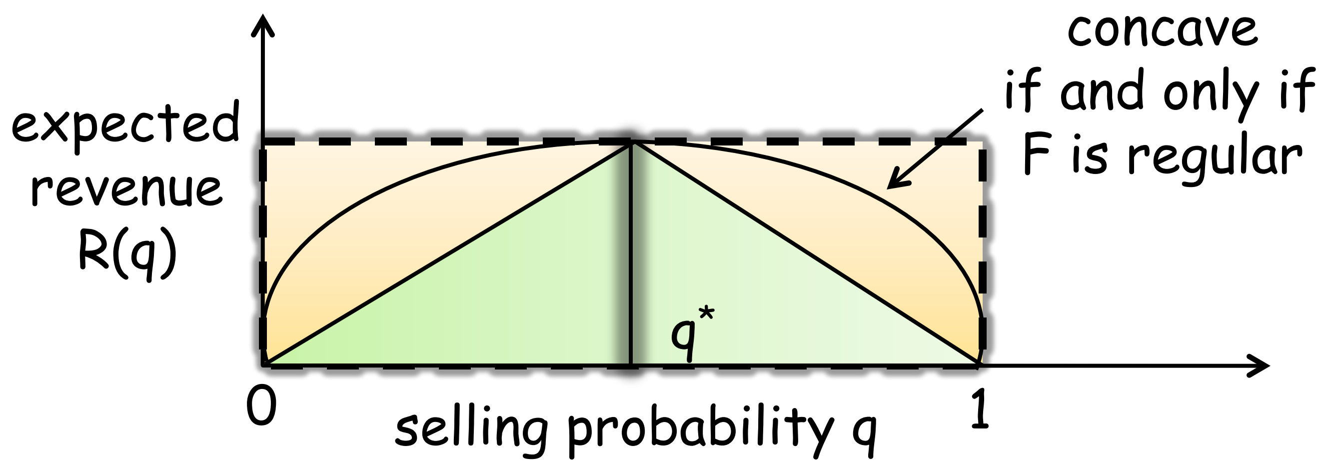

- opt revenue = R(q*)
- revenue of random reserve r (from F) = expected value of R(q) for q uniform in [0,1] = area under revenue curve ≥ ½ ∘ R(q*)

Recent Progress

BK theorem: the "prior-free" Vickrey auction with extra bidder as good as optimal (w.r.t. F) mechanism, no matter what F is.

More general "bicriteria bounds": [Hartline/Roughgarden EC 09], [Dughmi/Roughgarden/Sundararajan EC 09]

Prior-independent approximations: [Devanur/Hartline EC 09], [Dhangwotnotai/Roughgarden/Yan EC 10], [Hartline/Yan EC 11]

What's Next?

Take-home points:

- standard competitive analysis useless for worst-case revenue maximization
- but can get *simultaneous* competitive guarantee with all Bayesian-optimal auctions

Future Directions:

- thoroughly understand “single-parameter” problems, include non "downward-closed" ones
- non-i.i.d. settings
- multi-parameter? (e.g., combinatorial auctions)

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Epilogue

Higher-Level Moral: worst-case approximation guarantees as powerful "intellectual export" to other fields (e.g., game theory).

- many reasons for approximation (not just computational complexity)

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THANKS!