Quantifying Inefficiency in Games and Mechanisms

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Talk Themes

- many economic concepts directly relevant for reasoning about applications in computer science
  - Shapley value, correlated equilibria, etc.

- tools from computer science can yield new insights into basic economic models
  - transportation networks, cost-sharing, etc.
  - using approximation to reason about efficiency loss

- shared concern: theory to guide design
  - traditional approaches: axiomatic, optimization
  - here: minimize worst-case efficiency loss
Example: one unit of traffic wants to go from $s$ to $t$

- $c(x) = x$ (cost depends on congestion)
- $c(x) = 1$ (no congestion effects)

Question: what will selfish drivers do?
- assume everyone wants smallest-possible cost
- [Pigou 1920]
Equilibrium in Pigou’s Example

**Claim:** all traffic will take the top link.

**Reason:**
- $\varepsilon > 0 \Rightarrow$ traffic on bottom is envious
- $\varepsilon = 0 \Rightarrow$ equilibrium
  - all traffic incurs one unit of cost
Can We Do Better?

Consider instead: traffic split equally

Improvement:
• half of traffic has cost 1 (same as before)
• half of traffic has cost $\frac{1}{2}$ (much improved!)
• “price of anarchy” [Kousoupias/Papadimitriou 99] = $\frac{4}{3}$
Braess’s Paradox

Initial Network:

\[ \text{Cost} = 1.5 \]
Braess’s Paradox

Initial Network:

Augmented Network:

Cost = 1.5

Now what?
Braess’s Paradox

Initial Network:

Augmented Network:

Price of anarchy = 4/3 in augmented network  (again!)
Bad Example:

\[
\begin{array}{ccc}
  & \chi^d & \\
\hline
s & 1 & 1 - \epsilon \\
\hline
t & \epsilon & 0 \\
\end{array}
\]

(d large)

equilibrium has cost 1, \textit{min cost} \to 0

\[\Rightarrow\] price of anarchy unbounded as \(d \to \infty\)

\textbf{Goal:} weakest-possible conditions under which the price of anarchy is small.
When Is the Price of Anarchy Bounded?

Examples so far:

Hope: imposing additional structure on the cost functions helps
- worry: bad things happen in larger networks
Defn: affine cost function is of form $c_e(x) = a_e x + b_e$

Theorem: [Roughgarden/Tardos 00] for every network with affine cost functions:

$$\text{cost of eq flow} \leq \frac{4}{3} \times \text{cost of opt flow}$$
The Potential Function

Easy fact: [BMW 56] Nash flows minimize “potential function” \[ \sum_e \int_0^{f_e} c_e(x) dx \] (over all flows).

Proof: FOC + convexity.

Corollary: for affine cost functions:

- cost, potential functions differ by factor of \( \leq 2 \)
- gives upper bound of 2 on price on anarchy

\[
C(f) \leq 2 \times PF(f) \leq 2 \times PF(f^*) \leq 2 \times C(f^*)
\]
General Cost Functions

Theorem: [Roughgarden 02], [Correa/Schulz/Stier Moses 03] fix any set of cost functions. Then, a Pigou-like example --- 2 nodes, 2 links, 1 link w/ a constant cost function --- achieves the worst P.O.A.
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Take-away: worst-case inefficiency governed by cost function nonlinearity, not network structure.
Benefit of Overprovisioning

M/M/1 Cost Functions: \( c(x) = \frac{1}{u_e - x} \)

Suppose: network is overprovisioned by \( \beta > 0 \) (i.e., \( \beta \) fraction of each edge unused). \( c_e(f_e) \)

Then: Price of anarchy is at most \( \frac{1}{2}(1+1/\sqrt{\beta}) \)

• arbitrary network size/topology, traffic matrix

Moral: Even modest (10%) over-provisioning sufficient for near-optimal routing.
Outline

1. The price of anarchy in routing games.

2. Learnable equilibria: robust POA bounds.
   1. Connections to learning in games.
   2. POA bounds: the next generation.


5. Simple auctions for complex settings.
Meaning of a POA bound: *if* the game is at an equilibrium, *then* outcome is near-optimal.

Problem: what if can’t reach an equilibrium?

• non-existence (pure Nash equilibria)
• intractability (mixed Nash equilibria) [Daskalakis/Goldberg/Papadimitriou 06], [Chen/Deng/Teng 06], [Etessami/Yannakakis 07]
• hard to learn Nash equilibria [Hart/Mas-Colell 03], ...

Worry: fail to converge, so POA bound doesn’t apply.
Learnable Equilibria

**Fact:** simple strategies converge quickly to more permissive equilibrium sets.

- correlated equilibria: [Foster/Vohra 97], [Fudenberg/Levine 99], [Hart/Mas-Colell 00], ...

- coarse/weak correlated equilibria (of [Moulin/Vial 78]): [Hannan 57], [Littlestone/Warmuth 94], ...
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**Question:** are there good “robust” POA bounds, which hold more generally for such “easily learned” equilibria?

[Mirrokni/Vetta 04], [Goemans/Mirrokni/Vetta 05], [Awerbuch/Azar/Epstein/Mirrokni/Skopalik 08], [Christodoulou/Koutsoupias 05], [Blum/Even-Dar/Ligett 06], [Blum/Hajiaghayi/Ligett/Roth 08]
A Hierarchy of Equilibria

Recall: POA determined by *worst* equilibrium (only increases with the equilibrium set).
POA Bounds Without Convergence

Theorem: [Roughgarden 2009] most known POA bounds hold even for all coarse correlated equilibria.

Part I: [extension theorem] every POA bound proved for pure Nash equilibria in a prescribed way extends automatically, with no quantitative loss, to all no-regret outcomes.

• eludes non-existence/intractability critiques.

Part II: most known POA bounds were proved in this way (so extension theorem applies).
Extension Theorems

permissive equilibrium concept (e.g., no-regret outcomes)

what we care about
Extension Theorems

pure Nash equilibria
what’s easy to analyze

easier

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Extension Theorems

- Pure Nash equilibria: what’s easy to analyze
- POA extension theorem: easier
- Permissive equilibrium concept (e.g., no-regret outcomes): what we care about
Outline

1. The price of anarchy in routing games.

2. Learnable equilibria: robust POA bounds.

   1. Inefficiency in mechanism design.
   2. Designing to minimize worst-case efficiency loss.


5. Simple auctions for complex settings.
E.g., [Deb/Razzolini 99], [Moulin 99]

- player \( i \) has valuation \( v_i \) for winning
- welfare of \( S = v(S) - C(S) \) \([\text{where } v(S) = \sum_i v_i]\)
- \( C(\emptyset) = 0, C(S) = 1 \) if \( S \neq \emptyset \)

**Constraints:** want a dominant strategy IC + IR, budget-balanced mechanism, no positive transfers.

- [Green/Laffont 79] efficiency loss inevitable

**Design goal:** mechanism with minimum worst-case loss.
Equal-Share Mechanism

The Mechanism: (which is BB+DSIC, even GSP)

• collects bids ($b_i$ for each $i$)
• initialize $S =$ all players
• if $b_i \geq 1/|S|$ for all $i$ in $S$, halt.
• else drop a player $i$ with $b_i < 1/|S|$ and iterate
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**Efficiency loss:**

- set \(v_i = (1/i) - \varepsilon\) for \(i=1,2,...,k\)
- max welfare \(\approx \ln k - 1\)
- mechanism welfare = 0
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- max welfare $\approx \ln k - 1$
- mechanism welfare $= 0$
Worst-Case Efficiency Loss

Theorems: [Moulin/Shenker 01, Roughgarden/Sundararajan 06]

• $\approx (\ln k - 1)$ is worst-case welfare loss.
• analogous results for general submodular ("diminishing marginal costs") functions
  • ex: $C(S) = \text{concave function of player weights in } S$
• analog of equal-split: cost shares $= \text{Shapley values}$
• worst-case welfare loss $= \text{value of potential function defined in } [\text{Hart/Mas-Colell 90}]$
  • corresponds also to a simple worst-case example
  • never worse than for a public excludable good
  • less severe as $C$ gets “more linear”
Can We Do Better?

Cost-sharing method: assigns cost share $\chi(i, S)$ to each player $i$ in $S$ (for every set $S$). (e.g., Shapley value)

- constraints: budget-balance, “cross-monotonicity”

Question: which cost-sharing method minimizes the worst-case efficiency loss (over all valuation profiles)?

Theorem: [Moulin/Shenker 01, Roughgarden/Sundararajan 06] the Shapley value is optimal (any submodular cost fn).

Extensions: [Dobzinski et al 08], [Juarez 13], …
Motivation: *attribution problems.*

- marketer or advertiser compares Q1 vs Q2 revenue
- suppose several variables changed in Q2
  - better targeting
  - better matching algorithms
  - stronger economy
- what percentage of change to attribute to each variable?
  - essentially a budget-balanced cost-sharing problem!

**Theorem:** [Sun/Sundararajan 11] axiomatic justification of Aumann-Shapley value for multi-linear revenue fns.
Outline

1. The price of anarchy in routing games.
2. Learnable equilibria: robust POA bounds.

4. Optimal cost sharing in routing games.
   1. Designing to minimize the POA.
   2. More magical properties of the Shapley value.

5. Simple auctions for complex settings.
Routing Games, Revisited

Weighted routing games: (w/finite number of players)

- player $i$ has origin $s_i$, destination $t_i$, chooses an $s_i$-$t_i$ path on which to route $w_i$ units of traffic
- cost on edge $e$: $c_e(f_e)f_e$, where $f_e =$ total weight using $e$

Design question: how to share joint cost among users?

Traditional answer: proportional to players’ weights.
Can We Do Better?

Proportional Cost Sharing:
• corresponds to a FIFO queueing policy [Shenker 95]
• pure Nash equilibria need not exist [Rosenthal 73]
• worst-case POA well understood; modest if cost functions “close to linear” [Awerbuch/Azar/Epstein 05], [Christodoulou/Koutsoupias 05], [Aland et al 06]

Design Questions:
• can different cost shares restore pure Nash equilibria?
• can different cost shares reduce the worst-case POA?
Theorem: [Kollias/Roughgarden 11] sharing costs using a weighted Shapley value ([Shapley 53], [Kalai/Samet 87]) induces a potential game ([Monderer/Shapley 96]).

- also, worst-case POA of unweighted Shapley value only slightly bigger than with proportional cost-sharing
Restoring Pure Equilibria

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- also, worst-case POA of unweighted Shapley value only slightly bigger than with proportional cost-sharing

**Theorem:** [Gopalakrishnan/Marden/Wierman 13] these are the only cost-sharing methods guaranteed to induce a pure Nash equilibrium in every weighted routing game!

- a potential is *necessary* for guaranteed existence of PNE
- extension of [Chen/Roughgarden/Valiant 08]
Minimizing the POA

**Theorem:** [Gkatzelis/Kollias/Roughgarden 14] among all budget-balanced cost-sharing methods that guarantee existence of pure Nash equilibria, the Shapley value minimizes the worst-case POA in weighted routing games.

- slightly worse POA than with proportional cost-sharing

**Theorem:** [Gkatzelis/Kollias/Roughgarden 14] among all budget-balanced cost-sharing methods, proportional cost shares minimize the worst-case POA in weighted routing games.
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Motivating Question

**Question:** When can simple auctions perform well in complex settings?

**Example:** welfare maximization with multiple non-identical goods (combinatorial auctions).

- theoretically optimal: VCG mechanism
- simple: selling items separately
  - when is equilibrium welfare close to optimal?
- example interpretation: is package bidding essential to good combinatorial auction designs?
The POA of Simple Auctions

[Christodoulou/Kovacs/Schapira 08], [Lucier/Borodin 10],
[Paes Leme/Tardos 10], [Bhawalkar/Roughgarden 11],
[Hassidim/Kaplan/Mansour/Nisan 11], [Lucier/Paes Leme 11],
[Caragiannis/Kaklamanis/Kanellopoulos/Kyropoulou 11],
[Lucier/Singer/Syrgkanis/Tardos 11], [Markakis/Telelis 12],
[Paes Leme/Syrgkanis/Tardos 12], [Bhawalkar/Roughgarden 12],
[Feldman/Fu/Gravin/Lucier 13], [Syrgkanis/Tardos 13],
de Keijzer/Markakis/Schaefer/Telelis 13],
[Duetting/Henzinger/Starnberger 13],
[Babaioff/Lucier/Nisan/Paes Leme 13],
[Devanur/Morgenstern/Syrgkanis 13], …
The High-Order Bits

- incomplete-info games
  - i.e., uncertain payoffs
- mixed Bayes-Nash equilibria

what we care about
(e.g., for auctions)
The High-Order Bits

- **full-information games**
  - i.e., certain payoffs
  - pure Nash equilibria
  - what’s easy to analyze

- **incomplete-info games**
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  - mixed Bayes-Nash equilibria
  - what we care about (e.g., for auctions)

Easier
The High-Order Bits

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- **incomplete-info games**
  - i.e., uncertain payoffs
  - mixed Bayes-Nash equilibria
  - what we care about (e.g., for auctions)

- **POA extension theorem**
  - easier
The High-Order Bits

- extension theorems for Bayes-Nash equilibria: [Roughgarden 12], [Syrgkanis/Tardos 13]
Concluding Remarks

• reasoning about inefficiency through approximation gives new insights into fundamental economic models
  • try applying these ideas to your favorite model!

• good bounds for many games of interest, even for easy-to-learn equilibria

• crisp advice for designing mechanisms and systems
  • overprovisioning communication networks
  • the many magical properties of the Shapley value