Potentials and Approximation
(2008 Shapley Lecture)

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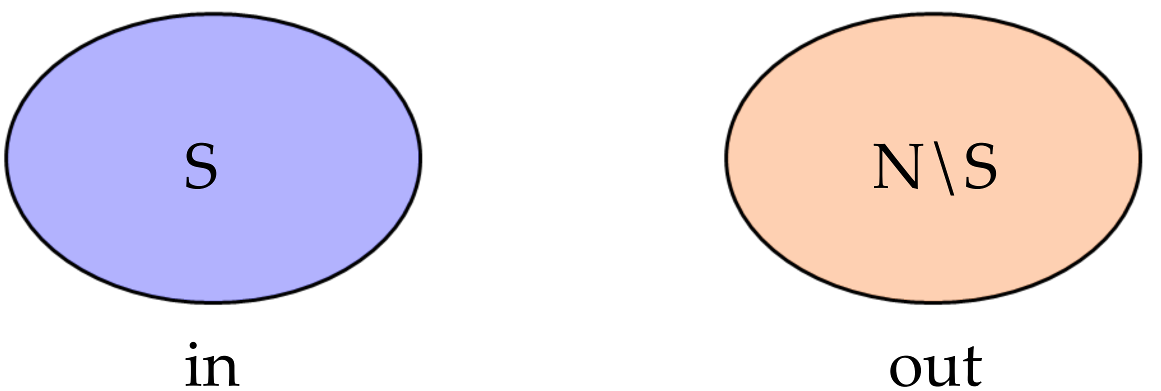
Fixed-Cost Participation Game

[Monderer/Shapley GEB 1996]:

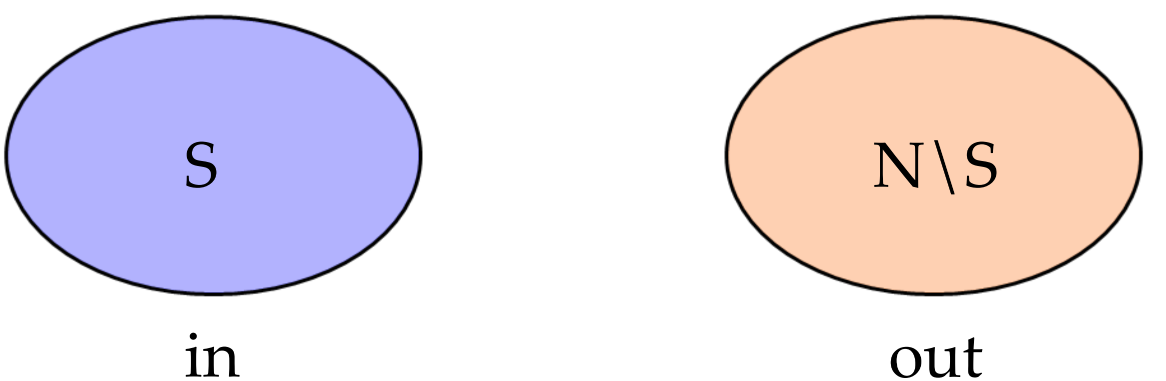
- n players, 2 strategies each ("in" or "out")
- i's cost of "out" = b_i (a constant)
- joint cost of "in" players S: $C(S) = 1$ (if $S \neq \emptyset$)
- "in" players split joint cost equally



General Participation Game

[Monderer/Shapley GEB 1996]:

- n players, 2 strategies each ("in" or "out")
- i's cost of "out" = b_i (a constant)
- joint cost of "in" players S: C(S)
- "value" φ splits joint cost $[\sum_{i \in S} \varphi(i, S) = C(S)]$



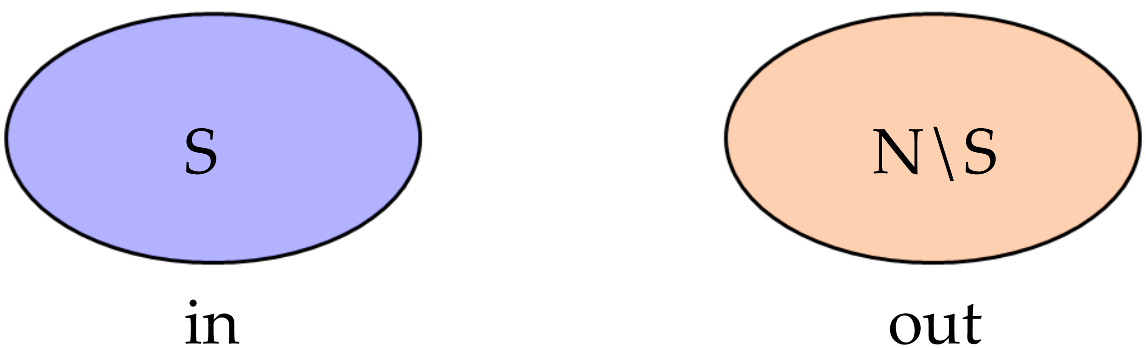
Potential for Fixed-Cost Game

Define: a potential function $P(S) = f(S) - \sum_{i \in S} b_i$

- where $f(S) = 1+1/2+1/3+\ldots+1/|S|$ [denoted $H_{|S|}$]

Key point: $\Delta P = \Delta cost_i$ for every player i

- Corollary 1: a pure Nash equilibrium exists
- Corollary 2: better-reply dynamics converge



General Potential Argument

Assume: $\varphi(i, S)$ is Shapley value of game (S,C)

Define: a potential function $P(S) = f(S) - \sum_{i \in S} b_i$

- where $f(S) = \sum_i \varphi(i, S_i)$
- S_i = first i players in a fixed, arbitrary ordering
- well-defined by [Hart/Mas-Colell Econometrica 89]

Again: $\Delta P = \Delta cost_i$ for every player i

  - same existence, convergence corollaries

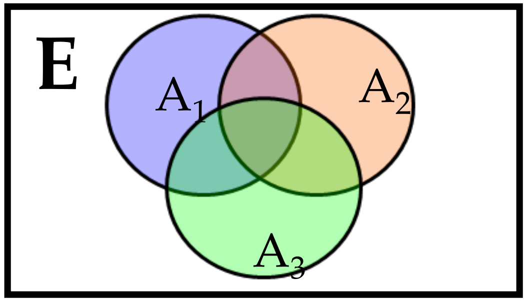
Talk Outline

- quantifying inefficiency in congestion games
  - governed by proximity of potential, objective fns
  - [Roughgarden/Tardos 02, Roughgarden 03]
- inefficiency in cost-sharing mechanisms
  - ascending auction as local search for potential fn
  - [Roughgarden/Sundararajan 06]
- which values always yield pure equilibria?
  - "concatenations" of weighted Shapley values
  - [Chen/Roughgarden/Valiant 08]

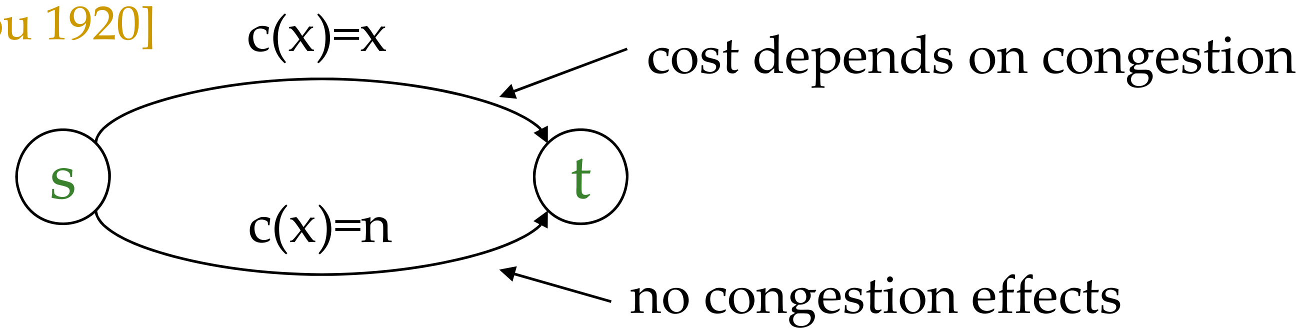
Congestion Games [Rosenthal 73]

Model: ground set E (resources, network links, etc.)

- players N, strategy sets = subsets of 2^E
- cost function c_e per e ϵ E
  - $c_e(x_e)$ = per-player cost (x_e players)
  - i's cost: $\sum_{e \in A} c_e(x_e)$



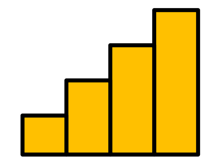
[Pigou 1920]



Congestion + Participation Games

Potential function: (S_e = players using e)

$$P(A_1,\ldots,A_n) = \sum_{e \in E} f_e(|S_e|) \quad [f_e(x) = \sum_{i=1}^{x} c_e(i)]$$



"Moral reason": view each e as participation game

- strategy A = games to participate in (all b_i's = 0)
- joint cost $C_e(S_e) = c_e(|S_e|)|S_e|$
- shared via Shapley value ($c_e(|S_e|)$ per player)

Nonatomic Congestion Games

- continuum of players (strategy sets $\subset 2^E$)
- cost function c_e per $e \in E$
  - $c_e(x_e)$ = per-player cost (x_e=fraction of players using e)
  - i's cost: $\sum_{e \in A} c_e(x_e)$
- potential function: $\sum_{e \in E} f_e(x_e)$ $[f_e(x_e) = \int_0^x c_e(y)dy]$

Assume: c_e's are continuous, nondecreasing.

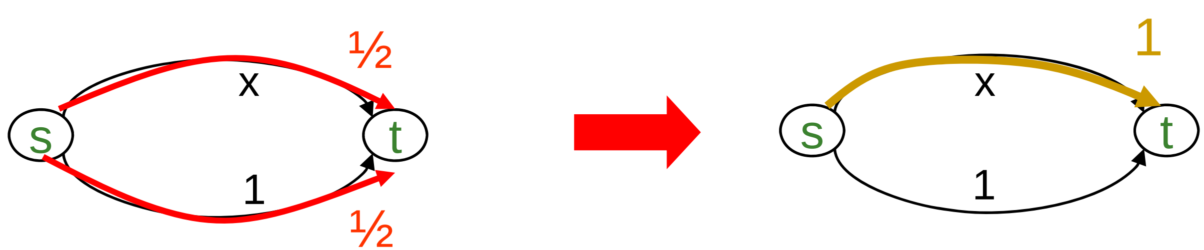
- equilibria are global potential minimizers, are payoff-equivalent [Wardrop 52], [Beckman/McGuire/Winsten 56]

The Price of Anarchy

Defn: price of anarchy of a game $= \dfrac{\text{obj fn value of equilibrium}}{\text{optimal obj fn value}}$

- definition from [Koutsoupias/Papadimitriou 99]

Example: POA = 4/3 in Pigou's example



Cost = 3/4 Cost = 1

Potentials + the Price of Anarchy

Example: affine cost fns $[c_e(x_e) = a_e x_e + b_e]$

Compare cost + potential function:

$$\text{cost}(\mathbf{X}) = \Sigma_e x_e \bullet c_e(x_e) = \Sigma_e [a_e x_e^2 + b_e x_e]$$

$$P(\mathbf{X}) = \Sigma_e \int_0^{x_e} c_e(y)dy = \Sigma_e [(a_e x_e^2)/2 + b_e x_e]$$

- cost, potential fns differ by factor of ≤ 2
- gives upper bound of 2 on price on anarchy:

$$C(x^{EQ}) \leq 2\times PF(x^{EQ}) \leq 2\times PF(x^{OPT}) \leq 2\times C(x^{OPT})$$

Price of Anarchy: Tight Bounds

Theorem: [Roughgarden/Tardos 02] POA is at most 4/3 in every nonatomic congestion game with affine cost fns. [Pigou's example is the worst!]

Theorem: [Roughgarden 03] fix any set of cost fns. Then, a Pigou-like example (2 nodes, 2 links, 1 link w/constant cost fn) achieves largest POA among all nonatomic congestion games.

- quartic functions: worst-case POA ≈ 2
- 10% extra "capacity": worst-case POA ≈ 2

Public Excludable Good

- player i has valuation v_i for winning
- surplus of S = v(S) - C(S) [where $v(S) = \Sigma_i v_i$]
- $c(ø)=0$, $c(S) = 1$ if $S \neq ø$

Constraints: want a dominant-strategy IC + IR, budget-balanced mechanism.

- [Green/Laffont 79]: efficiency loss inevitable

Design goal: mechanism with smallest-possible worst-case surplus loss (over all v).

The Shapley Value Mechanism

Shapley value mechanism: simulate ascending auction; use prices $1/|S|$ in iteration with remaining players S. [Moulin 99, Moulin/Shenker 01]

Fact: dominant-strategy IC + IR, budget-balanced.

- also "groupstrategyproof" (NTU coalitions)

Surplus loss: k players with $v_i = (1/i) - \varepsilon$

- mechanism's surplus = 0
- full surplus $\approx H_k - 1$

Efficiency Loss + Potentials

Interpretation: Shapley value mechanism as local search to maximize potential: v(S) - $H_{|S|}$

- recall surplus = v(S) - C(S)

Worst-case surplus loss: [assume optimal S is N]

- initially [S = U]: potential ≥ surplus – (H_n - 1)
- always [any S]: potential ≤ surplus
- potential only increases => worst-case surplus loss is (H_n - 1)

General Cost Functions

Fact: Shapley value mechanism is IR, IC, + BB for every submodular cost functions.

- minimizes worst efficiency loss among mechanisms based on ascending auctions [Moulin/Shenker 01]
- and strategyproof mechanisms satisfying "weak symmetry" [Dobzinski/Mehta/Roughgarden/Sundararajan 08]

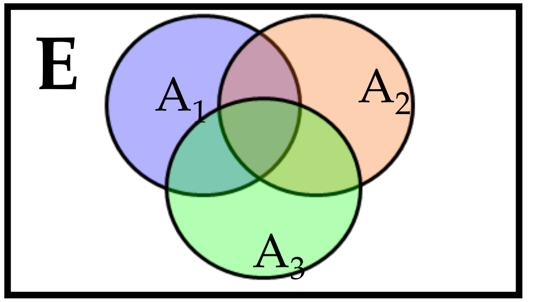
Non-submodular cost fns: [e.g., facility location]

- can't use Shapley value mechanism (not strategyproof)
- analyze efficiency loss via "order-dependent" potentials [Roughgarden/Sundararajan 06]

A Cost Allocation Game

Model: ground set E (resources, network links, etc.)

- each has fixed, unit cost
- (asymmetric) players N
- strategy sets $\subset 2^E$

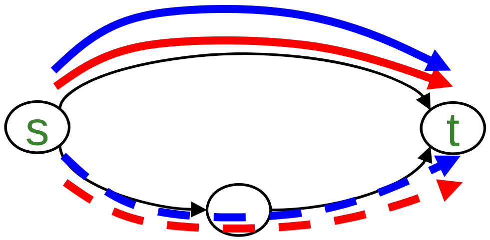


Design space: "value" φ s.t. $\sum_{i \in S} \varphi(i, S) = 1$ for all S

- Players + strategies + φ => full-info game G_φ

Note: get a congestion game (for any E + strategy sets) if and only if φ is the Shapley value.

An Example



[2 symmetric players]

[unit fixed-costs]

Examples:

- φ = Shapley: 2 PNE [both above or both below]
- φ = *sequential*: $\varphi(1, \{1,2\}) = 1$; $\varphi(2, \{1,2\}) = 0$
  - i.e., player 2 can free ride on player 1
  - unique PNE [both players above]

The Search for Pure Equilibria

Question: for what φ is G_φ guaranteed to have a pure-strategy Nash equilibrium (PNE)?

- should hold for every ground set + strategy sets
- original motivation: network protocol design

Examples:

- φ = Shapley [=> have a potential => have a PNE]
- also φ = *sequential* w.r.t. ordering π of N
  - $\varphi(i,S) = 1$ if i first player of S w.r.t. π, 0 otherwise
  - PNE exist (iterated removal of dominated strategies)

Potential for Weighted Shapley

Claim: [Shapley 53, Hart/Mas-Colell 89, Monderer/Shapley 96] if φ_w = weighted Shapley value (any $w > 0$), then G_φ always has a PNE.

Proof idea:

- underlying participation game has a weighted potential (i.e., $\Delta P = w_i \cdot \Delta c_i$ for every i)
- extends to all cost allocation games by adding
- building on [Kalai/Samet 87]: can view P as E[max{exponential RVs with $\lambda_i = 1/w_i$}]

Concatenation

Definition: For two values φ_1, φ_2 for disjoint player sets N_1,N_2 the *concatenation* of φ_1 and φ_2 is:

- if S contained in N_2, use φ_2
- else use φ_1 for players of $N_1 \cap S$, 0 for others

Notes:

- Sequential = concatenation of 1-player values.
- If φ_1,φ_2 always induce a PNE, so does concatenation.
- If φ_1,φ_2 always induce potentials, concatenation induces "lexicographically ordered potential".

Characterization

Theorem: [Chen/Roughgarden/Valiant 08] a value φ always induces a game G_φ with a PNE *if and only if* φ is the concatenation of $\varphi_{w1},\ldots,\varphi_{wk}$ for some weight vectors $w_1,\ldots,w_k > 0$.

Application: identify φ that minimizes worst-case equilibrium efficiency loss (over all induced G_φ).

- φ = Shapley is optimal in *directed* networks
- φ = sequential is optimal in *undirected* networks

Taste of Proof

1st Milestone: if a positive value φ always induces a game G_φ with a PNE, then is φ monotone: φ(i,S) only decreases with S.

Step 1: failures of monotonicity are symmetric (i makes j worse off => converse also holds).

- basic reason: else can encode matching pennies

Step 2: no (symmetric) failures of monotonicity.

- basic reason: otherwise contradict budget-balance (sum of all cost shares fixed)

Take-Home Points

Potential functions:

- historically used for existence of, converge to equilibria [Rosenthal 73, Monderer/Shapley 96]
- also imply worst-case efficiency loss guarantees
  - pure Nash equilibria in congestion games, etc.
  - budget-balance cost-sharing mechanisms

Approximation:

- second-best as interesting as first-best!
- designing for a good second-best solution