CS364B: Exercise Set #1

Due by the beginning of class on Wednesday, January 15, 2014

Instructions:

(1) Turn in your solutions to all of the following exercises directly to the TA (Okke). Please type your solutions if possible and feel free to use the LaTeX template provided on the course home page. Email your solutions to `cs364b-win1314-submissions@cs.stanford.edu`. If you prefer to hand-write your solutions, you can give it to Okke in person at the start of the lecture.

(2) Your solutions will be graded on a "check/minus/zero" system, with "check" meaning satisfactory and "minus" meaning needs improvement.

(3) Solve these exercises and write up your solutions on your own. You may, however, discuss the exercises verbally at a high level with other students. You may also consult any books, papers, and Internet resources that you wish. And of course, you are encouraged to contact the course staff (via Piazza or office hours) to clarify the questions and the course material.

(4) No late assignments will be accepted.

Lecture 1 Exercises

Exercise 1

Recall the English auction from lecture, with k identical goods and unit-demand bidders. An *action* of a player is a function from the iteration t and the auction's history-so-far (i.e., the sets $S_0, S_1, \ldots, S_{t-1}$ of active bidders in previous iterations) to a decision as to whether or not to remain in the auction. Once the bidder drops out of the auction, it cannot re-enter. A *strategy* is a function from a player's valuation to an action. *Sincere bidding* is the strategy that, at iteration t, remains in the auction if and only if the current price ϵt is at most the bidder's valuation v_i.

Prove that sincere bidding is an ϵ-dominant strategy for player i: no matter what strategies other players use, no unilateral deviation by i can increase its payoff by more than ϵ.

Exercise 2

Consider again the English auction with k identical goods and unit-demand bidders. Prove that if every bidder bids sincerely, then the auction's outcome has surplus within $k\epsilon$ of the maximum possible.

Exercise 3

Consider Scenario #2 from lecture: m non-identical goods and bidders with additive valuations. Prove that sincere bidding in m simultaneous English auctions is an $(m\epsilon)$-ex post Nash equilibrium. That is: for every valuation profile $\mathbf{v}$, if all bidders other than i bid sincerely, then no strategy of bidder i can ever improve over the payoff of sincere bidding by more than $m\epsilon$.

Exercise 4

Prove that applying the Revelation Principle[1] to an EPIC mechanism yields an equivalent direct-revelation DSIC mechanism.

[1] See CS364A, Lecture 4, Section 3.3.

More formally, consider bidders with valuation spaces $V_1, \ldots, V_n$. Let M be a mechanism with action sets $A_1, \ldots, A_n$ such that for each v_i there is a "sincere bidding action" $s_i(v_i) \in A_i$. Assume that the strategy profile induced by sincere bidding is an ex post Nash equilibrium of M. Prove that there is a direct-revelation DSIC mechanism M' such that, for every valuation profile $\mathbf{v}$, the direct-revelation outcome of M' is the same as the sincere-bidding outcome of M.

Lecture 2 Exercises

Exercise 5

Consider m non-identical goods and n unit-demand bidders. Call (M, q) an *ϵ-approximate Walrasian equilibrium* if unsold goods have price 0, every bidder i has non-negative utility $v_i(M(i)) - q(M(i))$, and every bidder receives a good within ϵ of its favorite: $v_i(M(i)) - q(M(i)) \geq v_i(\ell) - q(\ell) - \epsilon$ for every good ℓ.

Prove an approximate version of the First Welfare Theorem: if (M, q) is an ϵ-approximate Walrasian equilibrium, then the maximum-possible surplus of a matching is at most $\min\{m, n\} \cdot \epsilon$ more than that of M.

Exercise 6

Consider m non-identical goods and n unit-demand bidders. Prove the "mix and match" lemma: if (M^1, q^1) and (M^2, q^2) are Walrasian equilibria, then so are (M^2, q^1) and (M^1, q^2).

[Hint: use the First Welfare Theorem, which implies that M^1 and M^2 are both surplus-maximizing allocations.]

Exercise 7

Consider m non-identical goods and n unit-demand bidders. Prove that the Walrasian equilibrium price vectors form a *lattice*. That is, if p^1 and p^2 are WE price vectors, than so are the price vectors $p^{\min}$ and $p^{\max}$ obtained by taking the minimum or maximum of p^1 and p^2 (component-wise).

Exercise 8

Consider m non-identical goods and n unit-demand bidders. Suppose there is a surplus-maximizing allocation in which bidder i receives good j. Then, after duplicating the good j (i.e., adding a copy j' with $v_{kj} = v_{kj'}$ for all players k), there is still a surplus-maximizing allocation in which bidder i receives the good j.

[Hint: The intuition is that if you can improve surplus by reassigning i to free up both copies of good j for other bidders, then you can equally well reassign i to improve the surplus of the original allocation.

One way to make this precise is the following; you are also free to use a different argument. Let M and M' be the old and the new surplus-maximizing matchings, and assume that i is matched to neither j nor j' in M'. Prove that the symmetric difference of M and M' is a collection of vertex-disjoint cycles and paths. Prove that i must participate in such a path or cycle. What does the optimality of M' imply about the edge weights on these paths and cycles? Can you use such a path or cycle to improve the surplus of M when there is only one copy of j (a contradiction)?]