A PROOF OF TOPOLOGICAL COMPLETENESS FOR S4 IN (0,1)

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Completeness of modal logic $S4$ for all topological spaces as well as for real line $\mathbb{R}$, $\mathbb{R}^n$ and segments $(0,1)$ etc. (with $\Box$ interpreted as interior) was proved by McKinsey and Tarski [2]. Simplified proofs in [1], Section 5 and [4], Chapter 9 contain gaps. A new proof presented here combines the ideas in [3], [4] and [1] and provides a further simplification.

Let $\mathscr{B}$ be the set of all infinite binary sequences $\vec{b} = b_1 b_2 \ldots$ ($b_i \in \{0,1\}$) except identical 0^ω and sequences ending in a tail of 1's; let $\mathscr{B}_1 = \{\vec{b} \in \mathscr{B} \mid (\exists i)(\forall j > i)\vec{b}(j) = 0\}$, $\mathscr{B}_2 = \mathscr{B} \setminus \mathscr{B}_1$. One-to-one correspondence between $\mathscr{B}$ and real interval $(0,1)$ is given by $\mathbf{real}(\vec{b}) = \sum_{i=1}^{\infty} \vec{b}(i)2^{-i}$. Define $B(x) =$ the unique $\vec{b} \in \mathscr{B}$ such that $\mathbf{real}(\vec{b}) = x$. Let $\mathbf{K} = \langle W, R \rangle$ be a finite Kripke $S4$-model with root w_0. Let $\mathcal{K}$ be the topological space with the carrier W and the base of open sets $\mathcal{O}_w = \{w' \in W \mid Rww'\}$ for all $w \in W$. Define a labeling $\mathcal{W}$ of finite binary sequences $\bar{b} = b_1 b_2 \ldots b_n$, $n \geq 1$ by worlds $w \in W$ as follows. $\mathcal{W}(\emptyset) = w_0$; If $\mathcal{W}(\bar{b}) = w$, no extension of $\bar{b}$ is yet labeled and $w, w_1, \ldots, w_m$ are all R-successors of $w \in W$, then let $\mathcal{W}(\bar{b}0^i) = w$ for all $0 < i \leq 2m$, $\mathcal{W}(\bar{b}0^{2i-1}1) = w_i$ for $0 < i \leq m$, and $\mathcal{W}(\bar{b}0^{2i}1) = w$ for $0 \leq i < m$. For $\vec{b} \in \mathscr{B}$ let $\vec{b} \restriction n = \vec{b}(1)\vec{b}(2)\ldots\vec{b}(n)$, $\lambda(\vec{b}) =$ the least $n \geq 1$ $[(\forall i, j \geq n) R\mathcal{W}(\vec{b} \restriction i)\mathcal{W}(\vec{b} \restriction j)]$, $\rho(\vec{b}) = max(1,n)$, if $\vec{b} = \vec{b}(1)\vec{b}(2)\ldots\vec{b}(n)10^\omega \in \mathscr{B}_1$; $\rho(\vec{b}) = \lambda(\vec{b})$, if $\vec{b} \in \mathscr{B}_2$, and $\pi(x) = \mathcal{W}(B(x) \restriction \rho(B(x)))$ for $x \in (0,1)$.

THEOREM 0.1. *The function* $\pi : (0,1) \to \mathcal{K}$ *is a continuous and open mapping.*

PROOF SKETCH. Define $\delta(\vec{b}) = max(1,n)$, if $\vec{b} = \vec{b}(1)\vec{b}(2)\ldots\vec{b}(n)10^\omega \in \mathscr{B}_1$; $\delta(\vec{b}) =$ the least $n > \lambda(\vec{b})(\vec{b}(n) = 1$ and $\vec{b}(n-1) = 0)$, if $\vec{b} \in \mathscr{B}_2$. Then for any $x, y \in (0,1)$, $|x-y| < 2^{-(\delta(B(x))+2)}$ implies $R\pi(x)\pi(y)$. Hence π is continuous. On the other hand for every $x \in (0,1)$, $w \in W$ with $R\pi(x)w$ and any $\epsilon > 0$, there exists $y \in (0,1)$ such that $|y-x| < \epsilon$ and $\pi(y) = w$. Hence π is open. $\dashv$

Since it is well-known that the class of finite rooted Kripke models is complete for $S4$, we have

COROLLARY 0.1. *$S4$ is complete for topological semantics in* $(0,1)$.

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