Propositional Logic of Continuous Transformations in Cantor Space

Grigori Mints
Philosophy Department, Stanford University
Ting Zhang
Computer Science Department, Stanford University

1 Introduction

A well-known axiomatization of the basic notions of general topology in the form of propositional logic S4 was given by McKinsey and (for mathematically more interesting spaces like real line, real plane etc.) by McKinsey and Tarski. If only open sets are considered instead of arbitrary sets, one gets intuitionistic logic. L.E.J. Brower (of fixed point theorem) who created the background for this logic, stressed the connection of his principles with continuity considerations. This line of investigation leads to the theory of toposes and other connections with mainstream mathematics. A topological space X often acquires more interesting structures when it is the domain of a dynamical topological system, that is, a pair $\langle X, T\rangle$ where X is the topological space and T is a continuous transformation on X. Dynamical topological logic [4, 5] studies dynamical topological systems by logical means. We consider here propositional systems, since predicate extensions tend to be intractable, in particular non-axiomatizable. Propositional formulas are constructed from variables (atomic formulas) by Boolean connectives, necessity $\Box$ and a monadic operation $\bigcirc$. In a standard interpretation, variables represents subsets of X, Boolean connectives act in a natural way, $\Box$ is the interior and $\bigcirc$ is the pre-image under the operation T. Under this interpretation the axiom schema

$$\bigcirc\Box A \to \Box \bigcirc A \tag{C}$$

expresses continuity of T. The propositional system S4C includes S4, (C) and standard axioms relating $\bigcirc$ to Boolean connectives:

$$\bigcirc(A\&B) \iff \bigcirc A\& \bigcirc B, \qquad \bigcirc\neg A \iff \neg \bigcirc A.$$

Completeness of S4C for the class of all topological spaces has been proved in [1], in particular for finite spaces derived from Kripke models. These spaces do not satisfy topological separability axioms and are not very natural mathematically. We prove completeness of S4C for Cantor space, a space that is very popular in the theory of dynamical systems. P. Kremer pointed out that the real line is not complete for S4C.

In Section 2 we recapitulate relevant definitions and prove general results on embeddings of dynamic topological spaces. Corollary 2.1 ensures that a refuting assignment for a formula in a Kripke frame $\langle W, R, S\rangle$ can be translated to the Cantor space $\mathcal{B}$ provided there is a continuous and open map $\mathcal{W} : X \to W$ and a continuous transformation T on $\mathcal{B}$ satisfying *functoriality condition*:

$$\mathcal{W}T = S\mathcal{W} \qquad (\star)$$

The remaining part of the paper provides application of these results. Section 3 treats the "intuitionistic" case where transferring an operation S from a finite Kripke model to the Cantor space is straightforward. For a sequence $s \in \mathcal{B}$, $T(s)(n) = S(s(n))$. This construction does not extend immediately to the general case where *clusters* (non-singleton sets of R-equivalent worlds) can appear. No reasonable uniform choice of a representative of a whole cluster (as in [3]) seems to satisfy the functoriality condition $(\star)$, if a map $\mathcal{W} : \mathcal{B} \to W$ is defined as in [3] and T is defined coordinate-wise. Section 4 presents a more sophisticated construction for the Cantor space. Operation $T : \mathcal{B} \to \mathcal{B}$ is defined coordinate-wise with delays (Definition 4.5) to make T^{-1} defined on the range of T and traceable back in the sense that every chain

$$x \in \mathcal{B},\ T^{-1}(x),\ T^{-2}(x),\ \ldots$$

terminates at $T^{-n}(x) \notin \mathit{range}(T)$. After that it is possible to define $\mathcal{W}(x)$ for $x \notin \mathit{range}(T)$ as an R-stabilization point, and then to *define* $\mathcal{W}(x)$ for the remaining $x \in \mathcal{B}$ using the functoriality condition such that

$$\mathcal{W}(x) = S(\mathcal{W}(T^{-1}(x))).$$

2 Dynamic Topological Models

We use ***Int*** to denote the interior operator.

Definition 2.1 (Dynamic Topological Space) *A **dynamic topological space** is a pair*

$$\langle X, T\rangle,$$

where X is a topological space and T is a continuous function on X.

Definition 2.2 (Dynamic Topological Model) *A **dynamic topological model** is a triple*

$$\langle X, T, V\rangle,$$

where $\langle X, T\rangle$ is a dynamic topological space and V is a function assigning a subset of X to each propositional variable. The valuation V is extended to all S4C formulas as follows:

$$\begin{array}{rcl} V(\alpha \vee \beta) & = & V(\alpha) \cup V(\beta), \\ V(\alpha \,\&\, \beta) & = & V(\alpha) \cap V(\beta), \\ V(\bigcirc\alpha) & = & T^{-1}(V(\alpha)), \\ V(\neg\alpha) & = & X \setminus V(\alpha), \\ V(\Box\alpha) & = & \boldsymbol{Int}(V(\alpha)). \end{array}$$

We say that α is valid in a topological model M and write $M \models \alpha$ if and only if $V(\alpha) = X$.

Definition 2.3 (Dynamic Kripke Model) *A **dynamic Kripke frame** (for S4C) is a tuple $\boldsymbol{K} = \langle W, R, S\rangle$ where W is a non-empty set, R is a reflexive and transitive relation on W and $S : W \to W$ is a function monotonic with respect to R, i.e., wRw' implies $S(w)\, R\, S(w')$.*

*The elements in W are called **worlds**. We say that a world w is an R-**successor** of a world w' if wRw', and w is R-**equivalent** to w' (written $w \equiv_R w'$) if both wRw' and $w'Rw$. A dynamic Kripke frame is **rooted** if there exists a world $\mathbf{0}$ such that any world w in W is an R-successor of $\mathbf{0}$.*

*A **dynamic Kripke model** is a tuple $M = \langle W, R, S, V\rangle$ with $\langle W, R, S\rangle$ a dynamic Kripke frame and V a valuation function, which assigns a subset of worlds in W to every propositional variable. Validity relation $\models$ is defined recursively in the standard way. In particular,*

$$(M, w) \models \Box\alpha \quad \text{iff} \quad (M, w') \models \alpha \text{ for every } w' \text{ such that } wRw',$$

and

$$(M, w) \models \bigcirc\alpha \quad \text{iff} \quad (M, S(w)) \models \alpha.$$

*We say that a formula α is **valid** in M if and only if $(M, w) \models \alpha$ for every $w \in W$. A formula α is **valid** (written $\models \alpha$) if α is valid in every dynamic Kripke model.*

We can think of a dynamic Kripke frame as being a dynamic topological space by imposing a topology on it.

Definition 2.4 (Dynamic Kripke Space) *Let $\boldsymbol{K} = \langle W, R, S\rangle$ be a dynamic Kripke frame. The **dynamic Kripke space** on $\boldsymbol{K}$ is a dynamic topological space $\mathcal{K} = \langle\langle W, \mathcal{O}\rangle, S\rangle$ where $\langle W, \mathcal{O}\rangle$ is a topological space with the carrier W and open sets **closed** under R, i.e., for any $V \subseteq W$,*

$$V \in \mathcal{O} \quad \text{iff} \quad [(w \in V \text{ and } wRw') \text{ implies } w' \in V] \text{ for all } w, w' \in W.$$

This topology is given by basic neighborhoods:

$$W_w = \{w' \in W : wRw'\}$$

The soundness of above definition is well-known [2].

Theorem 2.1 *S is monotonic with respect to R if and only if S is continuous with respect to $\mathcal{O}$.*

It was shown by J. Davoren that $S4C$ is complete for finite rooted dynamic Kripke models [2].

Theorem 2.2 *For any $S4C$ formula α, $S4C \vdash \alpha$ if and only if α is valid in all finite rooted dynamic Kripke models.*

Definition 2.5 *Let $M_1 = \langle X_1, T_1\rangle$, $M_2 = \langle X_2, T_2\rangle$ be two dynamic topological spaces. We say a map $\mathcal{W}$ is a **dynamic topological functor** from M_1 to M_2 if*

1. *$\mathcal{W}$ is a continuous and open map from X_1 onto X_2, and*
2. *$\mathcal{W}(T_1(x)) = T_2(\mathcal{W}(x))$, that is, the following diagram commutes:*

$$\begin{array}{ccc} X_1 & \xrightarrow{\mathcal{W}} & X_2 \\ {\scriptstyle T_1}\downarrow & & \downarrow{\scriptstyle T_2} \\ X_1 & \xrightarrow[\mathcal{W}]{} & X_2 \end{array}$$

Lemma 2.1 *Let $M_1 = \langle X_1, T_1, V_1\rangle, M_2 = \langle X_2, T_2, V_2\rangle$ be two dynamic topological models. Suppose that $\mathcal{W} : M_1 \to M_2$ is a functor and for each propositional variable p,*

$$V_1(p) = \mathcal{W}^{-1}(V_2(p)).$$

Then

$$V_1(\alpha) = \mathcal{W}^{-1}(V_2(\alpha))$$

for any S4C-formula α.

Proof. By induction on α. The base case and induction steps for connectives $\vee, \&, \neg$ are straightforward. Now consider the remaining two cases: $\alpha \equiv \Box\beta$ and $\alpha \equiv \bigcirc\beta$.

- Case $\alpha \equiv \Box\beta$. We have

$$\begin{array}{rcll} V_1(\alpha) & = & V_1(\Box\beta) & \\ & = & \boldsymbol{Int}(V_1(\beta)) & \textit{by the definition of } V_1 \\ & = & \boldsymbol{Int}(\mathcal{W}^{-1}(V_2(\beta))) & \textit{by the induction hypothesis} \\ & = & \mathcal{W}^{-1}(\boldsymbol{Int}(V_2(\beta))) & \textit{by the continuity and openness of } \mathcal{W} \\ & = & \mathcal{W}^{-1}(V_2(\Box\beta)) & \textit{by the definition of } V_2 \\ & = & \mathcal{W}^{-1}(V_2(\alpha)). & \end{array}$$

- Case $\alpha \equiv \bigcirc\beta$. We need to show that $V_1(\bigcirc\beta) = \mathcal{W}^{-1}(V_2(\bigcirc\beta))$. Let $x \in X_1$. We have

$$\begin{array}{rcll} x \in V_1(\bigcirc\beta) & \Leftrightarrow & T_1(x) \in V_1(\beta) & \textit{by the definition of } V_1 \\ & \Leftrightarrow & T_1(x) \in \mathcal{W}^{-1}(V_2(\beta)) & \textit{by the induction hypothesis} \\ & \Leftrightarrow & \mathcal{W}(T_1(x)) \in V_2(\beta) & \\ & \Leftrightarrow & T_2(\mathcal{W}(x)) \in V_2(\beta) & \textit{since } \mathcal{W} \textit{ is a functor} \\ & \Leftrightarrow & \mathcal{W}(x) \in V_2(\bigcirc\beta) & \textit{by the definition of } V_2 \\ & \Leftrightarrow & x \in \mathcal{W}^{-1}(V_2(\bigcirc\beta)). & \end{array}$$

$\dashv$

Lemma 2.2 *Let $M_1 = \langle X_1, T_1, V_1\rangle, M_2 = \langle X_2, T_2, V_2\rangle$ be two dynamic topological models. Suppose that $\mathcal{W} : M_1 \to M_2$ is a functor and for each propositional variable p,*

$$V_1(p) = \mathcal{W}^{-1}(V_2(p)).$$

Then for any S4C-formula α,

$$M_2 \models \alpha \quad \textit{iff} \quad M_1 \models \alpha.$$

Proof. Suppose that $M_2 \models \alpha$, that is, $V_2(\alpha) = X_2$. By Lemma 2.1 $V_1(\alpha) = \mathcal{W}^{-1}(V_2(\alpha))$, and so $V_1(\alpha) = X_1$ as required. On the other hand suppose that $M_1 \models \alpha$, but $M_2 \not\models \alpha$, i.e., $V_2(\alpha) \neq X_2$. Since $\mathcal{W}$ is onto and $V_1(\alpha) = \mathcal{W}^{-1}(V_2(\alpha))$, we have $V_1(\alpha) \neq X_1$, that is, $M_1 \not\models \alpha$, a contradiction. $\dashv$

Corollary 2.1 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of dynamic Kripke models such that for every model $M_2 \in \mathcal{C}_2$ there is an $M_1 \in \mathcal{C}_1$ and a functor $\mathcal{W} : M_1 \to M_2$. Then, if $\mathcal{C}_2$ is complete for S4C, then $\mathcal{C}_1$ is also complete for S4C.*

Proof. If $M_2 \not\models \alpha$, then $M_1 \not\models \alpha$ by Lemma 2.2. $\dashv$

2.1 Root-preserving Dynamic Kripke Models

In this section we present a proof that an additional condition

$$T(\mathbf{0}) = \mathbf{0} \tag{1}$$

for dynamic Kripke models with a root $\mathbf{0}$ preserves completeness.

Theorem 2.3 ([2]) *A formula α is valid in all dynamic Kripke model iff it is valid in all models satisfying (1).*

Proof. To a given Kripke model M with a carrier W and a forcing relation $\models$, add a root $\mathbf{0}$ with the relation (1). Denote the new model by M' and its forcing relation by $\models'$. (The definition of $\mathbf{0} \models' p$ is not important.) By induction on formula α it is easy to prove (using that the accessibility relation R and the operation T are not changed for elements of W): for $w \in W$,

$$w \models \alpha \iff w \models' \alpha$$

Hence if α is refuted in M it is also refuted in M'. ⊣

From now on we assume condition (1) and another condition: a dynamic Kripke model has at least two worlds.

3 Completeness in a Simple Case

This section presents a simplified construction for the case where the Kripke frame does not contain clusters. We assume that a finite rooted dynamic Kripke frame $\boldsymbol{K} = \langle W, R, S\rangle$ is given such that

$$W = \{0, 1, \ldots, N-1\},\ N \geq 2.$$

Let

$$W^{mon} = \{s \in W^* \mid \forall n(s(n)Rs(n+1))\}$$

denotes the set of all R-monotone sequences of worlds in W.

Let $W^{f\ mon}$ be the set of all *finite* monotonic non-empty sequences of worlds in W. Let $\boldsymbol{lth}(b)$ denote the length of $b \in W^{f\ mon}$ so that

$$b = b(1) \ldots b(\boldsymbol{lth}(b)).$$

Define $\mathcal{W}^{\mathsf{f}} : W^* \to W$

$$\mathcal{W}^{\mathsf{f}}(b) = b(\boldsymbol{lth}(b)). \tag{2}$$

For W^{mon} consider *in this section* only the situation when R is *a partial ordering*, that is, wRw' and $w'Rw$ together implies $w = w'$. In this case every sequence $s \in W^{mon}$ stabilizes:

$$(\exists n_0)(\forall n \geq n_0)\ s(n) = s(n_0).$$

Denote minimal such n_0 by $n(s)$ and define $\mathcal{W} : W^\omega \to W$ by

$$\mathcal{W}(s) = s(n(s)).$$

Definition 3.1 *The topology in $W^{f\ mon}, W^{mon}$ is introduced in a standard way by the basic neighborhoods*

$$\mathcal{T}_b = \text{the set of sequences with a prefix } b.$$

Definition 3.2 *The operation S on a dynamic Kripke frame $\boldsymbol{K}$ is extended coordinate-wise to an operation $T^{\mathsf{f}} : W^{f\ mon} \to W^{f\ mon}$ and an operation $T : W^{mon} \to W^{mon}$, respectively.*

$$\begin{array}{lcll} T^{\mathsf{f}}(b(1)\dots b(n)) & = & S(b(1))\dots S(b(n)) & \text{for } b \in W^{f\ mon}, \\ (T(s))(n) & = & S(s(n)) & \text{for } s \in W^{mon}. \end{array}$$

Theorem 3.1 *1. Both operations $\mathcal{W}^{\mathsf{f}} : W^{f\ mon} \to W^{f\ mon}$ and $\mathcal{W} : W^{mon} \to W^{mon}$ are continuous and open.*

2. The pair $(\mathcal{W}^{\mathsf{f}}, T^{\mathsf{f}})$ for $W^{f\ mon}$ and the pair $(\mathcal{W}, T)$ for W^{mon} are functors from the respective spaces onto $\mathcal{K}$.

Proof.

1. It is easy to verify (cf. [3]) for $w = \mathcal{W}(b)$,

$$\begin{array}{lcl} \mathcal{W}^{\mathsf{f}}(\mathcal{T}_b) & = & W_w, \\ \mathcal{W}(\mathcal{T}_b) & = & W_w. \end{array}$$

2. For $b' = T^{\mathsf{f}}(b)$, it is easy to verify

$$\begin{array}{lcl} (T^{\mathsf{f}})^{-1}(\mathcal{T}_{b'}) & \supseteq & \mathcal{T}_b, \\ T^{-1}(\mathcal{T}_{b'}) & \supseteq & \mathcal{T}_b, \end{array}$$

using that T^{f} and T commute with concatenation $*$ of sequences in $W^{f\ mon}$ and W^{mon}, respectively:

$$\begin{array}{lcl} T^{\mathsf{f}}(b * c) & = & T^{\mathsf{f}}(b) * T^{\mathsf{f}}(c), \\ T(b * s) & = & T(b) * T(s). \end{array}$$

Functoriality relation $\mathcal{W}^{\mathsf{f}} T^{\mathsf{f}} = S \mathcal{W}^{\mathsf{f}}$ is easy for $W^{f\ mon}$: if $n = \boldsymbol{lth}(b)$, then $n = \boldsymbol{lth}(T(b))$ and hence

$$\mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b)) = (T^{\mathsf{f}}(b))(n) = S(b(n)) = S(\mathcal{W}^{\mathsf{f}}(b)).$$

For W^{mon} note that the inequality $n(T(s)) \leq n(s)$ holds for stabilization points. Hence

$$\mathcal{W}(T(s)) = (T(s))(n(T(s))) = (T(s))(n(s)) = S(s(n(s))) = S(\mathcal{W}(s))$$

as required. ⊣

Theorem 3.2 *S4C is complete for $W^{f\ mon}$, as well as for the partially ordered W^{mon}.*

Proof. By Corollary 2.1. ⊣

Unfortunately this proof does not work for the general case of W^{mon}. Let us try to define the map $\mathcal{W}$ in the same way as in [3]:

$\mathcal{W}(s)$=some fixed representative of a cluster C such that $s(n) \in C$ for all sufficiently large n.

Consider a two-element cluster $W = \{0, 1\}$ with $S(i) = 1 - i$ and choose 0 as a representative of the cluster. Then

$$\mathcal{W}(T(1^\omega)) = \mathcal{W}(0^\omega) = 0, \qquad S(\mathcal{W}(0^\omega)) = S(0) = 1.$$

4 Cantor Space and Kripke Models

4.1 Cantor Space

Let $\mathcal{T} = \{0,1\}^\omega$ be the full infinite binary tree where each node in the tree is identified by a finite prefix (binary-word) from the root Λ to it. We use b and $\boldsymbol{b}$ to denote finite prefixes and infinite paths respectively. We write $\mathcal{T}_{\langle b\rangle}$ for the subtree rooted at node b. Let $\mathcal{B}$ be the standard Cantor space represented by $\mathcal{T}$, where each element of $\mathcal{B}$ is identified with an infinite path (an infinite binary word). For each $\boldsymbol{b} \in \mathcal{B}$, $\boldsymbol{b}\restriction n$ denotes the prefix of length n, i.e., the finite prefix $\boldsymbol{b}\restriction n = \boldsymbol{b}(1)\ldots\boldsymbol{b}(n)$. We write $\boldsymbol{b}_1 \equiv_n \boldsymbol{b}_2$ if $\boldsymbol{b}_1\restriction n = \boldsymbol{b}_2\restriction n$. One can imagine adding the component $\boldsymbol{b}(0) = 0$ to account for the root Λ, but we do not do that.

We partition $\mathcal{B}$ into two classes. The first class consists of all paths which end with 0^ω. The second class consists of paths which either ends with 1^ω or contains infinitely many 0's as well as infinitely many 1's.

Formally let

$$\mathcal{B}_1 = \{\boldsymbol{b} \in \mathcal{B} \mid \boldsymbol{b} = b0^\omega \text{ for some } b \in \{0,1\}^*\}, \quad \mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1.$$

We say $\langle \mathcal{B}, T\rangle$ is a dynamic Cantor space if T is a continuous function on $\mathcal{B}$.

4.2 Proof Plan

We shall show the construction of an open and continuous map $\mathcal{W} : \mathcal{B} \to W$ and a continuous map $T : \mathcal{B} \to \mathcal{B}$ such that the functoriality condition $(\star)$ is satisfied. To achieve this, we first define $\mathcal{W}^{\mathsf{f}} : \{0,1\}^* \to W$ and $T^{\mathsf{f}} : \{0,1\}^* \to \{0,1\}^*$ such that

$$\mathcal{W}^{\mathsf{f}} T^{\mathsf{f}} = S\mathcal{W}^{\mathsf{f}}. \tag{$\sharp$}$$

Then $\mathcal{W}$ and T are obtained by uplifting $\mathcal{W}^{\mathsf{f}}$ and T^{f} from $\{0,1\}^*$ to $\mathcal{B}$.

The proof outline is as follows. First, we define $\mathcal{W}^{\mathsf{f}}$ as the labeling function induced from unwinding $\boldsymbol{K}$ into $\mathcal{B}$ [3]. Second, using the coordinate-wise bisimulation between $\mathcal{B}$ and $\boldsymbol{K}$, T^{f} is defined such that $(\sharp)$ holds, and if $b \prec b'$, then $T^{\mathsf{f}}(b) \prec T^{\mathsf{f}}(b')$. Third, T is obtained as the limit function of T^{f}, using the fact that an infinite word can be viewed as the limit of its prefixes in the increasing order (with respect to $\prec$). At last, we lift $\mathcal{W}^{\mathsf{f}}$ to $\mathcal{W}$ by defining

$$\mathcal{W}(\boldsymbol{b}) = \mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(\boldsymbol{b})), \tag{$\ddagger$}$$

where $\boldsymbol{ch}$ is the *choice function* which select a finite representative $\boldsymbol{ch}(\boldsymbol{b})$ for each infinite word $\boldsymbol{b}$. To establish $(\star)$, it suffices to show

$$\boldsymbol{ch}(T(\boldsymbol{b})) = T^{\mathsf{f}}(\boldsymbol{ch}(\boldsymbol{b})), \tag{\S}$$

because

$$\begin{array}{rlcll} & \mathcal{W}(T(\boldsymbol{b})) & = & S\mathcal{W}(b) & \\ \Leftrightarrow & \mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(T(\boldsymbol{b}))) & = & S(\mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(\boldsymbol{b}))) & \textit{by } (\ddagger) \\ \Leftrightarrow & \mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(T(\boldsymbol{b}))) & = & \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(\boldsymbol{ch}(\boldsymbol{b}))). & \textit{by } (\sharp) \end{array}$$

For this we introduce the notions *segment rank*, *landmark rank* and *T-rank*, and show that T^{-1} is a well-founded relation. Then we are able to define $\boldsymbol{ch}$ inductively, such that for $\boldsymbol{b} \in range(T^{-1})$

$$\boldsymbol{ch}(\boldsymbol{b}) = T^{\mathsf{f}}(\boldsymbol{ch}(T^{-1}(\boldsymbol{b}))),$$

from which (§) follows.

4.3 The Labeling Function $\mathcal{W}^{\mathsf{f}}$

Let $\boldsymbol{K} = \langle W, R, S \rangle$ be a finite dynamic Kripke frame with the root $\mathbf{0}$, and $\mathcal{K}$ be the corresponding dynamic Kripke space. We use an unwinding technique similar to [3] to label every node of $\mathcal{T}$ by a world in W.

Definition 4.1 (Unwinding and Labeling) *We define a labeling function $\mathcal{W}^{\mathsf{f}} : \{0,1\}^* \to W$ recursively.*

1. $\mathcal{W}^{\mathsf{f}}(\Lambda) = \mathbf{0}$.
2. *Let $b \in \{0,1\}^*$ be a node in $\mathcal{B}$. Suppose b is already labeled by a world w_0 (i.e., $\mathcal{W}^{\mathsf{f}}(b) = w_0$) which has m R-successors $w_0, w_1, \ldots, w_{m-1}$. Then*

$$\mathcal{W}^{\mathsf{f}}(b0^i) = w_0 \quad \text{for } 0 < i \leq m, \tag{3}$$

$$\mathcal{W}^{\mathsf{f}}(b0^i1) = w_i \quad \text{for } 0 \leq i < m. \tag{4}$$

$\mathcal{T}_b^W$ denotes the infinite binary tree with root b labeled by elements of W as above. $\mathcal{T}^W$ stands for the labeled full infinite binary tree.

Proposition 4.1 *Let $b \in \{0,1\}^*$ be a node in $\mathcal{B}$. If $\mathcal{W}^{\mathsf{f}}(b)$ has m R-successors (including itself), then for any $k \geq 0$,*

$$\mathcal{W}^{\mathsf{f}}(b) = \mathcal{W}^{\mathsf{f}}(b0^k), \tag{5}$$

$$\mathcal{W}^{\mathsf{f}}(b1) = \mathcal{W}^{\mathsf{f}}(b0^{mk}1). \tag{6}$$

Proof. The relation (5) is obvious. For (6) assume that $b = b'0^i$ $(i \geq 0)$ and the label $\mathcal{W}^{\mathsf{f}}(b1) = w_i$ was obtained from labeling beginning with the node b' with $\mathcal{W}^{\mathsf{f}}(b') = \mathcal{W}^{\mathsf{f}}(b) = w_0$. Then the labeling is repeated beginning with the node $b'0^m$ such that

$$\mathcal{W}^{\mathsf{f}}(b'0^m0^j1) = w_j, \; 0 \leq j < m.$$

By induction on k, we obtain $\mathcal{W}^{\mathsf{f}}(b'0^{mk}0^j1) = w_j$. In particular for $j = i$, we have

$$\mathcal{W}^{\mathsf{f}}(b0^{mk}1) = \mathcal{W}^{\mathsf{f}}(b'0^{mk}0^i1) = w_i = \mathcal{W}^{\mathsf{f}}(b1). \qquad \dashv$$

Proposition 4.2 *Let $\mathcal{W}^{\mathsf{f}}(b) = w$. Then for any $w' \in W$ with wRw' there exist infinitely many $k \geq 0$ such that $\mathcal{W}^{\mathsf{f}}(b0^k1) = w'$.*

Proof. Let $w, w_1, \ldots, w_{m-1}$ $(m \geq 1)$ be all R-successors of w. By Definition 4.1 all R-successors of w will be enumerated by the sequence

$$\langle \mathcal{W}^{\mathsf{f}}(b0^i1) \mid 0 \leq i < m \rangle.$$

In particular, assume $\mathcal{W}^{\mathsf{f}}(b0^{k'}1) = w'$ for some k' such that $0 \leq k' < m$. By Proposition 4.1, for any j such that $j \geq 0$, we have

$$\mathcal{W}^{\mathsf{f}}(b0^{mj+k'}1) = \mathcal{W}^{\mathsf{f}}(b0^{k'}1) = w'.$$

So any R-successors of w will appear infinitely often as a label for a node ending in 1. $\dashv$

Definition 4.2 (Monotonic Sequences) *An infinite sequence* $\boldsymbol{b}$ *of worlds in* W *is* ***monotonic*** *(with respect to* R*) if* $R\boldsymbol{b}(i)\boldsymbol{b}(j)$ *holds for any* $i < j$*. We write* W^{mon} *for the set of all monotonic sequences in* W^{ω}*.*

By Definition 4.1 each path in $x \in \mathcal{B}$ is labeled by a monotonic sequence in W^{mon}. We write

$$\boldsymbol{W}(x) = \lambda n.\mathcal{W}(x \upharpoonright n) = \mathcal{W}(x \upharpoonright 1)\mathcal{W}(x \upharpoonright 2) \cdots .$$

Since $\boldsymbol{K}$ is a finite frame, each sequence in W^{mon} has a tail consisting of R-equivalent worlds. Indeed $\neg\big[(\boldsymbol{W}(x))(n+1)R(\boldsymbol{W}(x))(n)\big]$ is possible only for $|W|-1$ values of n.

Definition 4.3 (Stabilization Point) *We say that* $x \in \mathcal{B}$ ***stabilizes*** *at a node* b *if* $\mathcal{W}^{\mathsf{f}}(b) \equiv_R \mathcal{W}^{\mathsf{f}}(b')$ *for any prefix* b' *of* x *such that* $b \preceq b'$*. Define*

$$\boldsymbol{stb}(x) = \text{ the least prefix } b \text{ of } x \text{ such that } \mathcal{W}^{\mathsf{f}}(b) \equiv_R \mathcal{W}^{\mathsf{f}}(b') \text{ for any } b' \succeq b.$$

The points $x = b0^{\omega}$ are dense in $\mathcal{B}$. The definition of the map $\mathcal{W} : \mathcal{B} \to W$ for such points will have a special form, which motivates the following definition.

Definition 4.4 (Stopping Point) *For* $x \in B$ *define*

$$\boldsymbol{stp}(x) = \begin{cases} \boldsymbol{stb}(x) & \text{if } x \in \mathcal{B}_2, \\ 0 & \text{if } x = 0^{\omega}, \\ b1 & \text{if } x = b10^{\omega} \text{ for some } b \in \{0,1\}^*. \end{cases}$$

Proposition 4.3 *For any* $x \in \mathcal{B}$*,* $\boldsymbol{stb}(x) \preceq \boldsymbol{stp}(x)$*.*

Proof. If $x \in \mathcal{B}_2$ then by Definition 4.4 $\boldsymbol{stp}(x) = \boldsymbol{stb}(x)$. If $x = 0^{\omega}$, then $\boldsymbol{stb}(x) = \Lambda \prec 0 = \boldsymbol{stp}(x)$. If $x = b10^{\omega}$, then x stabilizes at or before $b1$. ⊣

4.4 The Function T^{f}

We obtain the function T^{f} using the coordinate-wise bisimulation between $\mathcal{B}$ and $\boldsymbol{K}$.

Definition 4.5 *Define a map* $T^{\mathsf{f}} : \{0,1\}^* \to \{0,1\}^*$ *inductively as follows.*

1. $T^{\mathsf{f}}(\Lambda) = \Lambda$.
2. *Suppose for* $b \in \{0,1\}^*$*,* $T^{\mathsf{f}}(b)$ *is already defined and satisfies*

$$\mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b)) = S(\mathcal{W}^{\mathsf{f}}(b)). \tag{7}$$

Suppose that the world $w = \mathcal{W}^{\mathsf{f}}(b)$ *has* m R*-successors (including* w*). Define*

$$\begin{aligned} T^{\mathsf{f}}(b0) &= T^{\mathsf{f}}(b)0^{n+m}0, & (8) \\ T^{\mathsf{f}}(b1) &= T^{\mathsf{f}}(b)0^{n+m}1, & (9) \end{aligned}$$

where n *is the least positive natural number satisfying*

$$\begin{aligned} \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b)0^n0) &= S(\mathcal{W}^{\mathsf{f}}(b0)), & (10) \\ \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b)0^n1) &= S(\mathcal{W}^{\mathsf{f}}(b1)). & (11) \end{aligned}$$

To see that n exists, first note that $\mathcal{W}^{\mathsf{f}}(b0^i) = \mathcal{W}^{\mathsf{f}}(b)$ for any i, and so for any n, we have

$$S(\mathcal{W}^{\mathsf{f}}(b0)) = S(\mathcal{W}^{\mathsf{f}}(b)) = \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b)) = \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b)0^n0).$$

Next by monotonicity of S we know that both $S(\mathcal{W}^{\mathsf{f}}(b0))$ and $S(\mathcal{W}^{\mathsf{f}}(b1))$ are successors of $\mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b))$. Now apply Proposition 4.2.

Proposition 4.4 *For any node $b \in \{0,1\}^*$, $T^{\mathsf{f}}(b)$ is defined and (7) is satisfied.*

Proof. By induction on the structure of the binary tree. For the base case, note that since $\mathcal{W}^{\mathsf{f}}(\Lambda) = \mathbf{0}$ and $S(\mathbf{0}) = \mathbf{0}$, we have

$$\mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(\Lambda)) = S(\mathcal{W}^{\mathsf{f}}(\Lambda)).$$

Suppose $\mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b)) = S(\mathcal{W}^{\mathsf{f}}(b))$. To show

$$\begin{aligned} \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b0)) &= S(\mathcal{W}^{\mathsf{f}}(b0)), \\ \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b1)) &= S(\mathcal{W}^{\mathsf{f}}(b1)), \end{aligned}$$

by (8), (9), we ought to show

$$\mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b)0^{n+m}0) = S(\mathcal{W}^{\mathsf{f}}(b0)), \tag{12}$$
$$\mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(b)0^{n+m}1) = S(\mathcal{W}^{\mathsf{f}}(b1)), \tag{13}$$

where m, n are as in Definition 4.5. By Proposition 4.1, (12) follows from (10) and (13) follows from (11). ⊣

Proposition 4.5 *For any non-empty $b \in \{0,1\}^*$, $\boldsymbol{lth}(T^{\mathsf{f}}(b)) > \boldsymbol{lth}(b)$.*

Proof. By induction on $\boldsymbol{lth}(b)$ with the following induction step ($i = 0$ *or* 1):

$$\begin{aligned} \boldsymbol{lth}(T^{\mathsf{f}}(bi)) &= \boldsymbol{lth}(T^{\mathsf{f}}(b)0^{n+m}i) && \textit{by the definition} \\ &= \boldsymbol{lth}(T^{\mathsf{f}}(b)) + n + m + 1 \\ &\geq \boldsymbol{lth}(b) + n + m + 1 && \textit{by the induction hypothesis} \\ &> \boldsymbol{lth}(b) + 1 && \textit{since } n + m > 0 \\ &= \boldsymbol{lth}(bi). \end{aligned}$$

⊣

Proposition 4.6 *T^{f} is strictly monotonic with respect to the prefix order of nodes: if $b_1 \prec b_2$, then $T^{\mathsf{f}}(b_1) \prec T^{\mathsf{f}}(b_2)$.*

Proof. Immediate from Definition 4.5. ⊣

Proposition 4.7 *T^{f} is injective.*

Proof. Let $c, d \in \{0,1\}^*$ be two distinct finite prefixes. If $c \prec d$, then by Proposition 4.6, we have $T^{\mathsf{f}}(c) \prec T^{\mathsf{f}}(d)$. Similarly we have $T^{\mathsf{f}}(d) \prec T^{\mathsf{f}}(c)$ if $d \prec c$. If none of c, d is a prefix of the other, then let b be the maximum common prefix of c and d. Without loss of generality we assume that $b0$ is a prefix of c and $b1$ is a prefix of d. Then $T^{\mathsf{f}}(b0)$ is of the form $T^{\mathsf{f}}(b)0^k0$ and is a prefix of $T^{\mathsf{f}}(c)$ while $T^{\mathsf{f}}(b1)$ is of the form $T^{\mathsf{f}}(b)0^k1$ and is a prefix of $T^{\mathsf{f}}(d)$. So $T^{\mathsf{f}}(c) \neq T^{\mathsf{f}}(d)$. ⊣

Definition 4.6 (Landmark) *We call a node $b \in \{0,1\}^*$ a **landmark** if b is in the range of T^{f}. If $b \prec \boldsymbol{b} \in \mathcal{B}$, then we say b is a landmark of $\boldsymbol{b}$.*

Proposition 4.8 *Let $b \in \{0,1\}^*$. $b0$ is a landmark if and only if $b1$ is.*

Proof. By Definition 4.5, we can assume that $(T^{\mathsf{f}})^{-1}(b0) = b'0$ for some $b' \in \{0,1\}^*$. Again by Definition 4.5, T^{f} always maps a pair of siblings to a pair of siblings. Hence $T^{\mathsf{f}}(b'1) = b1$. The reverse direction is similarly obtained. ⊣

Proposition 4.9 *If none of $b0^i$ $(0 < i < n)$ is a landmark, then none of $b0^i1$ $(0 \leq i < n-1)$ is a landmark.*

Proof. By Proposition 4.8. ⊣

Proposition 4.10 *For $b \in \{0,1\}^*$ the nodes $T^{\mathsf{f}}(b)$ and b contain the same number of 1's and $T^{\mathsf{f}}(b)$ contains no consecutive 1's.*

Proof. It follows immediately from Definition 4.5 by induction on the length of b. ⊣

Note 4.1 *Suppose that $b \in \{0,1\}^*$ contains n 1's. We can view b as a concatenation of $n+1$ (possibly empty) segments of 0's separated by 1. The node $T^{\mathsf{f}}(b)$ is obtained from b by inserting a non-zero number of 0's into every such segment.*

Recall that our goal is to show the well-foundedness of T^{-1} (Definition 4.8). To achieve this, we need first show the well-foundedness of $(T^{\mathsf{f}})^{-1}$.

Definition 4.7 (Segment Rank) *A **segment** of a finite node $b \in \{0,1\}^*$ is any maximum segment of consecutive 0's in b. Define the **segment rank** of b by*

$$\boldsymbol{srk}(b) = \text{ the length of the shortest segment of } b.$$

Proposition 4.11 *For $b \in \{0,1\}^*$, $\boldsymbol{srk}(b) < \boldsymbol{srk}(T^{\mathsf{f}}(b))$.*

Proof. It follows from Note 4.1 that $T^{\mathsf{f}}(b)$ is obtained from b by adding a non-zero number of 0's into each segment of b. ⊣

Proposition 4.12 (Well-foundedness of $(T^{\mathsf{f}})^{-1}$) *For any $b \in \{0,1\}^*$ there is no infinite chain of the form*

$$b,\ (T^{\mathsf{f}})^{-1}(b),\ (T^{\mathsf{f}})^{-2}(b),\ \ldots,\ (T^{\mathsf{f}})^{-n}(b),\ \ldots$$

In other words, there exists the least n such that $(T^{\mathsf{f}})^{-n}(b)$ is not in the range of T^{f}.

Proof. It follows from Proposition 4.11 that $\boldsymbol{srk}((T^{\mathsf{f}})^{-1}(b)) < \boldsymbol{srk}(b)$ and that the segment rank for any finite prefix can not be negative. ⊣

4.5 The Function T

Note that an infinite word can be viewed as the limit of the increasing sequence of its prefixes. Hence we can define T as the limit function of T^{f}.

Definition 4.8 *Define a binary relation T on $\mathcal{B}$:*

$$\langle x, y\rangle \in T \Leftrightarrow \text{for any finite prefix } b \text{ of } x,\ T^{\mathsf{f}}(b) \text{ is a prefix of } y\ : (\forall b \prec x)\, T^{\mathsf{f}}(b) \prec y.$$

It is easily seen that $\langle 0^\omega, 0^\omega\rangle \in T$.

Proposition 4.13 *T is a function on $\mathcal{B}$.*

Proof. **(Uniqueness.)** Suppose $\langle x, y\rangle \in T$ and $\langle x, y'\rangle \in T$. We prove that any finite prefix of y is a prefix of y', and vice versa. Let $b \prec y$. Consider a prefix $b' \prec x$ of the same length as b. By Definition 4.8, $T^{\mathsf{f}}(b') \prec y, T^{\mathsf{f}}(b') \prec y'$, and by Proposition 4.5, $\boldsymbol{lth}(b) < \boldsymbol{lth}(T^{\mathsf{f}}(b'))$. So we have $b \prec T^{\mathsf{f}}(b') \prec y'$, $b \prec T^{\mathsf{f}}(b') \prec y$, and hence $y = y'$.
(Totality.) By Proposition 4.6, for every prefix b of x, $T^{\mathsf{f}}(b)$ is a prefix of one and the same infinite path of $\mathcal{B}$. ⊣

Proposition 4.14 *T is injective.*

Proof. Let $x, y \in \mathcal{B}$ be two infinite paths. Suppose that $x \neq y$ and let b be the maximum finite common prefix of x and y. Without loss of generality we assume that $b0 \prec x$ and $b1 \prec y$. Then by Proposition 4.7 $T^{\mathsf{f}}(b0) \neq T^{\mathsf{f}}(b1)$. Since $T^{\mathsf{f}}(b0) \prec T(x)$ and $T^{\mathsf{f}}(b1) \prec T(y)$, we have $T(x) \neq T(y)$. ⊣

Proposition 4.15 *None of paths in $\mathcal{T}_{\langle 1\rangle}$ is in the range of T.*

Proof. Since $T^{\mathsf{f}}(0)$, $T^{\mathsf{f}}(1)$ both begin with 0, every $y \in \mathit{range}(T)$ begins with 0. ⊣

Next we show the well-foundedness of T^{-1}.

Definition 4.9 (Landmark Rank) *Define the **landmark rank** of a finite prefix $b \in \{0,1\}^*$ by*

$$\boldsymbol{lrk}(b) = \text{the least } n \text{ such that } (T^{\mathsf{f}})^{-n}(b) \text{ is not in the range of } T^{\mathsf{f}}.$$

In particular $\boldsymbol{lrk}$ always exists by Proposition 4.12 and $\boldsymbol{lrk}(b) = 0$ if b is not in the range of T^{f}.

Proposition 4.16 *For any $b \in \{0,1\}^*$, we have $\boldsymbol{lrk}(b) < \boldsymbol{lrk}(T^{\mathsf{f}}(b))$.*

Proof. By Definition 4.9. ⊣

Proposition 4.17 *Let $y = T(x)$ for some $x \in \mathcal{B}$ and b be a landmark prefix of y. Then $(T^{\mathsf{f}})^{-1}(b) \prec x$.*

Proof. Let b, b' be two consecutive landmarks, i.e., $b \prec b'$ and any b'' between b and b' is not a landmark. By Definition 4.5, $(T^{\mathsf{f}})^{-1}(b') = (T^{\mathsf{f}})^{-1}(b)i$ for $i = 0, 1$. By Definition 4.8, we can enumerate *a*ll landmarks in y by

$$T^{\mathsf{f}}(b_0),\ T^{\mathsf{f}}(b_1),\ \ldots,\ T^{\mathsf{f}}(b_n),\ \ldots,$$

where $b_0, b_1, \ldots, b_n, \ldots$ enumerates all prefixes of x in the increasing order. So if b is a landmark prefix of y, then $(T^{\mathsf{f}})^{-1}(b) \prec x$. ⊣

Proposition 4.18 *If $y = T(x)$ for some $x \in \mathcal{B}$, then all finite prefixes of y ending with 1 are landmarks.*

Proof. Assume $b1 \prec y$ is not a landmark. Take the maximum prefix $c \prec x$ such that $T^{\mathsf{f}}(c) \prec b1$. Such c exists as $T^{\mathsf{f}}(\Lambda) = \Lambda$. One of $T^{\mathsf{f}}(ci)$ $(i = 0, 1)$ is a prefix of y. It is different from $b1$, since $b1$ is not a landmark; it is longer than $b1$, since c is maximal. By definition 4.5, we have $T^{\mathsf{f}}(ci) = T^{\mathsf{f}}(c)0^{n+m}i$, and hence $T^{\mathsf{f}}(ci) \prec b1 \prec T^{\mathsf{f}}(c)0^{n+m}i$. But this is contradictory, as the last 1 in $b1$ appears in a segment of 0's. ⊣

The contrapositive of the above proposition says that if $y \in \mathcal{B}$ contains a finite non-landmark prefix ending with 1, then y is not in the range of T. Similar to Proposition 4.12, we have

Proposition 4.19 (Well-foundedness of T^{-1}) *If $y \in \mathcal{B}$ contains 1, then there is no infinite chain of the form*

$$y,\ T^{-1}(y),\ T^{-2}(y),\ \ldots,\ T^{-n}(y),\ \ldots$$

I.e., there exists the least n such that $T^{-n}(y)$ is not in the range of T.

Proof. We prove by contradiction. Suppose there exists such an infinite chain. Let b be a prefix of y ending with 1. By Proposition 4.18, the existence of $T^{-1}(y)$ implies that b is a landmark and hence $(T^{\mathsf{f}})^{-1}(b)$ is defined. By Proposition 4.16, we have $\boldsymbol{lrk}((T^{\mathsf{f}})^{-1}(b)) < \boldsymbol{lrk}(b)$. Moreover by Proposition 4.10, we know that $(T^{\mathsf{f}})^{-1}(b)$ ends with 1. By Proposition 4.17, $(T^{\mathsf{f}})^{-1}(b)$ is a prefix of $T^{-1}(y)$, and hence $T^{-1}(y)$ contains 1. Repeating the above argument, we have an infinite sequence of the form

$$\boldsymbol{lrk}(b) > \boldsymbol{lrk}((T^{\mathsf{f}})^{-1}(b)) > \boldsymbol{lrk}((T^{\mathsf{f}})^{-2}(b)) > \ldots > \boldsymbol{lrk}((T^{\mathsf{f}})^{-n}(b)) > \ldots$$

which is impossible as a landmark rank can not be negative. ⊣

Proposition 4.19 proves the soundness of the following definition.

Definition 4.10 (T-Rank) *Define the T-**rank** of an infinite path $x \in \mathcal{B}$ by*

$$\boldsymbol{trk}(x) = \begin{cases} 0 & \text{if } x = 0^\omega \text{ or } x \notin range(T), \\ \text{the least } n > 0\ [T^{-n}(y) \notin range(T)] & \text{otherwise.} \end{cases}$$

4.6 The Embedding Function $\mathcal{W}$

To lift $\mathcal{W}^{\mathsf{f}}$ from $\{0,1\}^*$ to $\mathcal{B}$, we need to select a finite representative for each infinite path.

Definition 4.11 (Choice Function) *Define the **choice function** by*

$$\boldsymbol{ch}(x) = \begin{cases} \boldsymbol{stp}(x) & \text{if } \boldsymbol{trk}(x) = 0, \\ T^{\mathsf{f}}(\boldsymbol{ch}(T^{-1}(x))) & \text{if } \boldsymbol{trk}(x) > 0. \end{cases}$$

Proposition 4.20 *If $x \in \mathcal{B}_1$, then $\boldsymbol{ch}(x) = \boldsymbol{stp}(x)$.*

Proof. By induction on $\boldsymbol{trk}(x)$. If $\boldsymbol{trk}(x) = 0$, then the statement follows directly from the first clause of Definition 4.11. Suppose that $\boldsymbol{trk}(x) > 0$. Then x must contain 1, since $\boldsymbol{trk}(0^\omega) = 0$. Suppose that $x = b'10^\omega$ and let $y = T^{-1}(x)$. By Proposition 4.18 $b'1$ is a landmark. Let

$(T^{\mathsf{f}})^{-1}(b'1) = b1$ for some $b \in \{0,1\}^*$. Then $y = b10^\omega$ and hence $y \in \mathcal{B}_1$. Since $\boldsymbol{trk}(y) < \boldsymbol{trk}(x)$, by the induction hypothesis $\boldsymbol{ch}(y) = \boldsymbol{stp}(y)$. In summary, we have

$$\begin{array}{rcll} \boldsymbol{ch}(x) & = & T^{\mathsf{f}}(\boldsymbol{ch}(T^{-1}(x))) & \textit{by Definition 4.11} \\ & = & T^{\mathsf{f}}(\boldsymbol{ch}(y)) & \\ & = & T^{\mathsf{f}}(\boldsymbol{stp}(y)) & \textit{by Proposition 4.20} \\ & = & T^{\mathsf{f}}(b1) & \\ & = & T^{\mathsf{f}}((T^{\mathsf{f}})^{-1}(b'1)) & \\ & = & b'1 & \\ & = & \boldsymbol{stp}(x) & \end{array}$$

⊣

Proposition 4.21 *If $x \in \mathcal{B}_2$, then $\boldsymbol{ch}(x) \succeq \boldsymbol{stb}(x) = \boldsymbol{stp}(x)$.*

Proof. By induction on $\boldsymbol{trk}(x)$. If $\boldsymbol{trk}(x) = 0$, then the statement follows directly from Definition 4.4 and the first clause of Definition 4.11. Suppose $\boldsymbol{trk}(x) > 0$. Let $y = T^{-1}(x)$. It is easily seen that $y \in \mathcal{B}_2$, as $y \in \mathcal{B}_1$ implies $T(y) = x \in \mathcal{B}_1$. Since $\boldsymbol{trk}(y) < \boldsymbol{trk}(x)$, by the induction hypothesis, we have $\boldsymbol{ch}(y) \succeq \boldsymbol{stp}(y)$. We prove $\boldsymbol{ch}(x) \succeq \boldsymbol{stb}(x)$ by showing $\mathcal{W}^{\mathsf{f}}(b) \equiv_R \mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(x))$ for any $b \prec x$ such that $\boldsymbol{ch}(x) \prec b$. Since $x \in \mathcal{B}_2$, there exists a landmark b' of x such that $b \prec b'$. Let $d = (T^{\mathsf{f}})^{-1}(b')$. By monotonicity of T^{f} (Proposition 4.6) we have $\boldsymbol{ch}(y) \prec d$, since the opposite $d \preceq \boldsymbol{ch}(y)$ implies $b' = T^{\mathsf{f}}(d) \preceq T^{\mathsf{f}}(\boldsymbol{ch}(y)) = T^{\mathsf{f}}(\boldsymbol{ch}(T^{-1}(x))) = \boldsymbol{ch}(x)$, a contradiction. So $\mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(y)) \equiv_R \mathcal{W}^{\mathsf{f}}(d)$ since $\boldsymbol{ch}(y) \succeq \boldsymbol{stb}(y)$. In summary, we have

$$\begin{array}{rcll} \mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(x)) & = & \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(\boldsymbol{ch}(T^{-1}(x)))) & \textit{by Definition 4.11} \\ & = & \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(\boldsymbol{ch}(y))) & \\ & = & S(\mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(y))) & \textit{by Proposition 4.4} \\ & \equiv_R & S(\mathcal{W}^{\mathsf{f}}(d)) & \\ & = & S(\mathcal{W}^{\mathsf{f}}((T^{\mathsf{f}})^{-1}(b'))) & \\ & = & \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}((T^{\mathsf{f}})^{-1}(b'))) & \textit{by Proposition 4.4} \\ & = & \mathcal{W}^{\mathsf{f}}(b'). & \end{array}$$

Since $\boldsymbol{ch}(x) \prec b \prec b'$, by monotonicity of $\mathcal{W}^{\mathsf{f}}$ we have

$$\mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(x)) \equiv_R \mathcal{W}^{\mathsf{f}}(b) \equiv_R \mathcal{W}^{\mathsf{f}}(b')$$

as required. ⊣

Proposition 4.22 *For any $x \in \mathcal{B}$ we have $\boldsymbol{stb}(x) \preceq \boldsymbol{stp}(x) \preceq \boldsymbol{ch}(x)$.*

Proof. By Propositions 4.3, 4.20, 4.21. ⊣

Now we are ready to lift $\mathcal{W}^{\mathsf{f}}$ from $\{0,1\}^*$ to $\mathcal{B}$ using $\boldsymbol{ch}$.

Definition 4.12 (Embedding Function) *Define the **embedding function** $\mathcal{W} : \mathcal{B} \to W$ by*

$$\mathcal{W}(x) = \mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(x)).$$

Proposition 4.23 *Let $b \in \{0,1\}^*$. For any $x \in \mathcal{T}_b$ we have $\mathcal{W}^{\mathsf{f}}(b) R \mathcal{W}(x)$.*

Proof. If $b \preceq \boldsymbol{ch}(x)$, i.e., $\boldsymbol{ch}(x)$ is a node in $\mathcal{T}_b$, then $\mathcal{W}^{\mathsf{f}}(b) R \mathcal{W}(x)$ follows from monotonicity. If $\boldsymbol{ch}(x) \prec b$, by Proposition 4.22, $\boldsymbol{stb}(x) \preceq \boldsymbol{ch}(x)$. Hence $\mathcal{W}^{\mathsf{f}}(b) \equiv_R \mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(x)) = \mathcal{W}(x)$ by Definition 4.3. ⊣

5 Proof of Completeness

Lemma 5.1 *The function $\mathcal{W}$ is onto.*

Proof. Let $W = \{w_0, \ldots, w_{m-1}\}$ with the root $w_0 = \mathbf{0}$. By Definition 4.1 $\mathcal{W}^{\mathsf{f}}(0^i1) = w_i$ for $0 \le i < m$. Hence for $0 \le i < m$, we have

$$\begin{aligned}
\mathcal{W}(0^i10^\omega) &= \mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(0^i10^\omega)) && \textit{by Definition 4.12} \\
&= \mathcal{W}^{\mathsf{f}}(\boldsymbol{stp}(0^i10^\omega)) && \textit{by Proposition 4.20} \\
&= \mathcal{W}^{\mathsf{f}}(0^i1) && \textit{by Definition 4.4} \\
&= w_i.
\end{aligned}$$

Hence the range of $\mathcal{W}$ is W. ⊣

Lemma 5.2 *The function $\mathcal{W}$ is open and continuous. That is, for every $b \in \{0,1\}^*$*

$$\mathcal{W}(\mathcal{T}_b) = \{w : \mathcal{W}^{\mathsf{f}}(b)Rw\}.$$

Proof. Recall that the sets $\mathcal{T}_b$ constitute a basis of open sets in $\mathcal{B}$, and the sets $\{w : \mathcal{W}^{\mathsf{f}}(b)Rw\}$ constitute a basis of open sets in $\mathcal{K}$. Consider an arbitrary $b \in \{0,1\}^*$. Let $y \in \mathcal{T}_b$. By Proposition 4.23 $\mathcal{W}^{\mathsf{f}}(b)R\mathcal{W}(y)$. Hence $\mathcal{W}(\mathcal{T}_b) \subseteq \{w \mid \mathcal{W}^{\mathsf{f}}(b)Rw\}$. On the other hand, let w be such that $\mathcal{W}^{\mathsf{f}}(b)Rw$. By Proposition 4.2 there exists an $i \ge 0$ such that $\mathcal{W}^{\mathsf{f}}(b0^i1) = w$. Since $b0^i10^\omega \in \mathcal{B}_1$, by Proposition 4.20 we have

$$\mathcal{W}(b0^i10^\omega) = \mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(b0^i10^\omega)) = \mathcal{W}^{\mathsf{f}}(\boldsymbol{stp}(b0^i10^\omega)) = \mathcal{W}^{\mathsf{f}}(b0^i1) = w.$$

Hence $\mathcal{W}(\mathcal{T}_b) \supseteq \{w \mid \mathcal{W}^{\mathsf{f}}(b)Rw\}$. ⊣

Lemma 5.3 *T is a continuous function on $\mathcal{B}$.*

Proof. Suppose $y = T(x)$ for some $x \in \mathcal{B}$. Let $\mathcal{O}_y$ be an arbitrary basic open set containing y. It is easily seen that $\mathcal{O}_y$ has the form $\mathcal{T}_r$ where r is a prefix of y. Let b' be a landmark of y in $\mathcal{T}_r$. By Proposition 4.17 there exists a finite prefix b of x such that $T^{\mathsf{f}}(b) = b'$. By the monotonicity of T^{f} (Proposition 4.6) $\mathcal{T}_b$ (which is open and contains x) is mapped into $\mathcal{T}_r$ under T. It follows that T is continuous. ⊣

Theorem 5.1 *The embedding map $\mathcal{W} : \mathcal{B} \to W$ is a functor from $\langle \mathcal{B}, T\rangle$ onto $\mathcal{K} = \langle W, R, S\rangle$.*

Proof. By Lemmas 5.1, 5.2 and 5.3, we only need to show that $\mathcal{W}(T(x)) = S(\mathcal{W}(x))$ for any $x \in \mathcal{B}$. We have

$$\begin{aligned}
\mathcal{W}(T(x)) &= \mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(T(x))) && \textit{by Definition 4.12} \\
&= \mathcal{W}^{\mathsf{f}}(T^{\mathsf{f}}(\boldsymbol{ch}(x))) && \textit{by Definition 4.11} \\
&= S(\mathcal{W}^{\mathsf{f}}(\boldsymbol{ch}(x))) && \textit{by Proposition 4.4} \\
&= S(\mathcal{W}(x)). && \textit{by Definition 4.12}
\end{aligned}$$

⊣

Theorem 5.2 *S4C is complete for dynamic Cantor spaces.*

Proof. By Corollary 2.1 and Theorem 5.1. ⊣

References

[1] S. Artemov, J. Davoren and A. Nerode. Modal logics and topological semantics for hybrid systems. Technical Report MSI 97-05, Cornell University, June 1997 (available at http://www.cs.gc.cuny.edu/~sartemov/publications/MSI97-05.ps).

[2] J. M. Davoren. Modal logics for continuous dynamics. Ph.D. Thesis, Cornell University, 1998.

[3] G. Mints. A completeness proof for propositional S4 in Cantor space. In: Orlowska, Ewa (ed.), Logic at work. Essays dedicated to the memory of Helena Rasiowa. Heidelberg: Physica-Verlag. Stud. Fuzziness Soft Comput. 24, pp. 79-88, 1999.

[4] G. Mints, P. Kremer. Dynamic topological logic. Bulletin of Symbolic Logic **3** (1997), pp. 371-372.

[5] P. Kremer, G. Mints. Dynamic topological logic. Annals of Pure and Applied Logic **131** (2005), pp. 133-158.