Dynamical topological logic of Cantor space

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Dynamical topological logic [2] studies models (X, S), where X is a topological space, S a transformation on X. Propositional formulas are constructed from variables (atomic formulas) by Boolean connectives, necessity $\Box$ and a monadic operation $\circ$. Variables are interpreted by subsets of X, Boolean connectives act in a natural way, $\Box$ is the interior and $\circ$ is the pre-image under operation S. The axiom schema $\circ\Box A \rightarrow \Box \circ A$ called (C) expresses continuity of S. The propositional system S4C includes S4,(C) and axioms $\circ(A\&B) \iff \circ A\& \circ B$, $\circ\neg A \iff \neg \circ A$. J. Davoren [1] proved completeness of S4C for the class of all topological spaces, in particular for finite spaces derived from Kripke models which are not very natural mathematically. We prove completeness of S4C for Cantor space very popular in the theory of dynamical systems. P. Kremer pointed out that the real line is not complete for S4C. An elegant example was constructed by J. van Benthem. Let $\mathcal{B}$ be the Cantor space, that is the set $\{0,1\}^\omega$ of all infinitary binary sequences $\boldsymbol{b} = b(0), b(1), \ldots$. Let $\mathcal{B}_1 = \{\boldsymbol{b} \in \mathcal{B} : \boldsymbol{b} = b0^\omega$ for some $b \in \{0,1\}^*\}$, $\mathcal{B}_2 = \mathcal{B} - \mathcal{B}_1$.

By [1] every formula underivable in S4C is refutable in a finite Kripke frame $\langle W, R\rangle$ with the root w_0 and operation $S : W \rightarrow W$ satisfying $Rww' \Rightarrow R(Sw)(Sw')$. Let $\mathcal{K}$ be the topological space with the carrier W and topology determined by R. Define a map $\mathcal{W} : \{0,1\}^* \rightarrow W$ as follows. For the empty sequence Λ, $\mathcal{W}(\Lambda) = w_0$; if $\mathcal{W}(b) = w$ and w has R-successors $w, w_1, \ldots, w_{m-1}$, then $\mathcal{W}(b0^i) = w$ for $i = 1, \ldots, m$, while $\mathcal{W}(b0^i1) = w_i$ for $i = 0, \ldots, m-1$. Define $w \sim_R w'$ as $Rww'\&Rw'w$. Define the stabilization point of a sequence $x \in \mathcal{B}$: $\boldsymbol{stb}(x) =$ the least finite prefix b of x such that $\mathcal{W}(b') \sim_R W(b)$ for any longer prefix b' of x. Define a stopping point by $\boldsymbol{stp}(x) = \boldsymbol{stb}(x)$ if $x \in \mathcal{B}_2$, $\boldsymbol{stp}(x) = \Lambda$, if $x = 0^\omega$, and $\boldsymbol{stp}(x) = b1$, if $x = b10^\omega$ for some $b \in \{0,1\}^*$. It is possible to define a map $T' : \{0,1\}^* \rightarrow \{0,1\}^*$ such that $\mathcal{W}T' = S\mathcal{W}$. Moreower, T' extends to a continuous function $T : \mathcal{B} \rightarrow \mathcal{B}$, and every chain of the form $x, T^{-1}x, \ldots, T^{-n}x$ is finite. Define $\boldsymbol{rk}(x)$ to be 0 for $x = 0^\omega$ and the least n such that $T^{-n}x$ is not in the range of T otherwise. Define a finite representative of a sequence $x \in \mathcal{B}$ by induction on $\boldsymbol{rk}(x)$: $\boldsymbol{ch}(x) = \boldsymbol{stp}(x)$, if $\boldsymbol{rk}(x) = 0$, and $\boldsymbol{ch}(x) = T'(\boldsymbol{ch}(T^{-1}(x))$ otherwise. Define $f(x) = \mathcal{W}(\boldsymbol{ch}(x))$.

Theorem. (a)The function f is a continuous open embedding of $\mathcal{B}$ onto $\mathcal{K}$ satisfying condition: $fT = Sf$.

(b) S4C is complete for topological semantics in Cantor space.

References

[1] J.M. Davoren, Modal logics for continuous dynamics, Ph.D. Thesis, Cornell University, 1998

[2] G. Mints, P. Kremer, Dynamic topological logic, Bull.Symb. Logic v.3 no 3, 1997, 371-372