MAT. SBORNIK
TOM 103(145)(1977), No. 2

MATH. USSR SBORNIK
Vol. 32(1977), No. 2

THE PROBLEM OF SOLVABILITY OF EQUATIONS IN A FREE SEMIGROUP

UDC 519.4

G. S. MAKANIN

ABSTRACT. In this paper we construct an algorithm recognizing the solvability of arbitrary equations in a free semigroup.
Bibliography: 4 titles.

The study of equations in a free semigroup was begun by A. A. Markov. He constructed an algorithm recognizing the solvability of equations in two unknowns. Ju. I. Hmelevskiĭ [1] constructed an algorithm recognizing the solvability of equations in three unknowns and also the solvability of systems of equations, each of which contains at most two unknowns.

In this present paper we construct an algorithm recognizing the solvability of arbitrary equations in a free semigroup.

The problem of the solvability of systems of equations in a free semigroup easily reduces to that of the solvability of a single equation in a free semigroup (see Lemma 1.28 of [1]).

§1. Preliminary lemmas

Suppose Π is a free semigroup with finite alphabet of generators

$$a_1, \ldots, a_\omega. \tag{1.1}$$

By an *equation* in the free semigroup Π with unknowns

$$x_1, \ldots, x_n' \tag{1.2}$$

we mean an equality of words in the combined alphabet (1.1), (1.2):

$$\varphi(a_1, \ldots, a_\omega, x_1, \ldots, x_n) = \psi(a_1, \ldots, a_\omega, x_1, \ldots, x_n). \tag{1.3}$$

The alphabet (1.1) is called the *coefficient alphabet*. The empty word will be denoted by the symbol 1, the length of a word P by the symbol $\partial(P)$, and graphical equality of two words P and Q will be denoted by $P \overset{\circ}{=} Q$.

A list of words

$$X_1, \ldots, X_n \tag{1.4}$$

in the alphabet (1.1) is called a *solution* of equation (1.3) if

$$\varphi(a_1, \ldots, a_\omega, X_1, \ldots, X_n) \overset{\circ}{=} \psi(a_1, \ldots, a_\omega, X_1, \ldots, X_n).$$

AMS (MOS) subject classifications (1970). Primary 20M05; Secondary 02E10.

Consider the system of linear Diophantine equations

$$\begin{array}{l} \xi_{1,1}z_1+\xi_{1,2}z_2+\ \dots\ +\xi_{1,n}z_n=\eta_1,\\ \xi_{2,1}z_1+\xi_{2,2}z_2+\ \dots\ +\xi_{2,n}z_n=\eta_2,\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \xi_{p,1}z_1+\xi_{p,2}z_2+\ \dots\ +\xi_{p,n}z_n=\eta_n, \end{array} \tag{1.5}$$

where the z_i are variables assuming natural (nonnegative integral) values, the $\xi_{i,j}$ and η_i are integers, and for each $i = 1, \dots, p$ at least one $\xi_{i,j} \neq 0$.

Let m be the maximum of all the numbers $|\xi_{i,j}|$ and $|\eta_i|$, and let q be the number of nonzero coefficients among $\xi_{1,1}, \xi_{1,2}, \dots, \xi_{1,n}$. The numbers n, p, m, q will be called the *parameters* of system (1.5). By the *index* of system (1.5) we mean the number $q + (p - 1)n$.

We will say that a solution

$$z_1^0, \dots, z_n^0 \tag{1.6}$$

of system (1.5) is *smaller* than a solution

$$z_1', \dots, z_n', \tag{1.7}$$

if some component of solution (1.6) is smaller than the corresponding component of solution (1.7), and the remaining components of (1.6) do not exceed the corresponding components of (1.7). A solution (1.6) is called *minimal* if there is no smaller solution. By the *height* of a solution (1.6) we mean its largest component.

LEMMA 1.1. *If a system* (1.5) *with parameters n, p, m, q and index I has a solution, then the height h of a minimal solution satisfies the inequality*

$$h \leqslant 2^{(2^I-2)} m^{(2^I-1)}.$$

PROOF. The proof is by induction on the index I.

Basis of the induction. If $I = 1$, then the system has the form

$$\xi_{1,1}z_1+\xi_{1,2}z_2+\ \dots\ +\xi_{1,n}z_n=\eta_1,$$

where only one of the coefficients $\xi_{1,1}, \xi_{1,2}, \dots, \xi_{1,n}$ is different from zero. Suppose $\xi_{1,i} \neq 0$, and let (1.6) be a minimal solution of this system. Obviously, all components of this solution except the ith are zero, and $z_i^0 \leqslant |\eta_1| \leqslant m$. Consequently, $h \leqslant m$.

Inductive step. Suppose $I > 1$ and the lemma is true for any system of type (1.5) with index less than I. Let (1.6) be a minimal solution of system (1.5). We consider two cases.

Case 1. For certain i and j we have $\xi_{1,i} > 0$ and $\xi_{1,j} < 0$. Let

$$z_i^0 = -\xi_{1,j} z_i' + \mu_j,$$

$$z_j^0 = \xi_{1,i} z_j' + \mu_i,$$

where z_i', z_j', μ_j, μ_i are natural numbers, with $\mu_j < -\xi_{1,j}$ and $\mu_i < \xi_{1,i}$.

Consider the system of equations resulting from (1.5) by means of the substitution

$$z_i \to -\xi_{1,j} z_i + \mu_j,$$

$$z_j \to \xi_{1,i} z_j + \mu_i.$$

It is easy to see that to the minimal solution (1.6) of system (1.5) there corresponds a minimal solution $z_1', z_2', \dots, z_n'$ of the resulting system, with $z_k' = z_k^0$ for $k \neq i, j$.

If $z_i' \geqslant z_j'$, then to the resulting system of equations we apply the transformation

$z_i \to z_j + z_i$, and if $z_i' < z_j'$, the transformation $z_j \to z_i + z_j$.

As a result, we obtain a new system. The parameters n_1, p_1, m_1 and the index I_1 of the new system satisfy the inequalities

$$n_1 \leqslant n, \quad p_1 \leqslant p, \quad m_1 \leqslant 2m^2, \quad I_1 < I. \tag{1.8}$$

If the height h of solution (1.6) is larger than m, then the height h_1 of the minimal solution of the new system corresponding to (1.6) satisfies the inequality $h \leqslant 2mh_1$.

By the inductive assumption we have

$$h_1 \leqslant 2^{(2^{I_1}-2)} m_1^{(2^{I_1}-1)};$$

hence, in view of (1.8) and the estimate for h, either $h \leqslant m$ or

$$h \leqslant 2^{(2^{I_1+1}-2)} m^{(2^{I_1+1}-1)},$$

and since $I_1 + 1 \leqslant I$, it follows that

$$h \leqslant 2^{(2^{I}-2)} m^{(2^{I}-1)}.$$

Case 2. All of the coefficients $\xi_{1,1}, \ldots, \xi_{1,n}$ have the same sign.

By assumption, $\xi_{1,i} \neq 0$ for some i. Consequently, $z_i^0 \leqslant |\eta_1| \leqslant m$. In this case we apply to system (1.5) the transformation $z_i \to z_i^0$. As a result, we obtain a new system. The parameters n_1, p_1, m_1 and the index I_1 of the new system satisfy inequalities (1.8). If the height h of solution (1.6) is larger than m, then the height h_1 of the minimal solution of the new system corresponding to (1.6) satisfies the inequality $h \leqslant 2mh_1$. The rest is as in Case 1.

The lemma is proved.

A word P is called *simple* if there exists no word Q such that $P \equiv Q^m$ for $m > 1$.

Suppose P is a simple word and

$$A \equiv A_1 P A_2. \tag{1.9}$$

The occurrence of P in A distinguished in the representation (1.9) is called *stable* if P is an end of A_1 and a beginning of A_2. It follows easily from Lemma 1.2 of [1] that if P is simple and $PP \equiv B_1 P B_2$, then either B_1 or B_2 is empty. Therefore, no two distinct stable occurrences of a simple word P in a word A can intersect. Consequently, any word A is uniquely representable in the form

$$A \equiv D_1 P D_2 P D_3 \ldots D_h P D_{h+1},$$

where all stable occurrences of the simple word P have been distinguished.

A parametric word

$$C_1 P^{\lambda_1} C_2 P^{\lambda_2} C_3 \ldots C_m P^{\lambda_m} C_{m+1}, \tag{1.10}$$

where P is a simple word, $C_1, \ldots, C_{m+1}$ are arbitrary words, and $\lambda_1, \ldots, \lambda_m$ are parameters assuming positive integral values, is called *stable* if the words C_1 and $PC_2, \ldots, PC_m$ end with P, and the words $C_2P, \ldots, C_mP$ and C_{m+1} begin with P.

To each stable parametric word (1.10) we assign the word

$$C_1 P C_2 P C_3 \ldots C_m P C_{m+1}. \tag{1.11}$$

Let us distinguish in (1.11) all stable occurrences of P. It is easy to see that the stable occurrences of P in (1.11) are the occurrences of P already distinguished in (1.11) and the stable occurrences of P contained entirely within the subwords C_i. We will call these latter occurrences of P *complementary*.

To each complementary stable occurrence of P in (1.11) there corresponds in a natural way an occurrence of P in the parametric word (1.10) which is stable for any positive integral values of the parameters $\lambda_1, \ldots, \lambda_m$. We call these occurrences the *complementary* stable occurrences of P in the parametric word (1.10).

Obviously, any stable parametric word (1.10) is uniquely representable in the form

$$C_1'P^{L_1}C_2'P^{L_2}C_3' \ldots C_q'P^{L_q}C_{q+1}', \tag{1.12}$$

where $C_2', \ldots, C_q'$ are *nonempty* words, and $L_1, \ldots, L_q$ are nonzero linear forms in the variables $\lambda_1, \ldots, \lambda_m$ with nonnegative integral coefficients such that if we assign arbitrary positive integral values to $\lambda_1, \ldots, \lambda_m$, then in the resulting word

$$C_1'P^{L_1^0}C_2'P^{L_2^0}C_3' \ldots C_q'P^{L_q^0}C_{q+1}' \tag{1.13}$$

all stable occurrences of the word P in (1.13) are found precisely among the distinguished subwords $P^{L_1^0}, \ldots, P^{L_q^0}$.

Such a representation (1.12) of the stable parametric word (1.10) is called *complete.*

LEMMA 1.2. *If words A and B contain, respectively, q and r stable occurrences of a simple word P, then the word AB contains at least $q + r$ and at most $q + r + 3$ stable occurrences of P.*

PROOF. Let us distinguish the stable occurrences of the simple word P in A, B and AB. An occurrence of P which is stable in A (or in B) remains stable in AB. Each new (i.e. generated by juxtaposing A and B) stable occurrence of P must either contain the "cut" of the word AB (the boundary between A and B) in its interior or must be separated from this "cut" by at most $\partial(P) - 1$ letters. Consequently, on juxtaposing A and B we obtain at most three new stable occurrences of P. The lemma is proved.

A solution (1.4) of equation (1.3) is called *minimal* if $\partial(X_1X_2 \cdots X_n)$ is minimal in the class of all solutions of this equation.

By the *notational length* of equation (1.3) we mean the number

$$\partial(\varphi(a_1, \ldots, a_\omega, x_1, \ldots, x_n)) + \partial(\psi(a_1, \ldots, a_\omega, x_1, \ldots, x_n)).$$

A number s is called the *exponent of periodicity* of solution (1.4) if some component X_i contains a nonempty subword of the form P^s, but no component X_i of this solution contains any nonempty subword of the form Q^{s+1}.

The following lemma is a refinement of one proved by V. K. Bulitko [2].

LEMMA 1.3. *If d is the notational length of equation* (1.3), *then the exponent of periodicity of any minimal solution does not exceed*

$$(6d)^{2^{(2d^4)}} + 2.$$

PROOF. Suppose an equation (1.3) with notational length d has a minimal solution (1.4) with exponent of periodicity s, and let P be a simple word such that P^s occurs in some component of (1.4).

In each component X_i of solution (1.4) we distinguish all stable occurrences of P and write these components in the form

$$\begin{array}{l} X_1 \overset{\circ}{=} C_{1,1}P^{\mu(1,1)}C_{1,2}P^{\mu(1,2)} \ldots C_{1,q_1}P^{\mu(1,q_1)}C_{1,q_1+1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ X_n \overset{\circ}{=} C_{n,1}P^{\mu(n,1)}C_{n,2}P^{\mu(n,2)} \ldots C_{n,q_n}P^{\mu(n,q_n)}C_{n,q_n+1}, \end{array} \tag{1.14}$$

where all $C_{i,2}, \ldots, C_{i,q_i}$ are nonempty words and all $\mu(i,j)$ are positive integers.

Consider the parametric words

$$\begin{aligned} X_1' &\rightleftharpoons C_{1,1}P^{z_{1,1}}C_{1,2}P^{z_{1,2}} \ldots C_{1,q_1}P^{z_{1,q_1}}C_{1,q_1+1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_n' &\rightleftharpoons C_{n,1}P^{z_{n,1}}C_{n,2}P^{z_{n,2}} \ldots C_{n,q_n}P^{z_{n,q_n}}C_{n,q_n+1}, \end{aligned} \tag{1.15}$$

where the $C_{i,j}$ are taken from (1.14) and the $z_{i,j}$ are variables assuming positive integral values. It is easy to see that each parametric word in (1.15) is stable.

Substituting the parametric words (1.15) into (1.3), we obtain parametric words

$$\Phi \rightleftharpoons \varphi(a_1, \ldots, a_\omega, X_1', \ldots, X_n'),$$
$$\Psi \rightleftharpoons \psi(a_1, \ldots, a_\omega, X_1', \ldots, X_n').$$

Let us write down the complete representations of these parametric words:

$$\begin{aligned} \Phi &= D_1P^{L_{1,1}}D_2P^{L_{1,2}}D_3 \ldots D_{r_1}P^{L_{1,r_1}}D_{r_1+1}, \\ \Psi &= D_1'P^{L_{2,1}}D_2'P^{L_{2,2}}D_3' \ldots D_{r_2}'P^{L_{2,r_2}}D_{r_2+1}'. \end{aligned} \tag{1.16}$$

Here $D_2, \ldots, D_{r_1}$ and $D_2', \ldots, D_{r_2}'$ are nonempty words, and the $L_{i,j}$ are nonzero linear forms in the variables $z_{\alpha,\beta}$ with nonnegative coefficients.

We now substitute for the parameters $z_{\alpha,\beta}$ the numbers $\mu(\alpha, \beta)$ defined by (1.14). We obtain words

$$\Phi^0 = D_1P^{L_{1,1}^0}D_2P^{L_{1,2}^0}D_3 \ldots D_{r_1}P^{L_{1,r_1}^0}D_{r_1+1},$$
$$\Psi^0 = D_1'P^{L_{2,1}^0}D_2'P^{L_{2,2}^0}D_3' \ldots D_{r_2}'P^{L_{2,r_2}^0}D_{r_2+1}'.$$

All stable occurrences of P in Φ^0 are found among the $P^{L_{1,j}^0}$. All stable occurrences of P in Ψ^0 are found among the $P^{L_{2,j}^0}$. Since $\Phi^0 \doteq \Psi^0$, we obtain from the uniqueness of distinguishing stable occurrences the relations $r_1 = r_2$ and $D_i' \doteq D_i$ for $i = 1, \ldots, r_1 + 1$, and

$$L_{1,i}^0 = L_{2,i}^0 \quad (i = 1, \ldots, r_1). \tag{1.17}$$

Consequently, the representations (1.16) have the form

$$\Phi = D_1P^{L_{1,1}}D_2P^{L_{1,2}}D_3 \ldots D_rP^{L_{1,r}}D_{r+1},$$
$$\Psi = D_1P^{L_{2,1}}D_2P^{L_{2,2}}D_3 \ldots D_rP^{L_{2,r}}D_{r+1}.$$

Let $\mathfrak{L}$ be the system of linear Diophantine equations $L_{1,i} = L_{2,i}$ $(i = 1, \ldots, r)$. It follows from (1.17) that $\mathfrak{L}$ has the solution

$$z_{\alpha,\beta} = \mu(\alpha, \beta), \tag{1.18}$$

and the height of solution (1.18) is $s - 2$. The minimality of solution (1.4) implies that (1.18) is a minimal solution of $\mathfrak{L}$.

Consider the linear form $L_{q,i}$:

$$\xi_1 z_{i_1,j_1} + \ldots + \xi_p z_{i_p,j_p} + \xi_{p+1}, \tag{1.19}$$

where $p \geqslant 0$, $\xi_1 > 0, \ldots, \xi_p > 0$, $\xi_{p+1} \geqslant 0$, and $\xi_1 + \cdots + \xi_p + \xi_{p+1} > 0$. The number ξ_{p+1} is called the *free term* of the form $L_{q,i}$. The number $\xi_1 + \cdots + \xi_p$ is called the *number of variable occurrences* in $L_{q,i}$. The linear form (1.19) is called *trivial* if it has the form $z_{\alpha,\beta}$.

By Lemma 1.2, the number of complementary stable occurrences of P in the parametric words Φ and Ψ together does not exceed $3d$. Consequently, the sum of the free terms of all of the linear forms $L_{1,1}, \ldots, L_{1,r}, L_{2,1}, \ldots, L_{2,r}$ does not exceed $3d$.

Since in each parametric word X_j' (see (1.15)) in the linear form $L_{q,i}$ in (1.16) there can occur at most one z-variable, the number of variables in any linear form $L_{q,i}$ does not exceed d.

The number of nontrivial linear forms $L_{q,i}$ does not exceed d.

If the system $\mathfrak{L}$ contains an equality $z_{i,j} = z_{\alpha,\beta}$ or a pair of equalities $L_{p,r} = z_{i,j}$ and $L_{p,r} = z_{\alpha,\beta}$, then we identify the variables $z_{i,j}$ and $z_{\alpha,\beta}$. We make all such identifications and represent the resulting system in the form (1.5).

Thus we obtain a new system $\mathfrak{L}'$, whose parameters (see (1.5)) satisfy the following conditions:

$$n \leqslant d^2 + d \leqslant 2d^2, \quad p \leqslant \frac{d(d-1)}{2} + d \leqslant d^2, \quad m \leqslant 3d.$$

Since $q \leqslant n$, we have $I \leqslant 2d^4$.

Using Lemma 1.1, we obtain

$$s = h + 2 < (2m)^{2^I} + 2 \leqslant (6d)^{2^{(2d^4)}} + 2.$$

LEMMA 1.4. *Suppose*

$$X_{\lambda_1}, \ \ldots, \ X_{\lambda_i}, X_{\lambda_{i+1}}, \ \ldots, \ X_{\lambda_k} \quad (k > 0) \tag{1.20}$$

is an arbitrary sequence of words composed (possibly with repetitions) of given nonempty words $X_1, \ldots, X_n$. *Let* $B_1, \ldots, B_k$ *and* $C_1, \ldots, C_k$ *be nonempty words such that*

$$X_{\lambda_i} \mathrel{\overset{\circ}{=}} B_i C_i \quad (i = 1, \ \ldots, \ k), \tag{1.21}$$

$$B_{i+1} \mathrel{\overset{\circ}{=}} S_i B_i \quad (i = 1, \ \ldots, \ k-1) \tag{1.22}$$

for certain $S_1, \ldots, S_{k-1}$ *and*

$$C_i R_i \mathrel{\overset{\circ}{=}} C_{i+1} T_i \quad (i = 1, \ \ldots, \ k-1) \tag{1.23}$$

for certain $R_1, \ldots, R_{k-1}$ *and* $T_1, \ldots, T_{k-1}$. *If, in addition, the words* $S_1, \ldots, S_{k-1}$ *satisfy the condition*

$$j - i \geqslant (2n)^2 \Rightarrow \partial(S_i S_{i+1} \ \ldots \ S_{j-1} S_j) > 0, \tag{1.24}$$

then some word X_t *has the form* $X_t \mathrel{\overset{\circ}{=}} P^s Q$, *where* P *is a nonempty word and* $k \leqslant 4n^3(n+1)(s+2)$.

PROOF. Let us assume for simplicity that

$$\partial(X_1) \leqslant \ \ldots \ \leqslant \partial(X_i) \leqslant \partial(X_{i+1}) \leqslant \ \ldots \ \leqslant \partial(X_n).$$

Consider the sequence of words $B_1, \ldots, B_k$. In view of (1.22) and the fact that $C_1, \ldots, C_k$ are nonempty, we have

$$\partial(B_1) \leqslant \ \ldots \ \leqslant \partial(B_j) \leqslant \partial(B_{j+1}) \leqslant \ \ldots \ \leqslant \partial(B_k) < \partial(X_n).$$

Between the words X_i of unequal length we distribute all of the B_j monotonically with respect to increasing length. Obviously, there exist p words $B_{i+1}, \ldots, B_{i+p}$, where

$$p \geqslant \frac{k}{n+1}, \tag{1.25}$$

such that for all $j = i+1, \ldots, i+p$ either

$$\partial(X_q) \leqslant \partial(B_j) < \partial(X_{q+1}) \tag{1.26}$$

for some $q > 0$, or

$$\partial(B_j) < \partial(X_1), \tag{1.27}$$

and then we assume that $q = 0$.

To the words $B_{i+1}, \ldots, B_{i+p}$ there corresponds in the sequence of words (1.20) the subsequence

$$X_{\lambda_{i+1}}, \ \ldots, \ X_{\lambda_{i+p}}. \tag{1.28}$$

In view of either (1.26) or (1.27) and the fact that the words $C_1, \ldots, C_k$ are nonempty, all words of the sequence (1.28) belong to the subset of words $X_{q+1}, \ldots, X_n$.

Retaining for the sequence (1.28) the decomposition (1.21), we see that conditions (1.21)–(1.24) are satisfied and, moreover, for all $j = i+1, \ldots, i+p$ and all $h = q+1, \ldots, n$ we have

$$\partial(B_j) < \partial(X_h). \tag{1.29}$$

Assume that for some r in the interval $i+1 \leqslant r \leqslant i+p-1$ the word C_r is a proper beginning of C_{r+1}, i.e. $C_{r+1} \mathrel{\overline{\underline{\circ}}} C_r Q$, where $\partial(Q) > 0$. Then the word X_{r+1} is representable in the form

$$X_{r+1} \mathrel{\overline{\underline{\circ}}} X'_{r+1} Q,$$

where $\partial(X'_{r+1}) = \partial(B_{r+1} C_r) \geqslant \partial(B_r C_r) = \partial(X_r)$. Replace X_{r+1} by X'_{r+1} everywhere in (1.28). In view of (1.29), it is easy to construct for the resulting sequence a decomposition of the type (1.21) in such a way that (1.22)–(1.24) are satisfied. For this purpose it is necessary in (1.21) to shorten on the right by Q those words C_i for which X_{λ_i} is X_{r+1}.

By means of such replacements we transform, after a finite number of steps, the sequence (1.28) into a sequence

$$X^*_{\lambda_{i+1}}, \ \ldots, \ X^*_{\lambda_{i+p}}, \tag{1.30}$$

composed of words $X^*_{q+1}, \ldots, X^*_n$ (X^*_i is some nonempty beginning of X_i), where there exist nonempty words $B^*_1, \ldots, B^*_p$ and $C^*_1, \ldots, C^*_p$ such that

$$X^*_{\lambda_{i+j}} \mathrel{\overline{\underline{\circ}}} B^*_j C^*_j \quad (j = 1, \ \ldots, \ p), \tag{1.31}$$

$$B^*_{j+1} \mathrel{\overline{\underline{\circ}}} S^*_j B^*_j \quad (j = 1, \ \ldots, \ p-1), \tag{1.32}$$

$$C^*_j \mathrel{\overline{\underline{\circ}}} C^*_{j+1} T^*_j \quad (j = 1, \ \ldots, \ p-1) \tag{1.33}$$

and, in addition, the words $S^*_1, \ldots, S^*_{p-1}$ satisfy the condition

$$j - i \geqslant (2n)^2 \Rightarrow \partial(S^*_i S^*_{i+1} \ \ldots \ S^*_{j-1} S^*_j) > 0. \tag{1.34}$$

From the sequence (1.30) we construct the subsequence

$$X^*_{\mu_1}, \ \ldots, \ X^*_{\mu_m}, \tag{1.35}$$

choosing the terms of (1.30) with subscripts $\lambda_{i+1}, \lambda_{i+1+4n^2}, \lambda_{i+1+8n^2}, \lambda_{i+1+12n^2}, \ldots$. Then

(1.25) implies that

$$m \geqslant \frac{k}{4n^2(n+1)}. \tag{1.36}$$

Furthermore, retaining for the words of (1.35) the decomposition (1.31), we see that there exist nonempty words $B'_1, \ldots, B'_m$ and $C'_1, \ldots, C'_m$ in the alphabet (1.1) such that

$$\begin{aligned} &X^*_{\mu_i} \overset{\circ}{=} B'_i C'_i \quad (i=1, \ldots, m), \\ &B'_{i+1} \overset{\circ}{=} S'_i B'_i \quad (i=1, \ldots, m-1), \\ &C'_i \overset{\circ}{=} C'_{i+1} T'_i \quad (i=1, \ldots, m-1), \\ &\partial(S'_i) > 0 \quad (i=1, \ldots, m-1). \end{aligned} \tag{1.37}$$

The sequence (1.35) is composed of the words $X^*_{q+1}, \ldots, X^*_n$, i.e. at most n words figure in its construction. It follows from (1.36) that there exists a number g, where

$$g \geqslant \frac{k}{4n^3(n+1)}, \tag{1.38}$$

such that some word X^*_t occurs in the sequence (1.35) g times.

From (1.35) we construct the subsequence

$$X^*_{\nu_1}, \ldots, X^*_{\nu_g}, \tag{1.39}$$

choosing the terms of (1.35) which are equal to X^*_t. It follows from (1.37) that there exist nonempty words $B''_1, \ldots, B''_g$ and $C''_1, \ldots, C''_g$ in the alphabet (1.1) such that

$$X^*_{\nu_i} \overset{\circ}{=} B''_i C''_i \quad (i=1, \ldots, g), \tag{1.40}$$

$$\begin{aligned} &B''_{i+1} \overset{\circ}{=} S''_i B''_i \quad (i=1, \ldots, g-1), \\ &C''_i \overset{\circ}{=} C''_{i+1} T''_i \quad (i=1, \ldots, g-1) \\ &\partial(S''_i) > 0 \quad (i=1, \ldots, g-1). \end{aligned} \tag{1.41}$$

From (1.40) and (1.41) we easily obtain the following conditions:

$$\begin{aligned} &S''_1 X^*_t \overset{\circ}{=} X^*_t T''_1, \\ &S''_2 X^*_t \overset{\circ}{=} X^*_t T''_2, \\ &\quad \cdots\cdots\cdots \\ &S''_{g-1} X^*_t \overset{\circ}{=} X^*_t T''_{g-1}. \end{aligned} \tag{1.42}$$

$$\partial(S''_1 S''_2 \ldots S''_{g-1}) < \partial(X^*_t). \tag{1.43}$$

It is easy to see that if for certain words S, T and X we have the graphical equality $SX \overset{\circ}{=} XT$, then there exist words A, B and a natural number r such that $S \overset{\circ}{=} AB$, $T \overset{\circ}{=} BA$ and $X \overset{\circ}{=} (AB)^r A$.

From the conditions (1.42) we obtain

$$S''_i \overset{\circ}{=} A_i B_i, \quad T''_i \overset{\circ}{=} B_i A_i, \quad X^*_t \overset{\circ}{=} (A_i B_i)^{r_i} A_i \quad (i=1, \ldots, g-1). \tag{1.44}$$

It follows from (1.44) that

$$(A_i B_i)^{r_i} A_i \overset{\circ}{=} (A_{i+1} B_{i+1})^{r_{i+1}} A_{i+1} \quad (i=1, \ldots, g-2),$$

from which, using our condition (1.43) and also Lemma 2.3 of Chapter 1 of [3], it follows by induction that all of the A_iB_i are powers of the same simple word P.

Using (1.43) once again, we obtain that $X_t^* \overset{\circ}{=} P^sQ_1$, where $s \geqslant g-2$; hence it follows from (1.38) that there exist words P and Q, where $\partial(P) > 0$, such that $X_t \overset{\circ}{=} P^sQ$, where $s+2 \geqslant k/4n^3(n+1)$. The lemma is proved.

§2. Generalizations of the concept of an equation in a free semigroup

In this section, starting from equations in a free semigroup, we successively define a series of systems which lead us in a natural way to a definition of the central concept of this paper, namely a *generalized equation in a free semigroup*.

An equation (1.3) in a free semigroup is called *coefficient-free* if it contains no letters of the alphabet (1.1). Otherwise, (1.3) will simply be called a *coefficient equation*.

A solution (1.4) of equation (1.3) is called *positive* if all of the components $X_1, \ldots, X_n$ are nonempty words.

LEMMA 2.1. *For any coefficient equation Σ in a free semigroup it is possible to construct a list*

$$\Sigma^{(1)}, \ldots, \Sigma^{(r)} \tag{2.1}$$

of coefficient equations such that Σ has a solution with exponent of periodicity s if and only if at least one $\Sigma^{(i)}$ in (2.1) has a positive solution with exponent of periodicity s.

PROOF. Suppose $x_{i_1}, \ldots, x_{i_q}$ $(0 \leqslant q \leqslant n)$ is some set of unknowns in the alphabet (1.2). An equation $\Sigma^{x_{i_1}, \ldots, x_{i_q}}$ is called the *projection* of Σ with respect to $x_{i_1}, \ldots, x_{i_q}$ if it is obtained by deleting $x_{i_1}, \ldots, x_{i_q}$ from Σ.

Suppose Σ is an arbitrary equation in a free semigroup, and let

$$\Sigma^{(1)}, \ldots, \Sigma^{(r)} \tag{2.2}$$

be all possible projections of Σ. It is easy to see that Σ has a solution with exponent of periodicity s if and only if some $\Sigma^{(i)}$ in (2.2) has a positive solution with exponent of periodicity s. The lemma is proved.

Suppose a system Σ_1 is defined by the coefficient alphabet (1.1), the alphabet of unknowns (1.2), an equality

$$w_1w_2 \ldots w_p = w_{p+1}w_{p+2} \ldots w_{p+q}, \tag{2.3}$$

where the w_i are letters of the combined alphabet (1.1), (1.2) and at least one of the w_i belongs to (1.1), and the additional condition

$$\partial(x_i) > 0 \qquad (i=1, \ldots, n). \tag{2.4}$$

A list of nonempty words $X_1, \ldots, X_n$ in the alphabet (1.1) is called a solution of the system Σ_1 if it is a solution of equation (2.3).

We introduce new variables

$$\begin{gathered} l_1,\ l_2,\ \ldots,\ l_{p+q}, \\ r_1,\ r_2,\ \ldots,\ r_{p+q}, \\ t \end{gathered} \tag{2.5}$$

and equalities

$$\begin{aligned}
&l_1 = 1, && r_1 = w_1 \ldots w_p, && \\
&l_i = w_1 \ldots w_{i-1}, && r_i = w_i \ldots w_p && (i = 2, \ldots, p), \\
&l_{p+1} = w_1 \ldots w_p, && r_{p+1} = 1, && \\
&l_{p+i} = w_{p+1} \ldots w_{p+i-1}, && r_{p+i} = w_{p+i} \ldots w_{p+q} && (i = 2, \ldots, q),
\end{aligned} \tag{2.6}$$

$$l_i r_i = t \quad (i = 1, \ldots, p+q), \tag{2.7}$$

$$\begin{aligned}
&l_1 w_1 r_2 = l_2 w_2 r_3 = \ldots = l_p w_p r_{p+1} = l_1 w_{p+1} r_{p+2} \\
&= l_{p+2} w_{p+2} r_{p+3} = \ldots = l_{p+q-1} w_{p+q-1} r_{p+q} = l_{p+q} w_{p+q} r_{p+1} = t.
\end{aligned} \tag{2.8}$$

Suppose a system Σ_2 is defined by the coefficient alphabet (1.1), the variables (1.2), (2.5), and also condition (2.4) and equalities (2.6)–(2.8). Note that, by assumption, one of the w_i in (2.8) belongs to the alphabet (1.1). A table of words

$$\begin{gathered}
X_1, \ldots, X_n, \\
L_1, \ldots, L_{p+q}, \\
R_1, \ldots, R_{p+q}, \\
T
\end{gathered} \tag{2.9}$$

in the alphabet (1.1) is called a solution of the system Σ_2 if $X_1, \ldots, X_n$ are nonempty and as a result of substituting the components of the table into (2.6)–(2.8) we obtain graphical equalities. (A word denoted by a capital Roman letter with (without) a subscript is substituted for the corresponding small Roman letter with the same subscript (without a subscript).) The exponent of periodicity of solution (2.9) of Σ_2 will be defined not with respect to all components of the solution, but only with respect to $X_1, \ldots, X_n$. The next lemma follows easily from the definitions of Σ_1 and Σ_2.

LEMMA 2.2. *The system Σ_1 has a solution with exponent of periodicity s if and only if the system Σ_2 has such a solution.*

We introduce conditions on the lengths of the variables $l_1, \ldots, l_{p+q}$ and t:

$$\begin{gathered}
0 = \partial(l_1) < \partial(l_2) < \ldots < \partial(l_p) < \partial(l_{p+1}) = \partial(t), \\
\partial(l_1) < \partial(l_{p+2}) < \partial(l_{p+3}) < \ldots < \partial(l_{p+q-1}) < \partial(l_{p+q}) < \partial(l_{p+1}).
\end{gathered} \tag{2.10}$$

Suppose a system Σ_3 is defined by the coefficient alphabet (1.1), the variables (1.2) and (2.5), and also the conditions (2.10) and equalities (2.7) and (2.8).

The next lemma follows easily from the definitions of Σ_2 and Σ_3.

LEMMA 2.3. *The system Σ_2 has a solution with exponent of periodicity s if and only if the system Σ_3 has such a solution.*

We say that a variable x_g of the system Σ_3 has *valency* i if it occurs i times in (2.8). Let

$$\nu = k_1 + k_2 + 2k_3 + \ldots + (b-1) k_b,$$

where k_i is the number of x-variables having valency i in the system Σ_3.

We introduce the alphabet of variables

$$y_1, y_2, \ldots, y_\nu, y_{\nu+1}, y_{\nu+2}, \ldots, y_{2\nu} \tag{2.11}$$

satisfying the condition

$$y_i = y_{i+\nu} \qquad (i=1, \ldots, \nu). \tag{2.12}$$

We replace the system of equalities (2.8) by a new system

$$\begin{gathered} l_{\alpha(1)} y_1 r_{\beta(1)} = \ldots = l_{\alpha(\nu)} y_\nu r_{\beta(\nu)} = l_{\alpha(\nu+1)} y_{\nu+1} r_{\beta(\nu+1)} = \ldots = l_{\alpha(2\nu)} y_{2\nu} r_{\beta(2\nu)} \\ = l_{\alpha(2\nu+1)} a_{\psi(2\nu+1)} r_{\beta(2\nu+1)} = \ldots = l_{\alpha(2\nu+m)} a_{\psi(2\nu+m)} r_{\beta(2\nu+m)} = t, \end{gathered} \tag{2.13}$$

by successively replacing components involving the x_i by components involving the y_i as follows.

Assume that in previous replacements we have already used the variables $y_1, \ldots, y_h$ and $y_{\nu+1}, \ldots, y_{\nu+h}$ in (2.11).

If a variable x_g in (1.2) occurs in (2.8) in only one component $l_{i_1} x_g r_{j_1}$, then we replace this component by $l_{i_1} y_{h+1} r_{j_1}$ and $l_{i_1} y_{\nu+h+1} r_{j_1}$.

If a variable x_g in (1.2) occurs in (2.8) in two components $l_{i_1} x_g r_{j_1}$ and $l_{i_2} x_g r_{j_2}$, then we replace these components by $l_{i_1} y_{h+1} r_{j_1}$ and $l_{i_2} y_{\nu+h+1} r_{j_2}$.

If a variable x_g in (1.2) occurs in (2.8) in the components $l_{i_1} x_g r_{j_1}$, $l_{i_2} x_g r_{j_2}$, $l_{i_3} x_g r_{j_3}, \ldots$, then we replace these components by $l_{i_1} y_{h+1} r_{j_1}$, $l_{i_2} y_{\nu+h+1} r_{j_2}$, $l_{i_2} y_{h+2} r_{j_2}$, $l_{i_3} y_{\nu+h+2} r_{j_3}, \ldots$.

Suppose a system Σ_4 is defined by the coefficient alphabet (1.1), the variables (2.11) and (2.5), and also the conditions (2.10) and (2.12) and equalities (2.7) and (2.13). By a solution of Σ_4 we mean a table of words

$$\begin{gathered} Y_1, \ldots, Y_\nu, Y_{\nu+1}, \ldots, Y_{2\nu}, \\ L_1, \ldots, L_{p+q}, \qquad R_1, \ldots, R_{p+q}, \qquad T \end{gathered}$$

in the alphabet (1.1) satisfying (2.10), (2.12), (2.7) and (2.13). The exponent of periodicity of a solution of Σ_4 is defined with respect to the components $Y_1, \ldots, Y_{2\nu}$. The next lemma follows easily from the definitions of Σ_3 and Σ_4.

LEMMA 2.4. *The system Σ_3 has a solution with exponent of periodicity s if and only if the system Σ_4 has such a solution.*

Note that in (2.13) there occurs at least one coefficient a_i, i.e. $m > 0$. Delete from the coefficient alphabet (1.1) those coefficients which do not occur in (2.13). We obtain an alphabet

$$a_1, \ldots, a_k \qquad (k>0). \tag{2.14}$$

(We may assume that all deleted coefficients occur at the end of the list (1.1).)

Using the functions $\alpha(i)$ and $\beta(i)$ of equation (2.13), we introduce the equalities

$$l_{\alpha(i)} w'_i r_{\beta(i)} = t \quad (i=1, \ldots, 2\nu+m), \tag{2.15}$$

where by $w'_1, \ldots, w'_{2\nu+m}$ we mean $y_1, \ldots, y_{2\nu}, a_{\psi(2\nu+1)}, \ldots, a_{\psi(2\nu+m)}$, respectively.

Suppose a system Σ_5 is defined by the coefficient alphabet (2.14), the variables (2.11) and (2.5), and also the conditions (2.10) and (2.12) and equalities (2.7) and (2.15).

The next lemma follows easily from the definitions of Σ_4 and Σ_5.

LEMMA 2.5. *The system Σ_4 has a solution with exponent of periodicity s if and only if the system Σ_5 has such a solution.*

§3. The concept of a generalized equation

By a *generalized equation* Ω in a free semigroup Π we mean any system consisting of the following five parts.

1. A *coefficient alphabet.*

The coefficient alphabet of Ω is the alphabet of generators of the semigroup Π:

$$a_1, \dots, a_\omega \qquad (\omega>0). \tag{3.1}$$

The letters of alphabet (3.1) are called the coefficients of Ω.

2. A *table of word variables.*

The word variables of Ω are given by a table

$$\begin{array}{l} x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}, \\ l_1, \dots, l_\rho, \quad r_1, \dots, r_\rho, \quad t, \end{array} \tag{3.2}$$

where $n \geqslant 0$ and $\rho \geqslant 2$. For each $i = 1, \dots n$ the variable x_i is called the *dual* of x_{i+n}, and conversely. Let

$$\Delta(p) = \begin{cases} p+n, & \text{if } 1 \leqslant p \leqslant n, \\ p-n, & \text{if } n+1 \leqslant p \leqslant 2n. \end{cases}$$

Then for each $p = 1, \dots, 2n$ the dual of x_p is $x_{\Delta(p)}$. Note that $\Delta(\Delta(p)) = p$.

Each pair of duals x_i and x_{i+n} must satisfy the *equality of duals*:

$$x_i = x_{i+n} \qquad (i=1, \dots, n). \tag{3.3}$$

The variables $l_1, \dots, l_\rho$ are called the *boundaries* of the generalized equation. To each boundary l_i there corresponds a *boundary equality*:

$$l_i r_i = t \qquad (i=1, \dots, \rho). \tag{3.4}$$

3. A *boundary comparison table.*

By a boundary comparison table we mean a system of conditions on the variables $l_1, \dots, l_\rho, t$ consisting of an *initial equality* $l_1 = 1$, a *final equality* $l_\rho = t$, and a nonempty finite list of *inequalities* of the form $l_i < l_j$ which is closed with respect to transitivity (i.e. if the table contains $l_i < l_j$ and $l_j < l_k$, then it also contains $l_i < l_k$).

A boundary comparison table must satisfy the following conditions:

D.1. *A boundary comparison table contains the inequalities $l_1 < l_i$ for $i \neq 1$.*

D.2. *A boundary comparison table contains the inequalities $l_i < l_\rho$ for $i \neq \rho$.*

The expression "the comparison table contains the inequality $l_i \leqslant l_j$" means that either $i = j$ or the table contains the inequality $l_i < l_j$. The notation $l_i < l_j < l_k$ is an abbreviation for the conjunction of the two inequalities $l_i < l_j$ and $l_j < l_k$.

4. A *base situation table.*

The generalized equation Ω contains a function $\psi(i)$ defined on the set $\{2n + 1, \dots, 2n + m\}$ with values in the set $\{1, \dots, \omega\}$ and assuming *all* values in this set.

The variables $x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}$ and coefficients $a_{\psi(2n+1)}, \dots, a_{\psi(2n+m)}$ are called the *bases* of Ω and are denoted, respectively, by

$$w_1, \dots, w_n, w_{n+1}, \dots, w_{2n}, w_{2n+1}, \dots, w_{2n+m}. \tag{3.5}$$

We call (3.5) the *list of bases* of Ω. To each base w_i we associate a *situation equality*:

$$l_{\alpha(i)} w_i r_{\beta(i)} = t \qquad (i=1, \ldots, 2n+m), \tag{3.6}$$

where $\alpha(i)$ and $\beta(i)$ are functions defined on the set $\{1, \ldots, 2n+m\}$ with values in the set $\{1, \ldots, \rho\}$. The boundary comparison table contains the inequality

$$l_{\alpha(i)} < l_{\beta(i)} \quad (i=1, \ldots, 2n+m). \tag{3.7}$$

We call $l_{\alpha(i)}$ the *left boundary* of the base w_i, and $l_{\beta(i)}$ the *right boundary* of w_i. Equality (3.6) indicates the two boundaries between which the base w_i is situated.

Certain boundaries of the table (3.2)

$$l_{i_1}, \ldots, l_{i_\tau} \tag{3.8}$$

are fixed and are called *initial boundaries*. Initial boundaries are related to the situation of the bases $w_1, \ldots, w_{2n+m}$ by the following condition.

D.3. *The left boundary of each base w_i is an initial boundary.*

However, it is not necessarily true that every initial boundary is the left boundary of some base.

A boundary is called *terminal* if and only if it is the right boundary of some base. A particular boundary can be both initial and terminal. Initial and terminal boundaries are called *essential*, the others *inessential*.

5. A *list of boundary connections.*

The generalized equation Ω contains a finite (possibly empty) list of boundary connections.

Each *boundary connection* has the form

$$l_p, x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_k}, l_q, \tag{3.9}$$

where $k > 0$. The boundary l_p is called the *original* boundary of the boundary connection (3.9), and l_q the *concluding* boundary.

For each boundary connection (3.9) of Ω the boundary comparison table must contain the inequalities

$$l_{\alpha(\lambda_1)} < l_p < l_{\beta(\lambda_1)}, \tag{3.10}$$

$$l_{\alpha(\lambda_2)} \leqslant l_{\alpha(\Delta(\lambda_1))},$$

$$l_{\alpha(\lambda_{i+1})} \leqslant l_{\alpha(\Delta(\lambda_i))} \ (i = 1, \ldots, u-1)$$

$$l_{\alpha(\lambda_{j+1})} \geqslant l_{\alpha(\Delta(\lambda_j))} \ (j = u, \ldots, k-1) \tag{3.11}$$

for some u, where $1 \leqslant u \leqslant k$,

. . . .

$$l_{\alpha(\lambda_k)} \leqslant l_{\alpha(\Delta(\lambda_{k-1}))},$$

$$l_{\alpha(\Delta(\lambda_k))} < l_q < l_{\beta(\Delta(\lambda_k))}. \tag{3.12}$$

We say that the boundary connection (3.9) connects the boundary l_p to the boundary l_q by means of the *path*

$$x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_k}. \tag{3.13}$$

The number k is called the *length of the path* (3.13).

In addition to the notation (3.9) for a boundary connection we will also use

$$l_p,\ (x_{\lambda_1},\ x_{\Delta(\lambda_1)}),\ (x_{\lambda_2},\ x_{\Delta(\lambda_2)}),\ \ldots,\ (x_{\lambda_k},\ x_{\Delta(\lambda_k)}),\ l_q, \tag{3.14}$$

which we will call the *complete notation* for the boundary connection (3.9). In particular, the complete notation will be useful in dealing with conditions (3.10)–(3.12). It is easy to see that the notation (3.9) for a boundary connection is uniquely determined by the complete notation, and conversely.

The boundary connections and boundaries of the generalized equation Ω are related by the following conditions.

D.4. *Each inessential boundary is the original boundary of some boundary connection.*

D.5. *The concluding boundary of each boundary connection is the right boundary of some base.*([1])

It follows from D.4 and D.5 that each inessential boundary is connected to some essential boundary by means of a boundary connection.

Let us turn to the definition of a solution of a generalized equation.

We say that a table of words

$$\begin{array}{l} X_1,\ \ldots,\ X_n,\ X_{n+1},\ \ldots,\ X_{2n}, \\ L_1,\ \ldots,\ L_\rho,\quad R_1,\ \ldots,\ R_\rho,\quad T \end{array} \tag{3.15}$$

in the alphabet (3.1) satisfies the equality $A_1 = A_2$, where A_1 and A_2 are words in the alphabet (3.1), (3.2), if as a result of replacing each small Roman letter of the alphabet (3.2) by the word denoted in (3.15) by the corresponding capital letter we obtain a graphical equality. We say that the table of words (3.15) satisfies the inequality $l_i < l_j$ if $\partial(L_i) < \partial(L_j)$.

In the following definition it is convenient to use the complete notation for a boundary connection.

We say that the table of words (3.15) satisfies a given boundary connection (3.14) if there exist nonempty words

$$\begin{array}{l} B_1,\ \ldots,\ B_k, \\ C_1,\ \ldots,\ C_k \end{array} \tag{3.16}$$

in the alphabet (3.1) such that

$$X_{\lambda_i} \overset{\circ}{=} B_i C_i \quad (i = 1,\ \ldots,\ k), \tag{3.17}$$

$$L_p \overset{\circ}{=} L_{\alpha(\lambda_1)} B_1, \tag{3.18}$$

$$\begin{array}{l} L_{\alpha(\Delta(\lambda_1))} B_1 \overset{\circ}{=} L_{\alpha(\lambda_2)} B_2, \\ L_{\alpha(\Delta(\lambda_2))} B_2 \overset{\circ}{=} L_{\alpha(\lambda_3)} B_3, \\ \ldots\ldots\ldots\ldots\ldots \\ L_{\alpha(\Delta(\lambda_{k-1}))} B_{k-1} \overset{\circ}{=} L_{\alpha(\lambda_k)} B_k, \end{array} \tag{3.19}$$

$$L_{\alpha(\Delta(\lambda_k))} B_k \overset{\circ}{=} L_q. \tag{3.20}$$

([1]) *Added in proof.* To condition D.5 add: w_π, *where if for some i we have* $l_{\alpha(\lambda_{i+1})} > l_{\alpha(\Delta(\lambda_i))}$, *then* $l_{\alpha(\pi)} > l_{\alpha(\Delta(\lambda_k))}$.

By a *solution* of a generalized equation Ω we mean any table of words (3.15) which satisfies the equalities of duals (3.3), the boundary equalities (3.4), all conditions of the boundary comparison table, the base situation equalities (3.6), and all boundary connections of Ω.

By the *exponent of periodicity* of the solution (3.15) of Ω we mean the exponent of periodicity (in the sense of §1) of the list of words $X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}$.

LEMMA 3.1. *For any coefficient equation Σ in a free semigroup it is possible to construct a list of generalized equations*

$$\Omega_1, \ldots, \Omega_r \tag{3.21}$$

without boundary connections such that Σ has a solution with exponent of periodicity s if and only if at least one Ω_i in (3.21) *has such a solution.*

PROOF. In view of Lemmas 2.1–2.5, we can construct from Σ a list of systems of type Σ_5 such that Σ has a solution with exponent of periodicity s if and only if at least one system in this list has such a solution. It is easy to see that any system of type Σ_5 is a generalized equation without boundary connections. Indeed, a system Σ_5 (see §2) is defined by a coefficient alphabet (2.14), a table of word variables (2.11), (2.5), the equalities of duals (2.12), the boundary equalities (2.7), a boundary comparison table (2.10), and a base situation table (2.15).

In the sequel we will often use the following simple lemma.

LEMMA 3.2. *If a generalized equation Ω has a solution* (3.15), *then for any i and j such that $1 \leqslant i, j \leqslant \rho$ one of the two words L_i or L_j is a beginning of the other.*

The lemma is an immediate consequence of the definition of a solution of a generalized equation and the equalities (3.4).

§4. Normalized generalized equations

Any segment of the path of a boundary connection is called a *subpath* of that boundary connection. Consider a subpath

$$x_{\lambda_i}, x_{\lambda_{i+1}} \tag{4.1}$$

of the boundary connection (3.9). In the complete notation this subpath has the form

$$(x_{\lambda_i}, x_{\Delta(\lambda_i)}),\ (x_{\lambda_{i+1}}, x_{\Delta(\lambda_{i+1})}).$$

According to (3.11), either

$$\alpha(\lambda_{i+1}) = \alpha(\Delta(\lambda_i)) \tag{4.2}$$

or the boundary comparison table contains the inequality

$$l_{\alpha(\lambda_{i+1})} < l_{\alpha(\Delta(\lambda_i))}. \tag{4.3}$$

In the case (4.3) we say that there is a *shift* in the subpath (4.1), and in the case (4.2) that there is *no shift*. We say that there is no shift in a subpath $x_{\lambda_h}, \ldots, x_{\lambda_g}$ of the boundary connection (3.9) if there is no shift in any pair $x_{\lambda_i}, x_{\lambda_{i+1}}$ of this subpath.

Suppose the boundary connection (3.9) contains a subpath $\pi_1, x_{\lambda_h}, \pi_2, x_{\lambda_g}, \pi_3$ without a shift, where π_1, π_2 and π_3 are certain subpaths and $\lambda_h = \lambda_g$. If all variables x_i which occur in the subpath π_2, x_{λ_g} also occur among the variables of $\pi_1, x_{\lambda_h}, \pi_3$, then the subpath π_2, x_{λ_g} is called a *superfluous* subpath of the boundary connection (3.9).

LEMMA 4.1. *If the boundary connection* (3.9) *contains a subpath*

$$x_{\lambda_i}, \ \ldots, \ x_{\lambda_j} \tag{4.4}$$

without a shift and if $j - i \geqslant (2n)^2$, *then it contains a superfluous subpath.*

PROOF. The path of any boundary connection can involve at most $2n$ x-variables. The length of subpath (4.4) is at least $(2n)^2 + 1$; hence some variable x_t occurs in this subpath at least $2n + 1$ times. Represent (4.4) in the form

$$\pi', \ x_t, \ \pi_1, \ x_t, \ \pi_2, \ \ldots, \ x_t, \ \pi_{2n}, \ x_t, \ \pi'',$$

where the subpaths $\pi_1, \ldots, \pi_{2n}$ do not contain x_t. For any $r = 1, \ldots, 2n$, either the subpath π_r consists of variables occurring in the subpaths $\pi_1, \ldots, \pi_{r-1}, \pi_{r+1}, \ldots, \pi_{2n}$, and then the subpath π_r, x_t is superfluous in (3.9), or else the subpath π_r contains some x-variable which does not occur in those subpaths. Since the number of subpaths $\pi_1, \ldots, \pi_{2n}$ is greater than the number of variables occurring therein, the boundary connection (3.9) contains a superfluous subpath. The lemma is proved.

A variable x_p is called *matched* if its left boundary is the same as the left boundary of its dual $x_{\Delta(p)}$ (i.e. if $\alpha(p) = \alpha(\Delta(p))$).

Of special significance in a generalized equation are the boundary l_1 and those bases for which l_1 is the left boundary. Such bases are called *leading bases*. Let

$$|p| = \begin{cases} p, & \text{if } 1 \leqslant p \leqslant n, \\ p-n, & \text{if } n+1 \leqslant p \leqslant 2n. \end{cases}$$

Let x_ν be an unmatched leading variable. Suppose that, for any unmatched leading variable x_j different from x_ν, either the boundary comparison table contains the inequality $l_{\beta(j)} < l_{\beta(\nu)}$, or else $\beta(j) = \beta(\nu)$ and $|j| < |\nu|$. Then the variable x_ν is called the *carrier* of the generalized equation. Obviously, a generalized equation has at most one carrier.

A subpath $x_{\lambda_i}, x_{\lambda_{i+1}}$ of the boundary connection (3.9) is called a *loop* if $x_{\lambda_{i+1}}$ is the carrier of the generalized equation and $\lambda_i = \Delta(\lambda_{i+1})$. In the complete notation a loop has the form

$$(x_{\Delta(\nu)}, \ x_\nu), \quad (x_\nu, \ x_{\Delta(\nu)}),$$

where x_ν is the carrier of the generalized equation.

A matched variable x_{λ_p} of the boundary connection (3.9) is called a *majorized* variable of (3.9) in the following three cases: if $p = 1$, if $p = k$, or if for $1 < p < k$ the boundary comparison table contains at least one of the following two inequalities:

$$l_{\beta(\lambda_{p+1})} \leqslant l_{\beta(\Delta(\lambda_p))},$$

$$l_{\beta(\Delta(\lambda_{p-1}))} \leqslant l_{\beta(\lambda_p)}.$$

A generalized equation Ω is called *normalized* if the following conditions are satisfied:

N.1. *The boundary comparison table contains the conditions*

$$l_1 = 1, \quad l_i < l_{i+1} \ (i = 1, \ \ldots, \ \rho - 1), \quad l_\rho = t.$$

(We do not explictly list the transitive consequences of the inequalities.)

N.2. *No boundary connection contains superfluous subpaths.*

N.3. *If the generalized equation has exactly one leading variable, then its boundary connections contain no loops.*

N.4. *No boundary connection contains majorized variables.*

LEMMA 4.2. *Suppose a generalized equation Ω satisfying* N.2 *has a solution with exponent of periodicity s. Let $2n$ be the number of x-variables of Ω, and k the length of a path of some boundary connection. Then $k \leqslant 4n^3(n+1)(s+2)$.*

PROOF. Suppose Ω has the solution (3.15) and contains the boundary connection (3.9). The path of this boundary connection is composed of variables $x_1, \ldots, x_n$ and their duals. Since the solution (3.15) satisfies the equalities of duals (3.3), there exist nonempty words $X_1, \ldots, X_n$ in the alphabet (3.1) which (possibly with repetitions) make up a sequence

$$X_{\lambda_1}, X_{\lambda_2}, \ \ldots, \ X_{\lambda_k} \quad (k > 0) \tag{4.5}$$

for which, by definition of a solution of a generalized equation, there exist nonempty words $B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ such that

$$X_{\lambda_i} \overset{\circ}{=} B_i C_i \quad (i = 1, \ \ldots, \ k), \tag{4.6}$$

$$L_{\alpha(\Delta(\lambda_i))} B_i \overset{\circ}{=} L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i = 1, \ \ldots, \ k-1). \tag{4.7}$$

Since the solution (3.15) satisfies (3.11), we have

$$\partial(L_{\alpha(\lambda_{i+1})}) \leqslant \partial(L_{\alpha(\Delta(\lambda_i))}) \quad (i = 1, \ \ldots, \ k-1). \tag{4.8}$$

Since (3.15) satisfies the boundary equalities (3.4), it follows from the length inequalities (4.8) that

$$L_{\alpha(\Delta(\lambda_i))} \overset{\circ}{=} L_{\alpha(\lambda_{i+1})} S_i \quad (i = 1, \ \ldots, \ k-1) \tag{4.9}$$

for certain $S_1, \ldots, S_{k-1}$. Since (3.15) satisfies the base situation equalities (3.6), we have

$$L_{\alpha(\Delta(\lambda_i))} X_{\Delta(\lambda_i)} R_{\beta(\Delta(\lambda_i))} \overset{\circ}{=} L_{\alpha(\lambda_{i+1})} X_{\lambda_{i+1}} R_{\beta(\lambda_{i+1})} \quad (i = 1, \ \ldots, \ k-1). \tag{4.10}$$

From the graphical equalities (4.7) and (4.9) we obtain $B_{i+1} \overset{\circ}{=} S_i B_i$ for $i = 1, \ldots, k-1$. Since (3.15) satisfies the equalities of duals (3.3), it follows that $X_{\lambda_i} \overset{\circ}{=} X_{\Delta(\lambda_i)}$ for $i = 1, \ldots, k-1$. Substituting the values of X_{λ_i} and $X_{\lambda_{i+1}}$ from (4.6) into (4.10) and cancelling, using (4.7), we obtain

$$C_i R_{\beta(\Delta(\lambda_i))} \overset{\circ}{=} C_{i+1} R_{\beta(\lambda_{i+1})} \quad (i = 1, \ \ldots, \ k-1).$$

If there is a shift in the subpath $x_{\lambda_i}, x_{\lambda_{i+1}}$, then, by (4.3),

$$\partial(L_{\alpha(\lambda_{i+1})}) < \partial(L_{\alpha(\Delta(\lambda_i))}),$$

i.e. $\partial(S_i) > 0$. If there is no shift in the subpath $x_{\lambda_i}, x_{\lambda_{i+}}$, then, by (4.2),

$$\partial(L_{\alpha(\lambda_{i+1})}) = \partial(L_{\alpha(\Delta(\lambda_i))}),$$

i.e. $\partial(S_i) = 0$. According to condition N.2 and Lemma 4.1, any subpath $x_{\lambda_i}, \ldots, x_{\lambda_j}$ such that $j - i \geqslant (2n)^2$ has a shift. Consequently, if $j - i \geqslant (2n)^2$, then $\partial(S_i S_{i+1} \cdots S_{j-1} S_j) > 0$. Thus we can apply Lemma 1.4. Therefore, there exists in the sequence (4.5) a word X_t such that $X_t \overset{\circ}{=} P^{s_1} Q$, where P is a nonempty word and $k \leqslant 4n^3(n+1)(s_1+2)$. If s is the exponent of periodicity of solution (3.15), then $s_1 \leqslant s$. Consequently, $k \leqslant 4n^3(n+1)(s+2)$.

§5. Estimate of the number of normalized admissible generalized equations

A generalized equation Ω is called *elementary* if its coefficient alphabet consists of a single letter, and *nonelementary* if its coefficient alphabet contains more than one letter.

LEMMA 5.1. *There exists an algorithm for recognizing whether any elementary generalized equation has a solution.*

PROOF. Suppose the coefficient alphabet of a generalized equation Ω consists of the single letter a_1. A solution of such a generalized equation must be a table of words of the form (3.15), each of whose components is a power of a_1. Therefore, the conditions which must be satisfied by the components of the table reduce to certain formulas of the elementary theory of arithmetic without multiplication. The existence of the desired algorithm follows from the decidability of this theory (see [4], §2.5).

Let Ω^0 denote the elementary generalized equation obtained from a generalized equation Ω by replacing its coefficient alphabet by the single letter a_1 and replacing each coefficient a_i in the list of bases and in the base situation table by a_1.

LEMMA 5.2. *If the generalized equation Ω has a solution, then so does the corresponding generalized equation Ω^0.*

PROOF. Suppose Ω has a solution (3.15). We construct the table of words

$$a_1^{\partial(X_1)}, \ \ldots, \ a_1^{\partial(X_n)}, \ a_1^{\partial(X_{n+1})}, \ \ldots, \ a_1^{\partial(X_{2n})},$$

$$a_1^{\partial(L_1)}, \ \ldots, \ a_1^{\partial(L_\rho)}, \quad a_1^{\partial(R_1)}, \ \ldots, \ a_1^{\partial(R_\rho)}, \quad a_1^{\partial(T)}.$$

It is easy to see that this table is a solution of Ω^0. The lemma is proved.

An elementary generalized equation is called *true* if it has a solution.

A generalized equation Ω is called *false* in either of two cases: Ω is *false with respect to variables* if Ω^0 has no solution; Ω is *false with respect to coefficients* if for certain i and j, where $i \neq j$, the left boundary of the base a_i is the same as the left boundary of the base a_j.

LEMMA 5.3. *If a generalized equation is false, then it has no solution.*

PROOF. An equation that is false with respect to variables has no solution by Lemma 5.2.

Suppose an equation that is false with respect to coefficients has a solution. Then this solution must satisfy the base situation equalities

$$l_{p_1}a_ir_{p_2} = t, \quad l_{p_1}a_jr_{p_3} = t$$

with $i \neq j$, i.e. must satisfy the graphical equality

$$L_{p_1}a_iR_{p_2} \overset{\circ}{=} L_{p_1}a_jR_{p_3},$$

which is impossible. The lemma is proved.

Since an elementary equation Ω agrees with Ω^0, any elementary equation is either true or else false with respect to variables.

A nonelementary generalized equation is called *admissible* if it is not false.

The next lemma follows easily from Lemma 5.1 and the appropriate definitions.

LEMMA 5.4. *The set of all generalized equations consists of three mutually disjoint classes:*

true, false and admissible equations. There exists an algorithm for recognizing whether any generalized equation is true, false, or admissible.

LEMMA 5.5. *If a generalized equation* Ω *contains the two boundary connections*

$$l_p,\ x_{\lambda_1},\ \ldots,\ x_{\lambda_k},\ l_q, \tag{5.1}$$

$$l_h,\ x_{\lambda_1},\ \ldots,\ x_{\lambda_k},\ l_q \tag{5.2}$$

and the inequality $l_p < l_h$, *then it is false.*

PROOF. Suppose Ω^0 has the solution (3.15) in the alphabet a_1. This solution must satisfy both the boundary connection (5.1) and the boundary connection (5.2). Therefore, on one hand, there exist nonempty words (3.16) such that conditions (3.17)–(3.20) are satisfied, and, on the other, there exist nonempty words $B'_1, \ldots, B'_k$ and $C'_1, \ldots, C'_k$ such that

$$X_{\lambda_i} \doteq B'_i C'_i \quad (i = 1, \ldots, k),$$

$$L_h \doteq L_{\alpha(\lambda_1)} B'_1,$$

$$L_{\alpha(\Delta(\lambda_i))} B'_i \doteq L_{\alpha(\lambda_{i+1})} B'_{i+1} \quad (i = 1, \ldots, k-1),$$

$$L_{\alpha(\Delta(\lambda_k))} B'_k \doteq L_q.$$

Combining these equalities, we obtain $B_i \doteq B'_i$ for $i = 1, \ldots, k$; hence $L_p = L_h$. Since this solution satisfies $l_p < l_h$, it follows that $\partial(L_p) < \partial(L_h)$. This contradiction shows that Ω^0 has no solution, i.e. Ω is false. The lemma is proved.

The proofs of the next two lemmas are analogous to the proof of Lemma 5.5.

LEMMA 5.6. *If a generalized equation* Ω *contains the two boundary connections*

$$l_p,\ x_{\lambda_1},\ \ldots,\ x_{\lambda_k},\ l_q,$$

$$l_p,\ x_{\lambda_1},\ \ldots,\ x_{\lambda_k},\ l_r$$

and the inequality $l_q < l_r$, *then it is false.*

LEMMA 5.7. *If the boundary* l_1 *is an original or concluding boundary of some boundary connection of* Ω, *then* Ω *is false.*

As is evident from its definition (see §3), a generalized equation Ω contains a whole series of parameters.

The parameter ω uniquely determines the coefficient alphabet of Ω.

The parameter n determines the number of pairs of x-variables.

The parameter ρ determines the number of boundaries.

The list of inequalities in the boundary comparison table $l_{\alpha_1} < l_{\beta_1}, \ldots, l_{\alpha_u} < l_{\beta_u}$ is uniquely determined by the list of ordered pairs

$$(\alpha_1,\ \beta_1),\ \ldots,\ (\alpha_u,\ \beta_u). \tag{5.3}$$

The parameter m determines the number of bases which are not variables.

The functions $\psi(i)$, $\alpha(i)$ and $\beta(i)$ determine the base situation table of Ω.

The parameter τ determines the number of initial boundaries (see (3.8)).

The unordered list of numbers

$$i_1,\ \ldots,\ i_\tau, \tag{5.4}$$

chosen from $1, \ldots, \rho$, determines the initial boundaries.

In addition, let μ be the number of boundary connections of Ω, and let δ be the maximum length of the paths of all boundary connections of Ω.

Each boundary connection of the form (3.9) is uniquely determined by the vector

$$(p, \lambda_1, \ldots, \lambda_k, q). \tag{5.5}$$

Thus, a generalized equation Ω is uniquely determined by the parameters ω, n, ρ, m, τ, μ, δ, the functions $\psi(i)$, $\alpha(i)$, $\beta(i)$, the list of ordered pairs (5.3), the list of numbers (5.4), and a list of μ vectors of the form (5.5).

By the *principal parameters* of a generalized equation Ω we mean the parameters n, m, τ and δ, where n is the number of pairs of x-variables, m is the number of bases which are not variables, τ is the number of initial boundaries, and δ is the maximum length of the paths of all boundary connections.

LEMMA 5.8. *There exists a recursive function $F(n, m, \tau, \delta)$ such that the number of distinct normalized admissible generalized equations whose principal parameters are bounded by n_0, m_0, τ_0 and δ_0 does not exceed $F(n_0, m_0, \tau_0, \delta_0)$.*

PROOF. Since the function $\psi(i)$ defined on the set $\{2n+1, \ldots, 2n+m\}$ assumes *all* values of the set $\{1, \ldots, \omega\}$, it follows that

$$\omega \leqslant m. \tag{5.6}$$

The number of terminal boundaries of Ω does not exceed the number of bases of this equation, i.e. the number $2n + m$. If σ is the number of essential boundaries, then

$$\sigma \leqslant 2n+m+\tau. \tag{5.7}$$

The number of distinct paths of boundary connections whose length does not exceed δ is bounded by the number $(2n+1)^\delta$. According to condition D.5, the number of distinct concluding boundaries does not exceed the number of bases, i.e. the number $2n + m$. By Lemma 5.5, in a normalized admissible generalized equation the original boundary of any boundary connection is uniquely determined by the path of this connection and the concluding boundary. Thus, the total number of distinct boundary connections which can be defined in a normalized admissible equation does not exceed $(2n+1)^\delta(2n+m)$. In particular,

$$\mu \leqslant (2n+1)^\delta (2n+m). \tag{5.8}$$

By condition D.4 and Lemma 5.5, in a normalized admissible generalized equation the number of inessential boundaries does not exceed the number of boundary connections, and hence

$$\rho \leqslant \mu+\sigma. \tag{5.9}$$

From (5.7)–(5.9) we obtain

$$\rho \leqslant (2n+1)^\delta (2n+m)+2n+m+\tau. \tag{5.10}$$

The number of distinct functions $\psi(i)$ does not exceed ω^m.

The number of distinct functions $\alpha(i)$ and the number of distinct functions $\beta(i)$ is bounded by ρ^{2n+m}.

The boundary comparison table of a normalized admissible generalized equation is uniquely determined by the number ρ.

The number of distinct lists of initial boundaries does not exceed 2^ρ.

Two lists of boundary connections are distinct if one list contains a boundary connection not contained in the other. We have shown that the total number of distinct boundary connections does not exceed $(2n + 1)^{\delta}(2n + m)$. Consequently, the number of distinct lists of boundary connections does not exceed

$$2^{(2n+1)^{\delta}(2n+m)}.$$

We have proved that the parameters of a normalized admissible generalized equation are bounded by certain recursive functions of the principal parameters. Consequently, there exists a recursive function $F(n, m, \tau, \delta)$ such that the number of distinct normalized admissible generalized equations whose principal parameters are bounded by n_0, m_0, τ_0 and δ_0 does not exceed $F(n_0, m_0, \tau_0, \delta_0)$.

§6. Normalization of generalized equations

By the *length* of a given solution (3.15) of a generalized equation Ω we mean the number $\partial(T)$.

By the *vector of a base* w_i of a generalized equation Ω relative to a given solution (3.15) we mean that $\partial(T)$-dimensional vector $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$, in which there are $\partial(L_{\alpha(i)})$ zeros, followed by $\partial(X_i)$ ones if w_i is an x-variable or a single one if w_i is a coefficient, followed by $\partial(R_{\beta(i)})$ zeros.

By the *distribution vector of a solution* (3.15) we mean the vector equal to the sum of all vectors of the bases relative to (3.15). We order distribution vectors of the same dimension lexicographically.

By the *path index* of a generalized equation we mean the ordered pair (b, c), where b is the number of occurrences of duals of leading bases in the paths of all boundary connections and c is the sum of the lengths of all paths.(2) We order path indices lexicographically.

By the *index of a solution* (3.15) of a generalized equation Ω we mean the ordered triple

$$(\partial(T), (r_1, \dots, r_{\partial(T)}), (b, c)),$$

where $\partial(T)$ is the length of solution (3.15), $(r_1, \dots, r_{\partial(T)})$ is the distribution vector of solution (3.15), and (b, c) is the path index of the generalized equation. We order indices of solutions lexicographically, viewing them as 3-dimensional vectors.

LEMMA 6.1. *For any admissible generalized equation Ω it is possible to construct a list of generalized equations*

$$\Omega_1, \dots, \Omega_r \tag{6.1}$$

such that the following conditions are satisfied.

1) *Each Ω_i in* (6.1) *either is admissible and satisfies condition* N.1, *or else is false.*

2) *The principal parameters of each Ω_i in* (6.1) *do not exceed the corresponding parameters of Ω.*

3) *The path index of each Ω_i in* (6.1) *does not exceed the path index of Ω.*

4) *If Ω has a solution with index I and exponent of periodicity s, then some Ω_i in* (6.1) *has a solution with index I_1, where $I_1 \leqslant I$, and exponent of periodicity s. If some Ω_i in* (6.1) *has a solution, then so does Ω.*

PROOF. We will first prove that the boundary comparison table of an admissible generalized equation cannot simultaneously contain $l_i < l_j$ and $l_j < l_i$. Indeed, if the

(2) *Added in proof.* The path index should be defined as the vector $(\delta, c_{\delta}, c_{\delta-1}, \dots, c_1)$, where δ is as before and c_i is the number of boundary connections with path length i.

comparison table of a generalized equation Ω contains both of these inequalities, then, since Ω is admissible, the generalized equation Ω^0 has a solution (3.15), in a one-letter alphabet a_1, such that $\partial(L_i) < \partial(L_j)$ and $\partial(L_j) < \partial(L_i)$, which is impossible.

Boundaries l_i and l_j are called *comparable* if the boundary comparison table contains either $l_i < l_j$ or $l_j < l_i$, and they are called *noncomparable* otherwise.

We will prove the lemma by induction on the number of noncomparable pairs of boundaries.

Basis of the induction. Suppose that each pair of boundaries of Ω is comparable. Then the inequalities of the boundary comparison table can be written in the form $l_{q_i} < l_{q_{i+1}}$ $(i = 1, \dots, \rho - 1)$, where $q_1 = 1$ and $q_\rho = \rho$. (We do not explicitly list the transitive consequences.) Renumbering the boundaries as follows: $l_{q_i} \to l_i$ $(i = 2, \dots, \rho - 1)$, we obtain an admissible generalized equation satisfying condition N.1 for which all assertions of the lemma are satisfied trivially.

Inductive step. Assume the lemma is true for all admissible generalized equations with less than h noncomparable pairs of boundaries, where $h > 0$, and suppose Ω contains h noncomparable pairs.

Suppose the boundaries l_i and l_j of Ω are noncomparable. We construct a generalized equation Ω_1 by adding to the boundary comparison table of Ω the inequality $l_i < l_j$ and closing the resulting list of inequalities with respect to transitivity. We construct a generalized equation Ω_2 by adding to the boundary comparison table of Ω the inequality $l_j < l_i$ and closing the resulting list of inequalities with respect to transitivity. We construct a generalized equation Ω_3 by identifying in Ω the boundaries l_i and l_j (i.e. throughout the description of Ω we replace the variables l_j and r_j by l_i and r_i, and then delete one of each two coincident variables and conditions).

Since Ω is admissible, it is nonelementary. By construction, each of Ω_1, Ω_2 and Ω_3 is nonelementary; hence each is not true. By Lemma 5.4, each of Ω_1, Ω_2 and Ω_3 is either false or admissible.

The principal parameters of Ω_1, Ω_2 and Ω_3 agree with the corresponding principal parameters of Ω in all cases but one: if both boundaries l_i and l_j are initial in Ω, then the number of initial boundaries of Ω_3 is one less than the number of initial boundaries of Ω.

It is easy to see that the number of occurrences of duals of leading bases and the sum of the lengths of the paths of all boundary connections of each Ω_i in the list Ω_1, Ω_2, Ω_3 do not exceed the corresponding number and sum for Ω.

It is also easy to see that any table of words (3.15) which is a solution of some Ω_i in the list Ω_1, Ω_2, Ω_3 is at the same time a solution of Ω. If a table of words (3.15) is a solution of Ω, then this table is a solution of Ω_1 if $\partial(L_i) < \partial(L_j)$, a solution of Ω_2 if $\partial(L_j) < \partial(L_i)$, and a solution of Ω_3 if $\partial(L_i) = \partial(L_j)$.

By Lemma 5.4, we can distinguish the admissible equations among Ω_1, Ω_2 and Ω_3. Any admissible equation in this list contains less than h noncomparable pairs of boundaries. By the inductive assumption, for each such equation it is possible to construct a list of generalized equations for which all four conditions of the lemma are satisfied. Combining the lists thus constructed with the false equations among Ω_1, Ω_2 and Ω_3, we obtain the desired list of generalized equations.

LEMMA 6.2. *Suppose an admissible generalized equation Ω satisfying condition* N.1 *contains a boundary connection with a superfluous subpath. Then it is possible to construct a generalized equation Ω_1 such that the following conditions are satisfied.*

1) Ω_1 *is admissible and satisfies condition* N.1.

2) *The principal parameters of* Ω_1 *do not exceed the corresponding parameters of* Ω.

3) *The path index of* Ω_1 *is smaller than the path index of* Ω.

4) *If* Ω *has a solution with index* I *and exponent of periodicity* s, *then* Ω_1 *has a solution with index* I_1, *where* $I_1 < I$, *and exponent of periodicity* s. *If* Ω_1 *has a solution, then so does* Ω.

PROOF. Suppose the boundary connection

$$l_p,\ \pi_0,\ \pi_1,\ x_{\lambda_h},\ \pi_2,\ x_{\lambda_g},\ \pi_3,\ \pi_4,\ l_q, \tag{6.2}$$

where the π_i are subpaths, contains the subpath with no shift

$$\pi_1,\ x_{\lambda_h},\ \pi_2,\ x_{\lambda_g},\ \pi_3. \tag{6.3}$$

Suppose $\lambda_h = \lambda_g$ and the subpath

$$\pi_2,\ x_{\lambda_g} \tag{6.4}$$

is superfluous in the subpath (6.3) of the boundary connection (6.2). Delete from (6.2) the subpath (6.4). We obtain the sequence

$$l_p,\ \pi_0,\ \pi_1,\ x_{\lambda_h},\ \pi_3,\ \pi_4,\ l_q. \tag{6.5}$$

Since $\lambda_h = \lambda_g$, this sequence satisfies conditions (3.10)–(3.12) and can therefore be a boundary connection of Ω.

Let Ω_1 be the generalized equation obtained as a result of replacing (6.2) by (6.5) in Ω.

Since under the passage $\Omega \to \Omega_1$ the boundary comparison table of Ω is not affected, Ω_1 satisfies condition N.1. The parameters n, m, τ of Ω_1 agree with the n, m, τ of Ω, and the parameter δ of Ω_1 does not exceed the δ of Ω. The path index of Ω_1 is smaller than the path index of Ω.

Suppose Ω has a solution (3.15) with index I and exponent of periodicity s. We will prove that (3.15) is also a solution of Ω_1. It suffices to prove that (3.15) satisfies the boundary connection (6.5).

By assumption, (3.15) satisfies (6.2), i.e. there exist nonempty words $B_1, \dots, B_k$ and $C_1, \dots, C_k$ such that conditions (3.17)–(3.20) are satisfied for (6.2).

Since the subpath (6.3) has no shift, for any pair x_{λ_i}, $x_{\lambda_{i+1}}$ of this subpath we have

$$L_{\alpha(\lambda_{i+1})} \overset{\circ}{=} L_{\alpha(\Delta(\lambda_i))}; \tag{6.6}$$

hence, in view of (3.19), for any pair x_{λ_i}, $x_{\lambda_{i+1}}$ of (6.3) we have

$$B_i \overset{\circ}{=} B_{i+1}. \tag{6.7}$$

If the path π_3, π_4 is nonempty, then, by (3.19), for the boundary connection (6.2) we have

$$L_{\alpha(\Delta(\lambda_g))}B_g \overset{\circ}{=} L_{\alpha(\lambda_{g+1})}B_{g+1}. \tag{6.8}$$

Since, in view of (6.7), $B_h \overset{\circ}{=} B_g$ and $\lambda_h = \lambda_g$, it follows from (6.8) that

$$L_{\alpha(\Delta(\lambda_h))}B_h \overset{\circ}{=} L_{\alpha(\lambda_{g+1})}B_{g+1},$$

and hence (3.19) is satisfied for (6.5). Conditions (3.18) and (3.20) are obvious.

If the path π_3, π_4 is empty, then condition (3.20) for the boundary connection (6.2) says that

$$L_{\alpha(\Delta(\lambda_g))}B_g \overset{\circ}{=} L_q. \tag{6.9}$$

Since $B_h \doteq B_g$ and $\lambda_h = \lambda_g$, it follows from (6.9) that (3.20) is satisfied for (6.5). Conditions (3.18) and (3.19) are obvious.

We have proved that any table of words which is a solution of Ω is also a solution of Ω_1. Consequently, the solutions of Ω and of Ω_1 have the same exponent of periodicity, length and distribution vector. Since the path index of Ω_1 is smaller than the path index of Ω, the index of the solution of Ω_1 is smaller than I.

We will now prove that Ω_1 is admissible. Since Ω is admissible, it is nonelementary, not false with respect to variables, and not false with respect to coefficients. Under the passage $\Omega \to \Omega_1$ the coefficient alphabet and base situation table are not affected; hence Ω_1 is nonelementary and not false with respect to coefficients. Since Ω is admissible, Ω^0 has a solution. We can show, exactly as above for Ω, that any table of words (3.15), in the one-letter alphabet a_1, which is a solution of Ω^0 is also a solution of Ω_1^0. Consequently, Ω_1^0 has a solution and Ω_1 is not false with respect to variables.

Suppose Ω_1 has a solution (3.15). To prove that (3.15) is a solution of Ω it suffices to prove that it satisfies the boundary connection (6.2).

If the subpath π_3, π_4 of (6.5) is nonempty, there exist nonempty words

$$B_1, \ldots, B_h, B_{g+1}, \ldots, B_k,$$
$$C_1, \ldots, C_h, C_{g+1}, \ldots, C_k$$

such that

$$X_{\lambda_i} \doteq B_i C_i \quad (i=1, \ldots, h, g+1, \ldots, k) \tag{6.10}$$

and

$$\begin{gathered} L_p \doteq L_{\alpha(\lambda_1)} B_1, \\ L_{\alpha(\Delta(\lambda_i))} B_i \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i=1, \ldots, h-1), \\ L_{\alpha(\Delta(\lambda_h))} B_h \doteq L_{\alpha(\lambda_{g+1})} B_{g+1}, \\ L_{\alpha(\Delta(\lambda_j))} B_j \doteq L_{\alpha(\lambda_{j+1})} B_{j+1} \quad (j=g+1, \ldots, k-1), \\ L_{\alpha(\Delta(\lambda_k))} B_k \doteq L_q, \end{gathered} \tag{6.11}$$

and if the subpath π_3, π_4 is empty, there exist nonempty words $B_1, \ldots, B_h$ and $C_1, \ldots, C_h$ such that

$$X_{\lambda_i} \doteq B_i C_i \quad (i=1, \ldots, h) \tag{6.12}$$

and

$$\begin{gathered} L_p \doteq L_{\alpha(\lambda_1)} B_1, \\ L_{\alpha(\Delta(\lambda_i))} B_i \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i=1, \ldots, h-1), \\ L_{\alpha(\Delta(\lambda_h))} B_h \doteq L_q. \end{gathered} \tag{6.13}$$

By assumption, each variable in the subpath π_2, x_{λ_g} occurs among the variables of π_1, x_{λ_h}, π_3. We construct nonempty words $B_{h+1}, \ldots, B_g$ and $C_{h+1}, \ldots, C_g$ for the words $X_{\lambda_{h+1}}, \ldots, X_{\lambda_g}$ in such a way that the constructed decompositions coincide with the corresponding decompositions of the words in the subpath π_1, x_{λ_h}, π_3.

Since the subpath (6.3) has no shift, for any pair x_{λ_i}, $x_{\lambda_{i+1}}$ of this subpath we have $L_{\alpha(\lambda_{i+1})} \doteq L_{\alpha(\Delta(\lambda_i))}$. Consequently, for any pair x_{λ_i}, $x_{\lambda_{i+1}}$ of the subpath π_1, x_{λ_h}, π_3 we have $B_i \doteq B_{i+1}$, and hence, by construction, for any pair x_{λ_i}, $x_{\lambda_{i+1}}$ of (6.3) we have $B_i \doteq B_{i+1}$.

Therefore, using (6.11) if the subpath π_3, π_4 is nonempty or (6.13) if π_3, π_4 is empty, we obtain that conditions (3.17)–(3.20) are satisfied for the boundary connection (6.2); hence the table of words (3.15) satisfies (6.2).

LEMMA 6.3. *Suppose an admissible generalized equation* Ω *satisfying condition* N.1 *has a unique leading base and contains a boundary connection with a loop. Then it is possible to construct a list of generalized equations*

$$\Omega_1, \ldots, \Omega_r \tag{6.14}$$

such that the following conditions are satisfied.

1) *Each* Ω_i *in* (6.14) *is either admissible or false.*

2) *The principal parameters of each* Ω_i *in* (6.14) *do not exceed the corresponding parameters of* Ω.

3) *The path index of each* Ω_i *in* (6.14) *is smaller than the path index of* Ω.

4) *If* Ω *has a solution with index* I *and exponent of periodicity* s, *then some* Ω_i *in* (6.14) *has a solution with index* I_1, *where* $I_1 < I$, *and exponent of periodicity* s. *If some* Ω_i *in* (6.14) *has a solution, then so does* Ω.

PROOF. By hypothesis, Ω contains a boundary connection

$$l_p,\ \pi_1,\ x_{\Delta(\nu)},\ x_\nu,\ \pi_2,\ l_q, \tag{6.15}$$

where π_1 and π_2 are subpaths of the connection and x_ν is the carrier of the equation. We may assume that π_1 does not contain a loop.

If π_1 contains the variable $x_{\Delta(\nu)}$, then, since the unique leading base of Ω is the carrier x_ν, it follows from (3.11) that x_ν occurs immediately after $x_{\Delta(\nu)}$, i.e. the subpath π_1 contains a loop. Consequently, π_1 does not contain the dual of the carrier; hence π_1 does not contain the duals of the leading bases of Ω.

The subpath π_1 can be empty or nonempty. Accordingly, we consider two cases.

Case 1. π_1 is empty.

The subpath π_2 can also be empty or nonempty. Accordingly, we consider two subcases.

Case 1.1. π_2 is empty.

Since Ω is admissible, Ω^0 has some solution

$$\begin{gathered} X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}, \\ L_1, \ldots, L_\rho,\ R_1, \ldots, R_\rho,\ T \end{gathered} \tag{6.16}$$

in the one-letter alphabet a_1. This solution satisfies the boundary connection (6.15) with empty subpaths π_1 and π_2. Consequently, there exist nonempty words B_1, B_2, C_1 and C_2 such that

$$\begin{gathered} X_{\Delta(\nu)} \mathrel{\overset{\circ}{=}} B_1C_1,\quad X_\nu \mathrel{\overset{\circ}{=}} B_2C_2, \\ L_p \mathrel{\overset{\circ}{=}} L_{\alpha(\Delta(\nu))}B_1, \\ L_{\alpha(\nu)}B_1 \mathrel{\overset{\circ}{=}} L_{\alpha(\nu)}B_2, \\ L_{\alpha(\Delta(\nu))}B_2 \mathrel{\overset{\circ}{=}} L_q. \end{gathered} \tag{6.17}$$

It follows easily from (6.17) that $L_p \mathrel{\overset{\circ}{=}} L_q$. If $p \neq q$, then, in view of condition N.1, the solution (6.16) must satisfy $\partial(L_p) < \partial(L_q)$ or $\partial(L_q) < \partial(L_p)$, which is impossible. Therefore, $p = q$, and in our case the boundary connection (6.15) has the form

$$l_p,\ x_{\Delta(\nu)},\ x_\nu,\ l_p. \tag{6.18}$$

Delete (6.18) from Ω, and denote the resulting equation by Ω_1.

Obviously, any solution of Ω is a solution of Ω_1, and any solution of Ω^0 is a solution of Ω_1^0.

Therefore, since Ω is admissible and the passage $\Omega \to \Omega_1$ does not affect the coefficient alphabet and the base situation table, it follows that Ω_1 is admissible.

Under the passage $\Omega \to \Omega_1$ the parameters n, m and τ remain the same, δ may decrease, and the number of occurrences of duals of leading bases may decrease.

Suppose Ω_1 has a solution (3.15). We will prove that (3.15) is a solution of Ω. It suffices to prove that (3.15) satisfies the boundary connection (6.18).

Since Ω contains (6.18), we know by (3.12) that the boundary comparison table of Ω contains the inequalities

$$l_{\alpha(\Delta(\nu))} < l_p < l_{\beta(\Delta(\nu))}.$$

The inequalities are also contained in the comparison table of Ω_1. Consequently,

$$L_p \overset{\circ}{=} L_{\alpha(\Delta(\nu))} B_1,$$

$$X_{\Delta(\nu)} \overset{\circ}{=} B_1 C_1,$$

where B_1 and C_1 are certain nonempty words.

Defining $B_2 \rightleftharpoons B_1$ and $C_2 \rightleftharpoons C_1$, we see that all of the equalities (6.17) are satisfied; hence (3.15) satisfies the boundary connection (6.18).

Case 1.2. π_2 is nonempty.

In this case the boundary connection (6.15) has the form

$$l_p,\ x_{\Delta(\nu)},\ x_\nu,\ x_{\lambda_3},\ \pi_2',\ l_q. \tag{6.19}$$

Since Ω is admissible, Ω^0 has a solution (6.16) in the one-letter alphabet a_1. This solution satisfies (6.19), i.e. there exist nonempty words $B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ such that (3.17)–(3.20) are satisfied, i.e. $X_{\lambda_i} \overset{\circ}{=} B_i C_i$ $(i = 1, \ldots, k)$, where $\lambda_1 = \Delta(\nu)$ and $\lambda_2 = \nu$, and

$$\begin{gathered} L_p \overset{\circ}{=} L_{\alpha(\Delta(\nu))} B_1, \\ L_{\alpha(\nu)} B_1 \overset{\circ}{=} L_{\alpha(\nu)} B_2, \\ L_{\alpha(\Delta(\nu))} B_2 \overset{\circ}{=} L_{\alpha(\lambda_3)} B_3, \\ L_{\alpha(\Delta(\lambda_i))} B_i \overset{\circ}{=} L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i = 3, \ldots, k-1), \\ L_{\alpha(\Delta(\lambda_k))} B_k \overset{\circ}{=} L_q. \end{gathered} \tag{6.20}$$

Consequently, $L_p \overset{\circ}{=} L_{\alpha(\lambda_3)} B_3$. Since $X_{\lambda_3} \overset{\circ}{=} B_3 C_3$, where B_3 and C_3 are nonempty words, it follows from condition N.1 that the comparison table of Ω contains the inequalities

$$l_{\alpha(\lambda_3)} < l_p < l_{\beta(\lambda_3)}. \tag{6.21}$$

In view of (6.21), conditions (3.10)–(3.12) are satisfied for the sequence

$$l_p,\ x_{\lambda_3},\ \pi_2',\ l_q; \tag{6.22}$$

hence this sequence can be a boundary connection of Ω.

Thus, the system Ω_1 obtained as a result of replacing the boundary connection (6.19) of Ω by (6.22) is a generalized equation.

We will prove that any solution of Ω is a solution of Ω_1. It suffices to prove that a table of words (3.15) satisfying (6.19) also satisfies (6.22).

Suppose a table of words (3.15) satisfies the boundary connection (6.19). Then there exist nonempty words $B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ such that $X_{\lambda_i} \doteq B_i C_i$ $(i = 1, \ldots, k)$, where $\lambda_1 = \Delta(\nu)$ and $\lambda_2 = \nu$, and the equalities (6.20) are satisfied. These equalities imply that $L_p \doteq L_{\alpha(\lambda_3)} B_3$, from which it follows that (3.15) satisfies (6.22).

In exactly the same way we can prove that any solution of Ω^0 is a solution of Ω_1^0, from which it follows that Ω_1 is admissible.

Under the passage $\Omega \to \Omega_1$ the boundary comparison table remains the same, the parameters n, m and τ remain the same, and the parameter δ may decrease. The number of occurrences of duals of leading bases decreases.

Suppose Ω_1 has a solution (3.15). Then (3.15) satisfies the boundary connection (6.22), i.e. there exist nonempty words

$$\begin{gathered} B_3, \ \ldots, \ B_k, \\ C_3, \ \ldots, \ C_k \end{gathered} \tag{6.23}$$

such that $X_{\lambda_i} \doteq B_i C_i$ $(i = 3, \ldots, k)$ and

$$\begin{gathered} L_p \doteq L_{\alpha(\lambda_3)} B_3, \\ L_{\alpha(\Delta(\lambda_i))} B_i \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i = 3, \ \ldots, \ k-1), \\ L_{\alpha(\Delta(\lambda_k))} B_k \doteq L_q. \end{gathered} \tag{6.24}$$

According to (3.10), the boundary comparison table of Ω contains for the connection (6.19) the inequalities

$$l_{\alpha(\Delta(\nu))} < l_p < l_{\beta(\Delta(\nu))}.$$

We define a word B_1 by the equality $L_p \doteq L_{\alpha(\Delta(\nu))} B_1$. Next, define $B_2 \rightleftharpoons B_1$. Then $X_{\Delta(\nu)} \doteq B_1 C_1$ and $X_\nu \doteq B_2 C_2$ for certain nonempty words C_1 and C_2.

Add the words B_1, B_2, C_1 and C_2 to the list (6.23). The resulting list of words satisfies (6.20). Consequently, the table (3.15) satisfies the boundary connection (6.19); hence it is a solution of Ω.

Case 2. π_1 is nonempty.

In this case the boundary connection (6.15) has the form

$$l_p, \ x_{\lambda_1}, \ \ldots, \ x_{\lambda_{t-1}}, \ x_{\lambda_t}, \ x_{\Delta(\nu)}, \ x_\nu, \ \pi_2, \ l_q. \tag{6.25}$$

We will prove that the sequence

$$l_p, \ x_{\lambda_1}, \ \ldots, \ x_{\lambda_{t-1}}, \ x_{\lambda_t}, \ \pi_2, \ l_q \tag{6.26}$$

can be a boundary connection of Ω. It suffices to prove that if π_2 is nonempty and $x_{\lambda_{t+3}}$ is the first element of π_2, then the comparison table contains the inequality

$$l_{\alpha(\lambda_{t+3})} \leqslant l_{\alpha(\Delta(\lambda_t))},$$

and if π_2 is empty, then the comparison table contains the inequalities

$$l_{\alpha(\Delta(\lambda_t))} < l_q < l_{\beta(\Delta(\lambda_t))}.$$

In the first case the desired inequality holds because, in view of (3.11), for the pairs x_{λ_t}, $x_{\Delta(\nu)}$ and x_ν, $x_{\lambda_{t+3}}$ of (6.25) we have

$$l_{\alpha(\Delta(\nu))} \leqslant l_{\alpha(\Delta(\lambda_t))},$$

$$l_{\alpha(\lambda_{t+3})} \leqslant l_{\alpha(\Delta(\nu))}.$$

In the second case, since Ω is admissible and therefore Ω^0 has a solution (6.16) in the one-letter alphabet a_1, we have for certain nonempty words B_i and C_i the following equalities:

$$\begin{gathered} X_{\lambda_t} \doteq B_t C_t, \\ X_{\Delta(\nu)} \doteq B_{t+1} C_{t+1}, \\ X_\nu \doteq B_{t+2} C_{t+2}, \\ L_{\alpha(\Delta(\lambda_t))} B_t \doteq L_{\alpha(\Delta(\nu))} B_{t+1}, \\ L_{\alpha(\nu)} B_{t+1} \doteq L_{\alpha(\nu)} B_{t+2}, \\ L_{\alpha(\Delta(\nu))} B_{t+2} \doteq L_q. \end{gathered}$$

Consequently, $\partial(L_{\alpha(\Delta(\lambda_t))}) < \partial(L_q) < \partial(L_{\alpha(\Delta(\lambda_t))} X_{\lambda_t})$ and, in view of condition N.1, the comparison table contains the desired inequalities.

Let Ω' denote the generalized equation obtained from Ω by replacing the boundary connection (6.25) by (6.26).

We consider the two possibilities $\beta(\Delta(\lambda_t)) \leqslant \beta(\Delta(\nu))$ and $\beta(\Delta(\nu)) < \beta(\Delta(\lambda_t))$.

Case 2.1. $\beta(\Delta(\lambda_t)) \leqslant \beta(\Delta(\nu))$.

We will prove that in this case the equation Ω' is the desired one.

Since Ω is nonelementary, Ω' is nonelementary. Consequently Ω' is not true; hence, by Lemma 5.4, it is either admissible or false.

Suppose Ω has a solution (3.15) with index I and exponent of periodicity s. Then (3.15) satisfies the boundary connection (6.25). Hence there exist nonempty words B_i and C_i such that

$$\begin{gathered} X_{\lambda_t} \doteq B_t C_t, \\ X_{\Delta(\nu)} \doteq B_{t+1} C_{t+1}, \\ X_\nu \doteq B_{t+2} C_{t+2}, \\ L_{\alpha(\Delta(\lambda_t))} B_t \doteq L_{\alpha(\Delta(\nu))} B_{t+1}, \\ L_{\alpha(\nu)} B_{t+1} \doteq L_{\alpha(\nu)} B_{t+2}, \\ L_{\alpha(\Delta(\nu))} B_{t+2} \doteq A_1, \end{gathered} \tag{6.27}$$

where A_1 is $L_{\alpha(\lambda_{t+3})} B_{t+3}$ if $\pi_2 = x_{\lambda_{t+3}}, \pi_2'$ and $X_{\lambda_{t+3}} \doteq B_{t+3} C_{t+3}$ and is L_q if π_2 is empty. These equalities easily imply that (3.15) satisfies (6.26) and is therefore a solution of Ω'.

It is easy to see that the path index decreases under the passage $\Omega \to \Omega'$.

Suppose Ω' has a solution (3.15). Then (3.15) satisfies the boundary connection (6.26). Consequently, there exist nonempty words B_i and C_i such that

$$\begin{gathered} X_{\lambda_t} \doteq B_t C_t, \\ L_{\alpha(\Delta(\lambda_t))} B_t \doteq A_1, \end{gathered}$$

where A_1 is $L_{\alpha(\lambda_{t+3})}B_{t+3}$ if $\pi_2 = x_{\lambda_{t+3}}, \pi_2'$ and $X_{\lambda_{t+3}} \doteq B_{t+3}C_{t+3}$, and is L_q if π_2 is empty.

By assumption, $\beta(\Delta(\lambda_t)) \leqslant \beta(\Delta(\nu))$. According to (3.11), for the pair $x_{\lambda_t}, x_{\Delta(\nu)}$ of the boundary connection (6.25) we have $\alpha(\Delta(\nu)) \leqslant \alpha(\Delta(\lambda_t))$. It follows from these two inequalities that we can determine nonempty words B_{t+1} and C_{t+1} such that

$$L_{\alpha(\Delta(\lambda_t))}B_t \doteq L_{\alpha(\Delta(\nu))}B_{t+1},$$

$$X_{\lambda_{t+1}} \doteq B_{t+1}C_{t+1}.$$

We now define $B_{t+2} \rightleftharpoons B_{t+1}$ and $C_{t+2} \rightleftharpoons C_{t+1}$. Then all of the equalities (6.27) are satisfied; hence (3.15) satisfies (6.25), from which it follows that (3.15) is a solution of Ω.

Case 2.2. $\beta(\Delta(\nu)) < \beta(\Delta(\lambda_t))$.

We will construct generalized equations

$$\Omega_0, \Omega_1, \ldots, \Omega_{t-1}, \Omega_1^*, \ldots, \Omega_{t-1}^*, \tag{6.28}$$

using the generalized equation Ω' and the boundary connection (6.25).

We obtain Ω_0 from Ω' by adding the boundary $l_{\rho+1}$, the inequalities

$$l_\rho < l_{\rho+1} < l_{\beta(\lambda_1)} \tag{6.29}$$

and the boundary connection

$$l_{\rho+1}, x_{\lambda_1}, \ldots, x_{\lambda_t}, l_{\beta(\Delta(\nu))}. \tag{6.30}$$

We obtain Ω_i ($i = 1, \ldots, t-1$) from Ω' by adding the boundary $l_{\rho+1}$, the inequalities

$$l_{\beta(\Delta(\lambda_i))} < l_{\rho+1} < l_{\beta(\lambda_{i+1})} \tag{6.31}$$

and the boundary connection

$$l_{\rho+1}, x_{\lambda_{i+1}}, \ldots, x_{\lambda_t}, l_{\beta(\Delta(\nu))}. \tag{6.32}$$

We obtain Ω_i^* ($i = 1, \ldots, t-1$) from Ω' by adding the inequality

$$l_{\beta(\Delta(\lambda_i))} < l_{\beta(\lambda_{i+1})} \tag{6.33}$$

and the boundary connection

$$l_{\beta(\Delta(\lambda_i))}, x_{\lambda_{i+1}}, \ldots, x_{\lambda_t}, l_{\beta(\Delta(\nu))}. \tag{6.34}$$

We will prove that the boundary connections (6.30), (6.32) and (6.34) satisfy conditions (3.10)–(3.12).

Condition (3.10) for the connection (6.30) follows from (6.29) and condition (3.10) for the connection (6.25). Condition (3.10) for the connection (6.32) follows from (6.31), condition (3.7) for the pair $l_{\alpha(\Delta(\lambda_i))}, l_{\beta(\Delta(\lambda_i))}$, and condition (3.11) for the pair $l_{\alpha(\lambda_{i+1})}, l_{\alpha(\Delta(\lambda_i))}$. Condition (3.10) for the connection (6.34) follows from (6.33), condition (3.7) for the pair $l_{\alpha(\Delta(\lambda_i))}, l_{\beta(\Delta(\lambda_i))}$, and condition (3.11) for the pair $l_{\alpha(\lambda_{i+1})}, l_{\alpha(\Delta(\lambda_i))}$.

Condition (3.11) for the connections (6.30), (6.32) and (6.34) follows from condition (3.11) for the connection (6.25).

Condition (3.12) for the connections (6.30), (6.32) and (6.34) follows from the condition defining Case 2.2 and from the condition

$$l_{\alpha(\Delta(\lambda_i))} < l_{\beta(\lambda_{i+1})},$$

which holds for any boundary connection of an admissible equation.

Each equation in (6.28) is nonelementary and therefore not true, hence, by Lemma 5.4, either false or admissible.

The principal parameters of the constructed equations agree with those of Ω. We consider the new boundary l_{p+1} to be inessential, hence not initial. The presence of the boundary connections (6.30), (6.32) and (6.34) guarantees condition D.4. Condition D.5 is obvious.

The path index of each equation in (6.28) is smaller than the path index of Ω. Indeed, the number of occurrences of duals of leading bases in the paths of all boundary connections of Ω' is less than the corresponding number for Ω, since the loop eliminated in the passage $\Omega \to \Omega'$ contains the variable $x_{\Delta(\nu)}$, and the carrier is a leading base. At the very beginning of the lemma we proved that the subpath $x_{\lambda_1}, \ldots, x_{\lambda_t}$ of (6.25) contains no duals of leading bases. Consequently, addition of the boundary connections (6.30), (6.32) and (6.34) to Ω' does not increase the number of duals of leading bases.

Suppose Ω has a solution (3.15). We will prove that this table of words is a solution of some equation in the list (6.28).

We proved in Case 2.1 that any table of words which is a solution of Ω is also a solution of Ω'. Therefore, it suffices to prove that (3.15) satisfies either (6.29) and (6.30), or (6.31) and (6.32) for some $i = 1, \ldots, t - 1$, or (6.33) and (6.34) for some $i = 1, \ldots, t - 1$.

Since (3.15) satisfies the connection (6.25), there exist nonempty words $B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ such that

$$X_{\lambda_i} \doteq B_i C_i \quad (i = 1, \ldots, k), \tag{6.35}$$

$$\begin{gathered} L_p \doteq L_{\alpha(\lambda_1)} B_1, \\ L_{\alpha(\Delta(\lambda_i))} B_i \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i = 1, \ldots, t-1), \\ L_{\alpha(\Delta(\lambda_t))} B_t \doteq L_{\alpha(\lambda_{t+1})} B_{t+1}, \\ L_{\alpha(\Delta(\lambda_{t+1}))} B_{t+1} \doteq L_{\alpha(\lambda_{t+2})} B_{t+2}, \\ L_{\alpha(\Delta(\lambda_j))} B_j \doteq L_{\alpha(\lambda_{j+1})} B_{j+1} \quad (j = t+2, \ldots, k-1), \\ L_{\alpha(\Delta(\lambda_k))} B_k \doteq L_q, \end{gathered} \tag{6.36}$$

where x_ν is the carrier of the equation and

$$\begin{gathered} \lambda_{t+1} = \Delta(\nu), \\ \lambda_{t+2} = \nu. \end{gathered} \tag{6.37}$$

By the assumption defining the case,

$$L_{\beta(\Delta(\lambda_t))} \doteq L_{\beta(\Delta(\nu))} K_t \tag{6.38}$$

for some nonempty word K_t. Consequently,

$$L_{\alpha(\Delta(\lambda_t))} B_t C_t \doteq L_{\alpha(\Delta(\nu))} B_{t+1} C_{t+1} K_t,$$

and, by (6.36),

$$C_t \doteq C_{t+1} K_t. \tag{6.39}$$

We denote the word C_{t+1} by H.

We consider two possibilities: $t = 1$ and $t > 1$.

Case 2.2.1. $t = 1$.

In this case the equation Ω_0 has a solution. Indeed, define

$$L_{p+1} \rightleftarrows L_{\alpha(\lambda_t)} B_t H.$$

Next, using (6.35) and (6.39), decompose the word X_{λ_t} into nonempty words $B_t H$ and K_t.

As a result, the following equalities and inequalities are satisfied:

$$X_{\lambda_t} \doteq B_t H K_t,$$
$$L_{\rho+1} \doteq L_{\alpha(\lambda_t)} B_t H,$$
$$L_{\alpha(\Delta(\lambda_t))} B_t H \doteq L_{\beta(\Delta(\nu))},$$
$$\partial(L_p) < \partial(L_{\rho+1}) < \partial(L_{\beta(\lambda_t)}).$$

Consequently, in this case the table of words (3.15) satisfies the new inequalities

$$l_p < l_{\rho+1} < l_{\beta(\lambda_1)}$$

and the new boundary connection

$$l_{\rho+1},\ x_{\lambda_t},\ l_{\beta(\Delta(\nu))},$$

i.e. it is a solution of Ω_0.

Case 2.2.2. $t > 1$.

Comparing the lengths of the words in the pairs

$$\begin{array}{ccc} L_{\beta(\Delta(\lambda_1))} & \text{and} & L_{\alpha(\lambda_2)} B_2 H, \\ L_{\beta(\Delta(\lambda_2))} & \text{and} & L_{\alpha(\lambda_3)} B_3 H, \\ \cdots & & \cdots \\ L_{\beta(\Delta(\lambda_{t-1}))} & \text{and} & L_{\alpha(\lambda_t)} B_t H, \end{array}$$

we see that either each left word is longer than the corresponding right word, or there exists a largest i such that the ith left word is shorter than the ith right word or equal to it in length. Accordingly, we divide this case into three subcases.

Case 2.2.2.1.

$$\begin{gathered} \partial(L_{\beta(\Delta(\lambda_1))}) > \partial(L_{\alpha(\lambda_2)} B_2 H), \\ \partial(L_{\beta(\Delta(\lambda_2))}) > \partial(L_{\alpha(\lambda_3)} B_3 H), \\ \cdots\cdots\cdots \\ \partial(L_{\beta(\Delta(\lambda_{t-1}))}) > \partial(L_{\alpha(\lambda_t)} B_t H). \end{gathered} \tag{6.40}$$

It follows from (6.36), (6.39), and (6.40), by induction on the number of conditions (6.40), that

$$C_j \doteq H K_j \quad (j = 1, \ldots, t-1) \tag{6.41}$$

for certain nonempty words $K_1, \ldots, K_{t-1}$. Define

$$L_{\rho+1} \rightleftharpoons L_{\alpha(\lambda_1)} B_1 H. \tag{6.42}$$

Using (6.39) and (6.41), we decompose the words X_{λ_j} ($j = 1, \ldots, t$) into subwords B_j' and K_j, where $B_j' \doteq B_j H$. From the first equality in (6.36), (6.42), the first equality in (6.41), and the fact that H and K_1 are nonempty we obtain

$$\partial(L_p) < \partial(L_{\rho+1}) < \partial(L_{\beta(\lambda_1)}).$$

Since, according to (6.37), $\lambda_{t+1} = \Delta(\nu)$, we obtain from (6.42) and (6.36) the following graphical equalities:

$$L_{\rho+1} \doteq L_{\alpha(\lambda_1)} B_1 H,$$
$$L_{\alpha(\Delta(\lambda_i))} B_i H \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} H \quad (i = 1, \ldots, t-1),$$
$$L_{\alpha(\Delta(\lambda_t))} B_t H \doteq L_{\beta(\Delta(\nu))}.$$

Consequently, the table of words (3.15) satisfies the inequalities

$$l_p < l_{p+1} < l_{\beta(\lambda_1)}$$

and the boundary connection

$$l_{p+1},\ x_{\lambda_1},\ x_{\lambda_2},\ \ldots,\ x_{\lambda_t},\ l_{\beta(\Delta(\nu))},$$

i.e. it is a solution of Ω_0.

Case 2.2.2.2. For some $i \leqslant t-1$,

$$\begin{aligned} &\partial(L_{\beta(\Delta(\lambda_i))}) < \partial(L_{\alpha(\lambda_{i+1})}B_{i+1}H),\\ &\partial(L_{\beta(\Delta(\lambda_{i+1}))}) > \partial(L_{\alpha(\lambda_{i+2})}B_{i+2}H),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &\partial(L_{\beta(\Delta(\lambda_{t-1}))}) > \partial(L_{\alpha(\lambda_t)}B_tH). \end{aligned} \tag{6.43}$$

As in Case 2.2.2.1, we obtain

$$C_j \overset{\circ}{=} HK_j \quad (j=i+1,\ \ldots,\ t-1) \tag{6.44}$$

for certain nonempty words $K_{i+1}, \ldots, K_{t-1}$. It follows from the first condition in (6.43) that $H \overset{\circ}{=} C_iM_i$. Define

$$L_{p+1} \rightleftarrows L_{\alpha(\lambda_{i+1})}B_{i+1}H. \tag{6.45}$$

Using (6.39) and (6.44), we decompose the words X_{λ_j} ($j = i+1, \ldots, t$) into subwords B'_j and K_j, where $B'_j \overset{\circ}{=} B_jH$. From the first inequality in (6.43), the first equality in (6.44), and the fact that H and K_{i+1} are nonempty we obtain the inequalities

$$\partial(L_{\beta(\Delta(\lambda_i))}) < \partial(L_{p+1}) < \partial(L_{\beta(\lambda_{i+1})}).$$

Since, according to (6.37), $\lambda_{t+1} = \Delta(\nu)$, we obtain from (6.45) and (6.36) the following graphical equalities:

$$\begin{gathered} L_{p+1} \overset{\circ}{=} L_{\alpha(\lambda_{i+1})}B_iH,\\ L_{\alpha(\Delta(\lambda_j))}B_jH \overset{\circ}{=} L_{\alpha(\lambda_{j+1})}B_{j+1}H \quad (j=i+1,\ \ldots,\ t-1),\\ L_{\alpha(\Delta(\lambda_t))}B_tH \overset{\circ}{=} L_{\beta(\Delta(\nu))}. \end{gathered}$$

Consequently, the table of words (3.15) satisfies the inequalities

$$l_{\beta(\Delta(\lambda_i))} < l_{p+1} < l_{\beta(\lambda_{i+1})}$$

and the boundary connection

$$l_{p+1},\ x_{\lambda_{i+1}},\ \ldots,\ x_{\lambda_t},\ l_{\beta(\Delta(\nu))},$$

i.e. it is a solution of Ω_i.

Case 2.2.2.3. For some $i \leqslant t-1$,

$$\begin{aligned} &\partial(L_{\beta(\Delta(\lambda_i))}) = \partial(L_{\alpha(\lambda_{i+1})}B_{i+1}H),\\ &\partial(L_{\beta(\Delta(\lambda_{i+1}))}) > \partial(L_{\alpha(\lambda_{i+2})}B_{i+2}H),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &\partial(L_{\beta(\Delta(\lambda_{t-1}))}) > \partial(L_{\alpha(\lambda_t)}B_tH). \end{aligned} \tag{6.46}$$

As in Case 2.2.2.1, we obtain

$$C_j \overset{\circ}{=} HK_j \quad (j=i+1,\ \ldots,\ t-1) \tag{6.47}$$

for certain nonempty words $K_{i+1}, \dots, K_{t-1}$. It follows from the first condition in (6.46) that $H \doteq C_i$. Using (6.39) and (6.47), we decompose the words X_{λ_j} $(j = i+1, \dots, t)$ into subwords B'_j and K_j, where $B'_j \doteq B_j H$. From the equality in (6.46), the first equality in (6.47), and the fact that K_{i+1} is nonempty we obtain the inequality

$$\partial(L_{\beta(\Delta(\lambda_i))}) < \partial(L_{\beta(\lambda_{i+1})}).$$

From (6.36) and (6.37) we obtain

$$\begin{gathered} L_{\beta(\Delta(\lambda_i))} \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} H, \\ L_{\alpha(\Delta(\lambda_j))} B_j H \doteq L_{\alpha(\lambda_{j+1})} B_{j+1} H \quad (j = i+1, \dots, t-1), \\ L_{\alpha(\Delta(\lambda_t))} B_t H \doteq L_{\beta(\Delta(\nu))}. \end{gathered}$$

Consequently, the table of words (3.15) satisfies the inequality

$$l_{\beta(\Delta(\lambda_i))} < l_{\beta(\lambda_{i+1})}$$

and the boundary connection

$$l_{\beta(\Delta(\lambda_i))}, x_{\lambda_{i+1}}, \dots, x_{\lambda_t}, l_{\beta(\Delta(\nu))},$$

i.e. it is a solution of Ω_i.

Suppose some equation in the list (6.28) has a solution $\mathfrak{D}$. Then obviously Ω' has a solution which either agrees with $\mathfrak{D}$ or is obtained from it by deleting the component $L_{\rho+1}$. Suppose this solution of Ω' is the table of words (3.15).

Then (3.15) satisfies the boundary connection (6.26), i.e. there exist nonempty words

$$\begin{gathered} B_1, \dots, B_{t-1}, B_t, B_{t+3}, \dots, B_k, \\ C_1, \dots, C_{t-1}, C_t, C_{t+3}, \dots, C_k \end{gathered} \tag{6.48}$$

such that

$$\begin{gathered} X_{\lambda_i} \doteq B_i C_i \quad (i = 1, \dots, t, t+3, \dots, k), \\ L_p \doteq L_{\alpha(\lambda_1)} B_1, \\ L_{\alpha(\Delta(\lambda_i))} B_i \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i = 1, \dots, t-1), \\ L_{\alpha(\Delta(\lambda_t))} B_t \doteq L_{\alpha(\lambda_{t+3})} B_{t+3}, \\ L_{\alpha(\Delta(\lambda_i))} B_i \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i = t+3, \dots, k-1), \\ L_{\alpha(\Delta(\lambda_k))} B_k \doteq L_q. \end{gathered} \tag{6.49}$$

We want to prove that (3.15) is a solution of Ω, and for this purpose it suffices to prove that it satisfies the boundary connection (6.25).

To prove that (3.15) satisfies (6.25) we must find nonempty words $B_{t+1}, B_{t+2}, C_{t+1}$ and C_{t+2} such that $X_{\lambda_i} \doteq B_i C_i (i = t+1, t+2)$ and the equalities (6.36) hold. In view of (6.49) and (6.37), we need only prove that

$$\partial(L_{\beta(\Delta(\nu))}) > \partial(L_{\alpha(\Delta(\lambda_t))} B_t). \tag{6.50}$$

We will prove that if the table of words $\mathfrak{D}$ satisfies (6.30), (6.32) or (6.34), then (6.50) holds.

Suppose $\mathfrak{D}$ satisfies (6.30). Then there exist nonempty words $B'_1, \dots, B'_t$ and

$C'_1, \ldots, C'_t$ such that

$$
\begin{gathered}
X_{\lambda_i} \doteq B'_i C'_i \quad (i=1, \ldots, t), \\
L_{\rho+1} \doteq L_{\alpha(\lambda_1)} B'_1, \\
L_{\alpha(\Delta(\lambda_i))} B'_i \doteq L_{\alpha(\lambda_{i+1})} B'_{i+1} \quad (i=1, \ldots, t-1), \\
L_{\alpha(\Delta(\lambda_t))} B'_t \doteq L_{\beta(\Delta(\nu))}.
\end{gathered} \tag{6.51}
$$

According to (6.29), $L_{\rho+1} \doteq L_p C$ for some nonempty word C.

Comparing (6.49) and (6.51), we see successively that for each $i = 1, \ldots, t$ we have $B'_i \doteq B_i C$; hence (6.50) holds.

Suppose $\mathfrak{D}$ is a solution of Ω_i. Then it satisfies a boundary connection of type (6.32), and hence there exist nonempty words $B'_{i+1}, \ldots, B'_t$ and $C'_{i+1}, \ldots, C'_t$ such that

$$
\begin{gathered}
X_{\lambda_j} \doteq B'_j C'_j \quad (j=i+1, \ldots, t), \\
L_{\rho+1} \doteq L_{\alpha(\lambda_{i+1})} B'_{i+1}, \\
L_{\alpha(\Delta(\lambda_j))} B'_j \doteq L_{\alpha(\lambda_{j+1})} B'_{j+1} \quad (j=i+1, \ldots, t-1), \\
L_{\alpha(\Delta(\lambda_t))} B'_t \doteq L_{\beta(\Delta(\nu))}.
\end{gathered} \tag{6.52}
$$

According to (6.31), $L_{\rho+1} \doteq L_{\alpha(\Delta(\lambda_i))} B_i C$ for some nonempty word C. From this equality and (6.49) we obtain

$$L_{\rho+1} \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} C.$$

Comparing (6.49) and (6.52), we see successively that for each $j = i + 1, \ldots, t$ we have $B'_j \doteq B_j C$; hence (6.50) holds.

Suppose the table of words (3.15) obtained from $\mathfrak{D}$ as indicated above is a solution of Ω_i^*. Then it satisfies a boundary connection of type (6.33); hence there exist nonempty words $B'_{i+1}, \ldots, B'_t$ and $C'_{i+1}, \ldots, C'_t$ such that

$$
\begin{gathered}
X_{\lambda_j} \doteq B'_j C'_j \quad (j=i+1, \ldots, t), \\
L_{\beta(\Delta(\lambda_i))} \doteq L_{\alpha(\lambda_{i+1})} B'_{i+1}, \\
L_{\alpha(\Delta(\lambda_j))} B'_j \doteq L_{\alpha(\lambda_{j+1})} B'_{j+1} \quad (j=i+1, \ldots, t-1), \\
L_{\alpha(\Delta(\lambda_t))} B'_t \doteq L_{\beta(\Delta(\nu))}.
\end{gathered} \tag{6.53}
$$

Since $L_{\beta(\Delta(\lambda_i))} \doteq L_{\alpha(\Delta(\lambda_i))} B_i C_i$ and, according to (6.49), $L_{\beta(\Delta(\lambda_i))} \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} C_i$, we see successively, on comparing (6.49) and (6.53), that for all $j = i + 1, \ldots, t$ we have $B'_j \doteq B_j C_i$; hence (6.50) holds.

LEMMA 6.4. *Suppose an admissible generalized equation Ω satisfying condition* N.1 *has a unique leading base and contains a boundary connection with a loop. Then it is possible to construct a list of generalized equations $\Omega_1, \ldots, \Omega_p$ such that the following conditions are satisfied.*

1) *Each Ω_i either is admissible and satisfies condition* N.1, *or else is false.*

2) *The principal parameters of each Ω_i do not exceed the corresponding parameters of Ω.*

3) *The path index of each Ω_i is less than the path index of Ω.*

4) *If Ω has a solution with index I and exponent of periodicity s, then some Ω_i has a solution with index I_1, where $I_1 < I$, and exponent of periodicity s. If some Ω_i has a solution, then so does Ω.*

PROOF. For the given equation Ω we construct, in accordance with Lemma 6.3, a list of equations (6.14), each of which is either admissible or false. Using the algorithm of Lemma 5.4, we distinguish the admissible equations in the list (6.14), and for each admissible equation we construct, in accordance with Lemma 6.1, a list of equations, each of which either is admissible and satisfies condition N.1, or is false. Combining all of the constructed lists together with the false equations in (6.14), we obtain the desired list.

LEMMA 6.5. *Suppose an admissible generalized equation Ω satisfying condition* N.1 *contains a boundary connection with a majorized variable. Then it is possible to construct a generalized equation Ω_1 such that the following conditions are satisfied.*

1) Ω_1 *is admissible and satisfies condition* N.1.

2) *The principal parameters of Ω_1 do not exceed the corresponding parameters of Ω.*

3) *The path index of Ω_1 is less than the path index of Ω.*

4) *If Ω has a solution with index I and exponent of periodicity s, then Ω_1 has a solution with index I_1, where $I_1 < I$, and exponent of periodicity s. If Ω_1 has a solution, then so does Ω.*

PROOF. In accordance with the definition of a majorized variable we consider three cases.

Case 1. The majorized variable is the beginning of the path of a boundary connection.

In this case the given boundary connection has the form

$$l_p,\ x_{\lambda_1},\ \pi,\ l_q, \tag{6.54}$$

where the variable x_{λ_1} is matched with its dual and π is some subpath. We consider two subcases.

Case 1.1. π is empty.

Since Ω is admissible, Ω^0 has a solution (6.16), in the one-letter alphabet a_1, which satisfies the boundary connection (6.54) with empty subpath π. Consequently, there exist nonempty words B_1 and C_1 such that

$$L_p \overset{\circ}{=} L_{\alpha(\lambda_1)}B_1,$$
$$L_{\alpha(\Delta(\lambda_1))}B_1 \overset{\circ}{=} L_q.$$

Since the variable x_{λ_1} is matched with its dual, it follows that $L_{\alpha(\lambda_1)} \overset{\circ}{=} L_{\alpha(\Delta(\lambda_1))}$ and therefore $L_p \overset{\circ}{=} L_q$; and since Ω satisfies condition N.1, the connection (6.54) has the form

$$l_p,\ x_{\lambda_1},\ l_p. \tag{6.55}$$

Delete the connection (6.55) and denote the resulting equation by Ω_1.

Obviously, any solution of Ω is also a solution of Ω_1.

Any solution of Ω^0 is a solution of Ω_1^0. From this assertion, the fact that Ω is admissible, and the fact that the passage $\Omega \to \Omega_1$ does not affect the coefficient alphabet and the base situation table it follows that Ω_1 is admissible.

The passage $\Omega \to \Omega_1$ does not affect the boundary comparison table; hence Ω_1 satisfies condition N.1.

Under the passage $\Omega \to \Omega_1$ the parameters n, m and τ remain the same, the parameter δ may decrease, the sum of the lengths of the paths of all boundary connections decreases, and the number of occurrences of duals of leading bases in the paths of the boundary connections does not increase.

Suppose Ω_1 has a solution (3.15). According to condition (3.10) for the connection (6.55), the boundary comparison table contains the inequalities $l_{\alpha(\lambda_1)} < l_p < l_{\beta(\lambda_1)}$. This means that it is possible to choose a decomposition $X_{\lambda_1} \doteq B_1 C_1$, where B_1 and C_1 are nonempty, such that

$$l_{\alpha(\lambda_1)} < l_p < l_{\beta(\lambda_1)}.$$

Consequently, the table of words (3.15) satisfies the boundary connection (6.55) and is a solution of Ω.

Case 1.2. π is nonempty.

In this case the connection (6.54) has the form

$$l_p,\ x_{\lambda_1},\ x_{\lambda_2},\ \pi',\ l_q, \tag{6.56}$$

where the variable x_{λ_1} is matched. Delete the majorized variable x_{λ_1} from (6.56). Since Ω is admissible, the sequence

$$l_p,\ x_{\lambda_2},\ \pi',\ l_q \tag{6.57}$$

can be a boundary connection of Ω. Let Ω_1 denote the equation obtained from Ω by replacing (6.56) by (6.57).

Suppose Ω has a solution (3.15). Then (3.15) satisfies the boundary connection (6.56), i.e. there exist nonempty words $B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ such that

$$L_p \doteq L_{\alpha(\lambda_1)} B_1,$$
$$L_{\alpha(\Delta(\lambda_1))} B_1 \doteq L_{\alpha(\lambda_2)} B_2,$$
$$L_{\alpha(\Delta(\lambda_i))} B_i \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i=2, \ldots, k-1),$$
$$L_{\alpha(\Delta(\lambda_k))} B_k \doteq L_q.$$

Since x_{λ_1} is matched with its dual, it follows that $L_{\alpha(\lambda_1)} \doteq L_{\alpha(\Delta(\lambda_1))}$ and therefore that $L_p \doteq L_{\alpha(\lambda_2)} B_2$.

Thus, the table of words (3.15) satisfies the boundary connection (6.57); hence it satisfies Ω_1.

Suppose Ω_1 has a solution (3.15). Then (3.15) satisfies the boundary connection (6.57), i.e. there exist nonempty words $B_2, \ldots, B_k$ and $C_2, \ldots, C_k$ such that

$$L_p \doteq L_{\alpha(\lambda_2)} B_2,$$
$$L_{\alpha(\Delta(\lambda_i))} B_i \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i=2, \ldots, k-1),$$
$$L_{\alpha(\Delta(\lambda_k))} B_k \doteq L_q.$$

According to condition (3.10) for the connection (6.56), the boundary comparison table contains the inequalities

$$l_{\alpha(\lambda_1)} < l_p < l_{\beta(\lambda_1)}.$$

Consequently, we can construct nonempty words B_1 and C_1 such that

$$L_p \doteq L_{\alpha(\lambda_1)} B_1,$$
$$X_{\lambda_1} \doteq B_1 C_1.$$

Then $L_{\alpha(\Delta(\lambda_1))} B_1 \doteq L_{\alpha(\lambda_2)} B_2$; hence (3.15) satisfies (6.56) and is a solution of Ω.

Case 2. The majorized variable is the end of the path of the given boundary connection.

In this case the connection has the form

$$l_p,\ \pi,\ x_{\lambda_k},\ l_q, \tag{6.58}$$

where the variable x_{λ_k} is matched with its dual and π is some subpath.

If π is empty, we are in Case 1.1. Suppose (6.58) has the form

$$l_p,\ \pi',\ x_{\lambda_{k-1}},\ x_{\lambda_k},\ l_q. \tag{6.59}$$

Since Ω is admissible (in view of the fact that Ω^0 has a solution), it follows easily that the sequence

$$l_p,\ \pi',\ x_{\lambda_{k-1}},\ l_q \tag{6.60}$$

can be a boundary connection of Ω. Let Ω_1 denote the equation obtained from Ω by replacing (6.59) by (6.60).

Suppose Ω has a solution (3.15). Then (3.15) satisfies the boundary connection (6.59), i.e. there exist nonempty words $B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ such that

$$L_p \overset{\circ}{=} L_{\alpha(\lambda_1)}B_1,$$

$$L_{\alpha(\Delta(\lambda_i))}B_i \overset{\circ}{=} L_{\alpha(\lambda_{i+1})}B_{i+1} \quad (i=1,\ \ldots,\ k-2),$$

$$L_{\alpha(\Delta(\lambda_{k-1}))}B_{k-1} \overset{\circ}{=} L_{\alpha(\lambda_k)}B_k,$$

$$L_{\alpha(\Delta(\lambda_k))}B_k \overset{\circ}{=} L_q.$$

Since the variable x_{λ_k} is matched, we obtain

$$L_{\alpha(\lambda_k)} \overset{\circ}{=} L_{\alpha(\Delta(\lambda_k))};$$

hence

$$L_{\alpha(\Delta(\lambda_{k-1}))}B_{k-1} \overset{\circ}{=} L_q.$$

The table of words (3.15) satisfies the boundary connection (6.60) and is therefore a solution of Ω_1.

Suppose Ω_1 has a solution (3.15). Then

$$L_{\alpha(\Delta(\lambda_{k-1}))}B_{k-1} \overset{\circ}{=} L_q.$$

According to condition (3.11) for the connection (6.59), the comparison table contains the inequality

$$l_{\alpha(\lambda_k)} \leqslant l_{\alpha(\Delta(\lambda_{k-1}))};$$

hence $\partial(L_{\alpha(\lambda_k)}) \leqslant \partial(L_{\alpha(\Delta(\lambda_{k-1}))})$. According to condition (3.12) for the connection (6.59), the comparison table contains the inequality

$$l_q < l_{\beta(\Delta(\lambda_k))};$$

hence $\partial(L_q) < \partial(L_{\alpha(\Delta(\lambda_k))}X_{\Delta(\lambda_k)})$. Since the variable x_{λ_k} is matched, we obtain

$$L_{\alpha(\Delta(\lambda_k))} \overset{\circ}{=} L_{\alpha(\lambda_k)}.$$

The desired nonempty words B_k and C_k are determined from the equalities

$$L_{\alpha(\Delta(\lambda_{k-1}))}B_{k-1} \overset{\circ}{=} L_{\alpha(\lambda_k)}B_k,$$

$$X_{\lambda_k} \overset{\circ}{=} B_k C_k.$$

The table of words (3.15) satisfies the boundary connection (6.59) and is therefore a solution of Ω.

Case 3. The majorized variable is neither the beginning nor the end of the path of the given boundary connection.

In this case the given connection has the form

$$l_p,\ \pi_1,\ x_{\lambda_{p-1}},\ x_{\lambda_p},\ x_{\lambda_{p+1}},\ \pi_2,\ l_q, \tag{6.61}$$

where the variable x_{λ_p} is matched with its dual, π_1 and π_2 are certain subpaths, and the comparison table contains at least one of the following two inequalities:

$$l_{\beta(\lambda_{p+1})} \leqslant l_{\beta(\Delta(\lambda_p))}, \tag{6.62}$$

$$l_{\beta(\Delta(\lambda_{p-1}))} \leqslant l_{\beta(\lambda_p)}. \tag{6.63}$$

According to condition (3.11) for the connection (6.61), the comparison table contains the inequalities

$$l_{\alpha(\lambda_p)} \leqslant l_{\alpha(\Delta(\lambda_{p-1}))},$$

$$l_{\alpha(\lambda_{p+1})} \leqslant l_{\alpha(\Delta(\lambda_p))}.$$

Since the variable x_{λ_p} is matched, the comparison table contains the inequality

$$l_{\alpha(\lambda_{p+1})} \leqslant l_{\alpha(\Delta(\lambda_{p-1}))}.$$

Therefore, the sequence

$$l_p,\ \pi_1,\ x_{\lambda_{p-1}},\ x_{\lambda_{p+1}},\ \pi_2,\ l_q \tag{6.64}$$

can be a boundary connection of Ω. Let Ω_1 denote the equation obtained from Ω by replacing (6.61) by (6.64).

Suppose Ω has a solution (3.15). Then

$$\begin{gathered} L_{\alpha(\Delta(\lambda_{p-1}))}B_{p-1} \overset{\circ}{=} L_{\alpha(\lambda_p)}B_p, \\ L_{\alpha(\Delta(\lambda_p))}B_p \overset{\circ}{=} L_{\alpha(\lambda_{p+1})}\, B_{p+1}. \end{gathered} \tag{6.65}$$

Since x_{λ_p} is matched, we obtain

$$L_{\alpha(\Delta(\lambda_{p-1}))}B_{p-1} \overset{\circ}{=} L_{\alpha(\lambda_{p+1})}B_{p+1}.$$

The table of words (3.15) satisfies the boundary connection (6.64) and is therefore a solution of Ω_1.

Suppose Ω_1 has a solution (3.15). Then

$$L_{\alpha(\Delta(\lambda_{p-1}))}B_{p-1} \overset{\circ}{=} L_{\alpha(\lambda_{p+1})}B_{p+1}.$$

According to condition (3.11) for the connection (6.61), the comparison table contains the inequality

$$l_{\alpha(\lambda_p)} \leqslant l_{\alpha(\Delta(\lambda_{p-1}))}. \tag{6.66}$$

In view of (6.66) and either of the conditions (6.62) or (6.63), we can construct nonempty words B_p and C_p such that the equalities (6.65) hold and $X_{\lambda_p} = B_pC_p$. It follows that (3.15) satisfies the connection (6.61) and is therefore a solution of Ω.

Normalization Lemma. *There exists an algorithm which produces, for any admissible* *generalized equation Ω, a list of generalized equations*

$$\Omega_1, \ldots, \Omega_q \tag{6.67}$$

such that the following conditions are satisfied.

1) *Each Ω_i in (6.67) is either normalized admissible or false.*

2) *The principal parameters of each Ω_i in (6.67) do not exceed the corresponding parameters of Ω.*

3) *If Ω has a solution with index I and exponent of periodicity s, then some Ω_i in (6.67) has a solution with index I_1, where $I_1 \leqslant I$, and exponent of periodicity s. If some Ω_i in (6.67) has a solution, then so does Ω.*

PROOF. Suppose Ω is an admissible generalized equation. In accordance with Lemma 6.1, we construct a list of equations (6.1), each of which either is admissible and satisfies condition N.1 or is false, and which satisfy conditions 2) and 3) of the Normalization Lemma. If some Ω_j in (6.1) contains a boundary connection with a superfluous subpath or a boundary connection with a loop under the condition that this equation has a unique leading base, or a boundary connection with a majorized variable, then, applying Lemma 6.2, 6.4, or 6.5 respectively, we construct for Ω_j a list of generalized equations, each of which either is admissible and satisfies condition N.1 or is false, and which are related to Ω_j by conditions analogous to conditions 2) and 3) of the Normalization Lemma. Replace Ω_j in the list (6.1) by the constructed list. The new list is composed of admissible equations satisfying condition N.1 and false equations, and satisfies conditions 2) and 3) of the Normalization Lemma. If the new list contains an admissible equation which is not normalized, repeat the above operation. Since each application of this operation decreases the path index of the transformed equation, we obtain, after a finite number of steps, the desired list of equations.

§7. Transformation of generalized equations

In an admissible generalized equation the boundary l_1 is essential. Indeed, according to Lemma 5.7 it cannot be an original boundary of an admissible equation. Consequently, in view of condition D.4, the boundary l_1 cannot be inessential.

An admissible equation is not elementary and not false with respect to coefficients. Therefore, among the bases of an admissible equation are coefficients a_i and a_j, where $i \neq j$, such that the left boundary of a_i is not the same as the left boundary of a_j. According to condition D.3, the left boundaries of a_i and a_j are initial, hence essential. Thus, an admissible equation contains at least two essential boundaries.

A boundary l_w is called *second essential* if any boundary l_i such that the comparison table contains the inequalities $l_1 < l_i < l_w$ is inessential. It is easy to see that in a normalized admissible generalized equation a second essential boundary is uniquely determined. In the sequel, l_w will always denote the second essential boundary of the equation under consideration.

Suppose Ω is a normalized admissible generalized equation, and let

$$w_{\gamma_1}, \ldots, w_{\gamma_v} \tag{7.1}$$

be a list of all its leading bases. (Recall that a leading base is one whose left boundary is l_1.)

If Ω has carrier x_ν, then any leading base w_i different from the carrier such that the comparison table contains the inequality $l_{\beta(i)} \leqslant l_{\beta(\nu)}$ is called a *transfer* base.

MAIN LEMMA. *There exists an algorithm which converts any normalized admissible generalized equation Ω into a generalized equation Ω_1 satisfying the following three assertions.*

ASSERTION 1. *The principal parameters n, m, τ of Ω_1 do not exceed the corresponding parameters of Ω.*

ASSERTION 2. *If Ω has a solution with index I and exponent of periodicity s, then Ω_1 has a solution with index I_1 and exponent of periodicity s_1, where $I_1 < I$ and $s_1 \leqslant s$.*

ASSERTION 3. *If Ω_1 has a solution, then so does Ω.*

PROOF. According to the nature of the leading bases and the position of the second essential boundary l_w, we distinguish the following seven types of normalized admissible generalized equations.

Type I. Ω has no carrier.

Type II. There is a carrier x_ν, there are no transfer bases, and $w = \beta(\nu) = 2$, where w is the subscript of the second essential boundary.

Type III. There is a carrier x_ν, there are no transfer bases, and $w = \beta(\nu) > 2$.

Type IV. There is a carrier x_ν, there are no transfer bases, $w < \beta(\nu)$, and there are matched variables among the leading bases.

Type V. There is a carrier x_ν, there are no transfer bases, $w < \beta(\nu)$, and there are no matched variables among the leading bases.

Type VI. There is a carrier x_ν and a transfer base w_π such that $\beta(\pi) < \beta(\nu)$.

Type VII. There is a carrier x_ν and a transfer base w_π such that $\beta(\pi) = \beta(\nu)$.

It is easy to see that these types account for all normalized admissible generalized equations.

We divide the proof of the lemma into seven cases, one for each type.

In each case we proceed by the following scheme. We first define a *transformation* of an equation of given type. The result of this transformation turns out to be a system Ω_1. We prove that Ω_1 is a generalized equation and then prove the three assertions of the lemma.

TYPE I. Ω *has no carrier.*

Since Ω has no carrier, the list of leading bases of Ω contains no unmatched variables. Consequently, the list of leading bases of Ω contains only coefficients and matched variables. The list of leading bases may be empty.

Suppose the list of leading bases contains the coefficients

$$w_{\xi_1}, \ \ldots, \ w_{\xi_b}. \tag{I.1}$$

Since Ω is admissible, the coefficients in (I.1) are all equal, i.e. there exists a number p such that $w_{\xi_1} \doteq \cdots \doteq w_{\xi_b} \doteq a_p$. Moreover, the admissibility of Ω implies that the right boundary of any coefficient in (I.1) is l_2, i.e. the coefficients in (I.1) have the situations

$$l_1 w_{\xi_i} r_2 = t \quad (i = 1, \ \ldots, \ b). \tag{I.2}$$

By hypothesis, if the list of leading bases contains a variable, then it also contains its dual. The admissibility of Ω implies that the right boundary of any matched variable is the same as the right boundary of its dual. Let

$$x_{\eta_1}, \ \ldots, \ x_{\eta_c}, \ x_{\Delta(\eta_1)}, \ \ldots, \ x_{\Delta(\eta_c)} \tag{I.3}$$

be those leading variables having the situations

$$
\begin{aligned}
&l_1 x_{\eta_i} r_2 = t \quad (i = 1, \ldots, c), \\
&l_1 x_{\Delta(\eta_i)} r_2 = t \quad (i = 1, \ldots, c).
\end{aligned} \tag{I.4}
$$

The remaining leading variables

$$x_{\theta_1}, \ldots, x_{\theta_d}, x_{\Delta(\theta_1)}, \ldots, x_{\Delta(\theta_d)} \tag{I.5}$$

have the situations

$$
\begin{aligned}
&l_1 x_{\theta_i} r_{\beta(\theta_i)} = t \quad (i = 1, \ldots, d), \\
&l_1 x_{\Delta(\theta_i)} r_{\beta(\theta_i)} = t \quad (i = 1, \ldots, d),
\end{aligned} \tag{I.6}
$$

where the comparison table of Ω contains the inequalities

$$l_2 < l_{\beta(\theta_i)} \quad (i = 1, \ldots, d). \tag{I.7}$$

To define a transformation $\Omega \to \Omega_1$ of an equation Ω of Type I we will indicate what changes must be made in the description of Ω in order to arrive at Ω_1.

Note that the various parts of a generalized equation are closely related, so that a change in one part often implies changes in the others.

The transformation $\Omega \to \Omega_1$ of an equation of Type I consists of the following four steps.

1. Delete the coefficients in (I.1) from the list of bases (3.5), and the situations of these coefficients from the base situation table. This diminishes the domain of definition of the function $\psi(i)$. If all bases equal to the coefficient a_p occur in (I.1), then the alphabet (3.1) must be replaced by $a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_\omega$. Since Ω is nonelementary, we have $\omega \geqslant 2$; hence the new coefficient alphabet is nonempty.

As a result of this change in the base situation table, l_2 may no longer be a right boundary. We will prove that condition D.5 is preserved. Indeed, if this is not the case, then for some boundary connection (3.9) we have, according to (3.12), $l_{\alpha(\Delta(\lambda_k))} < l_2$, which, in view of condition N.1, implies that $\alpha(\Delta(\lambda_k)) = 1$. Then $x_{\Delta(\lambda_k)}$ is a leading variable and therefore matched. This means that x_{λ_k} is a majorized variable, which is impossible by condition N.4.

2. Delete the variables in (I.3) from the list of bases (3.5), and the situations of these variables from the base situation table. This diminishes the number of x-variables in the table of word variables.

We will prove that the variables in (I.3) and (I.5) do not occur in the paths of boundary connections. Assume that a variable x_{λ_i} of some boundary connection (3.9) belongs to the combined list (I.3), (I.5). Then, in view of condition N.4, we have $i < k$, and, according to (3.11), the variable $x_{\lambda_{i+1}}$ also belongs to the list (I.3), (I.5). Therefore, at least one of x_{λ_i}, $x_{\lambda_{i+1}}$ is majorized, which contradicts condition N.4.

In the same way we can prove that condition D.5 is preserved.

3. Replace the situations (I.6) of the variables (I.5) by

$$
\begin{aligned}
&l_2 x_{\theta_i} r_{\beta(\theta_i)} = t \quad (i = 1, \ldots, d), \\
&l_2 x_{\Delta(\theta_i)} r_{\beta(\theta_i)} = t \quad (i = 1, \ldots, d).
\end{aligned} \tag{I.8}
$$

The conditions (3.7) for (I.8) are guaranteed by (I.7). Since the variables in (I.5) do not occur in paths of boundary connections, the change of left boundaries for these variables does not violate conditions (3.10)–(3.12) for boundary connections.

Since new inessential boundaries do not appear and all right boundaries remain right boundaries, conditions D.4 and D.5 are preserved.

4. Delete l_1 and r_1 from the table of word variables. Also, delete all equalities and inequalities in which these variables occur, namely the boundary equality $l_1 r_1 = t$, the initial equality $l_1 = 1$, and all inequalities $l_1 < l_i$ in the boundary comparison table. Note that l_1 no longer occurs in the base situation table resulting from the third step of the transformation. It is easy to see that the first three steps preserve the admissibility of the equation; hence, by Lemma 5.7, the boundary l_1 can be neither original nor concluding for any boundary connection of the equation.

As the new initial equality we take $l_2 = 1$. In view of condition N.1, condition D.1 is satisfied.

We declare the boundary l_2 to be initial, whether or not the set (I.8) is empty.

It is easy to see that the resulting system Ω_1, up to a renumbering of the variables l_i and r_i, satisfies all of the requirements contained in the definition of a generalized equation.

Under the transformation $\Omega \to \Omega_1$ the number of pairs of x-variables does not increase, and the same is true of the number of bases which are not variables and the number of initial boundaries.

Suppose Ω has a solution (3.15) with index I and exponent of periodicity s. It is easy to see that the components of this solution satisfy the equalities

$$L_1 \stackrel{\circ}{=} 1,$$

$$L_2 R_2 \stackrel{\circ}{=} T,$$

$$L_1 X_{\eta_i} R_2 \stackrel{\circ}{=} T \quad (i = 1, \dots, c),$$

from which it follows that

$$\begin{aligned} &X_{\eta_i} \stackrel{\circ}{=} L_2 \quad (i = 1, \dots, c), \\ &X_{\Delta(\eta_i)} \stackrel{\circ}{=} L_2 \quad (i = 1, \dots, c). \end{aligned} \tag{I.9}$$

Moreover,

$$\begin{aligned} X_{\theta_i} &\stackrel{\circ}{=} L_2 X'_{\theta_i} \quad (i = 1, \dots, d), \\ X_{\Delta(\theta_i)} &\stackrel{\circ}{=} L_2 X'_{\Delta(\theta_i)} \quad (i = 1, \dots, d), \\ L_i &\stackrel{\circ}{=} L_2 L'_i \quad (i = 2, \dots, \rho), \\ T &\stackrel{\circ}{=} L_2 T' \end{aligned} \tag{I.10}$$

for certain X'_{θ_i}, $X'_{\Delta(\theta_i)}$, L'_i and T'.

We obtain a solution of Ω_1 by deleting the components $X_{\eta_1}, \dots, X_{\eta_c}$, $X_{\Delta(\eta_1)}, \dots, X_{\Delta(\eta_c)}$, L_1 and R_1 from the table (3.15) and shortening the components in (I.10) on the left by the word L_2. The exponent of periodicity of the resulting solution is less than or equal to s, and the index of this solution of Ω_1 is less than I, since $\partial(L_2) > 0$

and therefore $\partial(T') < \partial(T)$. Assertion 2 is proved for Type I.

If Ω_1 has a solution, then a solution of Ω can be constructed by adding the missing components and "building up" the appropriate components on the left by a_p if the set of coefficients (I.1) is nonempty, and by an arbitrary nonempty word if the set (I.1) is empty. Assertion 3 is proved for Type I,.

TYPE II. *In Ω there is a carrier x_ν, there are no transfer bases, and $w = \beta(\nu) = 2$, where w is the subscript of the second essential boundary.*

The list of leading bases of Ω contains no coefficients, since in an admissible equation a coefficient a_j which is a leading base has the situation $l_1 a_j r_2 = t$ and in this case would be a transfer base.

The subscript of the right boundary of any leading variable different from the carrier x_ν is greater than the subscript of the right boundary of the carrier (since Ω has no transfer bases), and hence any such variable is matched with its dual.

Thus, the list of leading bases of Ω consists of the carrier x_ν with the situation

$$l_1 x_\nu r_2 = t \tag{II.1}$$

and a (possibly empty) set of matched variables

$$x_{\theta_1}, \ \ldots, \ x_{\theta_d}, \ x_{\Delta(\theta_1)}, \ \ldots, \ x_{\Delta(\theta_d)} \tag{II.2}$$

with situations

$$\begin{aligned} &l_1 x_{\theta_i} r_{\beta(\theta_i)} = t \quad (i = 1, \ \ldots, \ d), \\ &l_1 x_{\Delta(\theta_i)} r_{\beta(\theta_i)} = t \quad (i = 1, \ \ldots, \ d), \end{aligned} \tag{II.3}$$

where the boundary comparison table contains the inequalities

$$l_{\beta(\nu)} < l_{\beta(\theta_i)} \quad (i = 1, \ \ldots, \ d). \tag{II.4}$$

The transformation $\Omega \to \Omega_1$ of an equation of Type II consists of replacing the situation (II.1) of the variable x_ν by the new situation

$$l_{\alpha(\Delta(\nu))} x_\nu r_{\beta(\Delta(\nu))} = t. \tag{II.5}$$

We make no other changes in Ω.

The initial boundaries of Ω are declared to be the initial boundaries of Ω_1. Condition D.3 is preserved in Ω_1, since the boundary $l_{\alpha(\Delta(\nu))}$ in Ω is initial, being the left boundary of the base $x_{\Delta(\nu)}$.

We will prove that no boundary connection contains variables of the set (II.2) in its path. Assume that in the boundary connection (3.9) the variable x_{λ_j} belongs to the set (II.2). In view of condition N.4, we have $j \neq k$. If $j < k$, then, according to (3.11), $x_{\lambda_{j+1}}$ is a leading variable, i.e. $x_{\lambda_{j+1}}$ is either the carrier x_ν or a variable of the set (II.2). In the first case, in view of (II.4), x_{λ_j} is majorized, and in the second, either x_{λ_j} or $x_{\lambda_{j+1}}$ is majorized. In either case we have a contradiction to condition N.4.

We will now prove that the carrier x_ν and its dual $x_{\Delta(\nu)}$ can occur in a boundary connection of Ω only in a loop.

Suppose the variable x_{λ_j} in the boundary connection (3.9) is $x_{\Delta(\nu)}$. If $j = k$, then, by (3.12), we have $l_{\alpha(\nu)} < l_q < l_{\beta(\nu)}$; hence, by (II.1), $l_1 < l_q < l_2$, contrary to condition N.1. If $j < k$, then, by (3.11), $x_{\lambda_{j+1}}$ is a leading variable. In view of condition N.4, $x_{\lambda_{j+1}}$ is unmatched. Then $x_{\lambda_{j+1}}$ is the carrier x_ν, i.e. the subpath $x_{\lambda_j}, x_{\lambda_{j+1}}$ is a loop.

Suppose the variable x_{λ_j} in the boundary connection (3.9) is x_ν. Then conditions (3.10) and (II.1) imply that $j > 1$. It follows easily from the admissibility of Ω that

$$l_{\alpha(\Delta(\lambda_{j-1}))} < l_{\beta(\lambda_j)}.$$

In view of conditions (II.1) and N.1, we have $\alpha(\Delta(\lambda_{j-1})) = 1$. In view of N.4, the variable $x_{\Delta(\lambda_{j-1})}$ is unmatched. Then this variable is the carrier, i.e. the subpath $x_{\lambda_{j-1}}, x_{\lambda_j}$ is a loop.

To verify condition D.5 we prove that l_2 (the right boundary of the carrier x_ν in Ω) cannot be the concluding boundary of any boundary connection, since otherwise, in view of (3.12), we would have $l_{\alpha(\Delta(\lambda_k))} < l_2$, and then, according to N.1, $x_{\Delta(\lambda_k)}$ is a leading variable. In view of N.4, this variable is unmatched. Then it is the carrier, contrary to (3.12). Therefore, condition D.5 is preserved under the transformation $\Omega \to \Omega_1$.

Since the transformation $\Omega \to \Omega_1$ changes the situation of the base x_ν, we must verify conditions (3.10)–(3.12) for the boundary connections containing x_ν. But x_ν occurs in boundary connections only in the loop $x_{\Delta(\nu)}, x_\nu$, and in this case conditions (3.10)–(3.12) are obviously satisfied.

Thus, the system Ω_1 satisfies all of the requirements pertinent to generalized equations.

Assertion 1 for equations of Type II is satisfied trivially, since the first three principal parameters do not change under the transformation $\Omega \to \Omega_1$.

Suppose Ω has a solution (3.15) with index I and exponent of periodicity s. We will prove that this table is a solution of Ω_1. The equalities of duals, the boundary equalities, and the boundary comparison table of Ω_1 agree with the analogous conditions for Ω. Since the table of words (3.15) satisfies the situation equality of the base $x_{\Delta(\nu)}$ of Ω and the equality of duals, it also satisfies the situation equality (II.5) of the base x_ν\of Ω_1. The situation equalities of the remaining bases of Ω_1 agree with the corresponding situation equalities of the bases of Ω.

It is easy to see that if the table (3.15) satisfies some boundary connection of Ω, then, with the same decompositions of the words x_{λ_i}, it satisfies the same boundary connection of Ω_1. Indeed, since x_ν occurs in a boundary connection only in a loop, it suffices to verify (3.19) for the pair of words $X_{\Delta(\nu)}, X_\nu$, this condition having the form

$$L_{\alpha(\nu)} B_i \overset{\circ}{=} L_{\alpha(\nu)} B_{i+1}.$$

But this is true, since in Ω we have $L_1 B_1 \overset{\circ}{=} L_1 B_{i+1}$.

The exponent of periodicity of solution (3.15) of Ω and of the same solution of Ω_1 are obviously the same. The index of solution (3.15) of Ω_1 is smaller than the index of solution (3.15) of Ω, since the fact that the carrier of Ω is unmatched implies that in the vector of the base x_ν relative to the solution (3.15) of Ω_1 the ones are shifted to the right in comparison with the vector of the base x_ν relative to solution (3.15) of Ω. Assertion 2 is proved for Type II.

Suppose Ω_1 has a solution (3.15). Then, in view of (II.3) and (II.4), we have the equalities

$$\begin{gathered} X_{\theta_i} \overset{\circ}{=} L_2 X'_{\theta_i} \quad (i = 1, \ldots, d), \\ X_{\Delta(\theta_i)} \overset{\circ}{=} L_2 X'_{\Delta(\theta_i)} \quad (i = 1, \ldots, d), \\ L_i \overset{\circ}{=} L_2 L'_i \quad (i = 2, \ldots, \rho), \\ R_1 \overset{\circ}{=} L_2 R'_1, \\ T \overset{\circ}{=} L_2 T' \end{gathered} \tag{II.6}$$

for certain X'_{θ_i}, $X'_{\Delta(\theta_i)}$, L'_i, R'_1, T'. Let

$$x_{\eta_i} \quad (i=1,\ \ldots,\ 2n-2d)$$

be the set of all x-variables not in the set (II.2). We construct the desired solution of Ω from the solution (3.15) of Ω_1 as follows:

$$\begin{aligned}
&X''_{\theta_i} \rightleftharpoons X_\nu X'_{\theta_i} \quad (i=1,\ \ldots,\ d),\\
&X''_{\Delta(\theta_i)} \rightleftharpoons X_\nu X'_{\Delta(\theta_i)} \quad (i=1,\ \ldots,\ d),\\
&X''_{\eta_i} \rightleftharpoons X_{\eta_i} \quad (i=1,\ \ldots,\ 2n-2d),\\
&L''_1 \rightleftharpoons 1,\\
&L''_i \rightleftharpoons X_\nu L'_i \quad (i=2,\ \ldots,\ \rho),\\
&R''_1 \rightleftharpoons X_\nu R'_1,\\
&R''_i \rightleftharpoons R_i \quad (i=2,\ \ldots,\ \rho),\\
&T'' \rightleftharpoons X_\nu T'.
\end{aligned} \tag{II.7}$$

The equalities of duals, the boundary equalities, the conditions of the boundary comparison table, and the base situation equalities can be verified directly from (II.6) and (II.7).

Since the variables of the set (II.2) do not occur in the paths of boundary connections and x_ν occurs only in a loop, we see, on replacing the words L_i by the words L''_i in (3.18)–(3.20), that conditions (3.18)–(3.20) for the boundary connections of Ω are satisfied with the same B_i and C_i with which they were satisfied in Ω_1. Consequently, the table of words (II.7) is a solution of Ω. Assertion 3 is proved for Type II.

TYPE III. *In Ω there is a carrier x_ν, there are no transfer bases, and $w = \beta(\nu) > 2$, where w is the subscript of the second essential boundary.*

Since Ω has no transfer bases, as in Type II, the list of leading bases consists of the carrier x_ν with situation

$$l_1 x_\nu r_{\beta(\nu)} = t \tag{III.1}$$

and a (possibly empty) set of matched variables

$$x_{\theta_1},\ \ldots,\ x_{\theta_d},\ x_{\Delta(\theta_1)},\ \ldots,\ x_{\Delta(\theta_d)} \tag{III.2}$$

with situations

$$\begin{aligned}
&l_1 x_{\theta_i} r_{\beta(\theta_i)} = t \quad (i=1,\ \ldots,\ d),\\
&l_1 x_{\Delta(\theta_i)} r_{\beta(\theta_i)} = t \quad (i=1,\ \ldots,\ d),
\end{aligned} \tag{III.3}$$

where the boundary comparison table contains the inequalities

$$l_{\beta(\nu)} < l_{\beta(\theta_i)} \quad (i=1,\ \ldots,\ d). \tag{III.4}$$

By hypothesis, the set of inessential boundaries

$$l_2,\ \ldots,\ l_{w-1} \tag{III.5}$$

is nonempty. According to condition D.4, each boundary in (III.5) is the original boundary of some boundary connection.

We will prove that for any boundary connection (3.9) whose original boundary is a boundary l_p in (III.5) the variable x_{λ_1} is the carrier x_ν. Indeed, according to condition (3.10) for the boundary connection under consideration we have $l_{\alpha(\lambda_1)} < l_p$, where $l_p < l_w$. By condition D.3, $l_{\alpha(\lambda_1)}$ is an initial boundary and therefore essential. Consequently, $\alpha(\lambda_1) = 1$, i.e. x_{λ_1} is a leading variable. The variable x_{λ_1} cannot be contained in the list (III.2) because of condition N.4. Therefore, x_{λ_1} is the carrier x_ν.

A boundary connection is called *primitive* if the length of its path is equal to 1.

It follows from condition N.1 and Lemma 5.6 that if Ω contains two primitive boundary connections with the same original boundary in the list (III.5), then these connections have the same concluding boundary.

The transformation $\Omega \to \Omega_1$ of an equation of Type III consists of the following.

Delete from the boundary comparison table all inequalities in which the boundaries $l_2, \ldots, l_{w-1}$ occur, and add the inequalities

$$l_{\alpha(\Delta(\nu))} < l_2 < l_3 < \ldots < l_{w-1} < l_{\beta(\Delta(\nu))}. \tag{III.6}$$

As usual, we close the boundary comparison table with respect to transitivity.

Since the boundaries in the list (III.5) are inessential, they do not occur in the base situation table.

As the initial boundaries of Ω_1 we take those boundaries which were initial in Ω.

We replace each boundary connection of

$$l_p,\ x_{\lambda_1},\ x_{\lambda_2},\ \ldots,\ x_{\lambda_k},\ l_q, \tag{III.7}$$

where $k > 1$ and l_p is in the list (III.5), by the boundary connection

$$l_p,\ \lambda_{\lambda_2},\ \ldots,\ x_{\lambda_k},\ l_q \tag{III.8}$$

and we add to the boundary comparison table the inequalities

$$l_{\alpha(\lambda_2)} < l_p < l_{\beta(\lambda_2)}, \tag{III.9}$$

which guarantee the fulfillment of the conditions (3.10) for the new boundary connections.

We exclude from the list of boundary connections each primitive connection

$$l_h,\ x_{\lambda_1},\ l_g, \tag{III.10}$$

where l_h is in the list (III.5), and we identify l_h with l_g.

It is easy to see that the system Ω_1 thus constructed is a generalized equation.

Assertion 1 for Type III is satisfied trivially, since the first three principal parameters of Ω_1 agree with the first three principal parameters of Ω.

Suppose Ω has a solution (3.15) with index I and exponent of periodicity s. For each $i = 2, \ldots, w-1$ we construct words L_i' and R_i' by means of the equalities

$$\begin{gathered} L_i' R_i' \doteq T, \\ \partial(L_i') = \partial(L_i) + \partial(L_{\alpha(\Delta(\nu))}). \end{gathered} \tag{III.11}$$

In view of (III.1) and $w = \beta(\nu)$, we have $L_w = X_\nu$, from which, in view of the equality of duals, we obtain

$$L_{\alpha(\Delta(\nu))} L_w \doteq L_{\beta(\Delta(\nu))}. \tag{III.12}$$

By condition N.1, $\partial(L_i) < \partial(L_w)$ for all $i = 2, \ldots, w-1$; hence $\partial(L_i) + \partial(L_{\alpha(\Delta(\nu))})$

$< \partial(T)$ and the equalities (III.11) uniquely determine L_i' and R_i' for all $i = 2, \ldots, w-1$.

We will prove that the table of words

$$\begin{gathered} X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}, \\ L_1, L_2', L_3', \ldots, L_{w-1}', L_w, \ldots, L_\rho, \\ R_1, R_2', R_3', \ldots, R_{w-1}', R_w, \ldots, R_\rho, T \end{gathered} \tag{III.13}$$

is a solution of Ω_1.

The equalities of duals, the boundary equalities, and the base situation equalities are satisfied in view of the conditions (III.11) and the fact that the transformation $\Omega \to \Omega_1$ introduces no changes in the base situation table.

The inequalities (III.6) follow from condition N.1 for Ω and conditions (III.11) and (III.12).

Suppose Ω contains a boundary connection (III.7). The table of words (3.15) satisfies this connection; hence there exist nonempty words $B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ such that

$$\begin{gathered} X_{\lambda_i} \doteq B_i C_i \quad (i = 1, \ldots, k), \\ L_p \doteq L_{\alpha(\lambda_1)} B_1, \\ L_{\alpha(\Delta(\lambda_1))} B_1 \doteq L_{\alpha(\lambda_2)} B_2, \\ L_{\alpha(\Delta(\lambda_i))} B_i \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i = 2, \ldots, k-1), \\ L_{\alpha(\Delta(\lambda_k))} B_k \doteq L_q. \end{gathered} \tag{III.14}$$

Since $\lambda_1 = \nu$, we have $\partial(L_{\alpha(\lambda_1)}) = 0$; and $L_p \doteq B_1$ in the conditions (III.14). By (III.11),

$$\partial(L_p') = \partial(B_1) + \partial(L_{\alpha(\Delta(\nu))}).$$

It follows from the third line of (III.14) that

$$\partial(L_p') = \partial(L_{\alpha(\lambda_2)}) + \partial(B_2).$$

Since L_p' and $L_{\alpha(\lambda_2)} B_2$ are beginnings of T and since B_2 and C_2 are nonempty, we have

$$\begin{gathered} L_p' \doteq L_{\alpha(\lambda_2)} B_2, \\ \partial(L_{\alpha(\lambda_2)}) < \partial(L_p') < \partial(L_{\beta(\lambda_2)}). \end{gathered}$$

Consequently, the table of words (III.13) satisfies the boundary connection (III.8) and the conditions (III.9).

Suppose Ω contains a primitive boundary connection (III.10). The table of words (3.15) satisfies this connection; hence there exist nonempty words (granting that $\lambda_1 = \nu$) B_1 and C_1 such that

$$\begin{gathered} L_h \doteq L_{\alpha(\nu)} B_1, \\ L_{\alpha(\Delta(\nu))} B_1 \doteq L_g. \end{gathered}$$

Since $\partial(L_{\alpha(\nu)}) = 0$, we have $\partial(L_g) = \partial(L_{\alpha(\Delta(\nu))}) + \partial(L_h)$, from which, in view of (III.11), we obtain $L_g \doteq L_h'$, which is consistent with the identification of boundaries.

Thus, we have proved that the table of words (III.13) is a solution of Ω_1. The exponent of periodicity of the solution (III.13) is equal to the exponent of periodicity of solution (3.15) of Ω. The index of solution (III.13) of Ω_1 is smaller than the index of solution (3.15) of Ω, since the lengths of the solutions and the distribution vectors of the solutions agree and, in view of condition D.4, the sum of the lengths of the paths of all boundary connections in Ω_1 is smaller than in Ω. Assertion 2 is proved for Type III.

Suppose Ω_1 has a solution (III.13). We construct a table of words (3.15), using the solution (III.13) and the equalities

$$L_iR_i \doteq T \quad (i=2, \ldots, w-1),$$
$$\partial(L_i) = \partial(L_i') - \partial(L_{\alpha(\Delta(\nu))}) \quad (i=2, \ldots, w-1). \tag{III.15}$$

We will prove that this table is a solution of Ω.

The equalities of duals, the boundary equalities, and the base situation equalities are satisfied in view of (III.15) and the fact that the transformation $\Omega \to \Omega_1$ introduces no changes in the base situation table. The conditions

$$\partial(L_1) < \partial(L_2) < \partial(L_3) < \ldots < \partial(L_{w-1}) < \partial(L_w)$$

follow from (III.6), (III.15) and the equality $\partial(L_w) = \partial(X_\nu)$, which in turn follows from (III.1) and $w = \beta(\nu)$.

Suppose Ω has a boundary connection (III.7). Then Ω_1 has the boundary connection (III.8), and the table of words (III.13) satisfies this connection. Consequently, there exist nonempty words $B_2, \ldots, B_k$ and $C_2, \ldots, C_k$ such that

$$X_{\lambda_i} \doteq B_iC_i \quad (i=2, \ldots, k),$$
$$L_p' \doteq L_{\alpha(\lambda_2)}B_2,$$
$$L_{\alpha(\Delta(\lambda_i))}B_i \doteq L_{\alpha(\lambda_{i+1})}B_{i+1} \quad (i=2, \ldots, k-1), \tag{III.16}$$
$$L_{\alpha(\Delta(\lambda_k))}B_k \doteq L_q.$$

In view of (III.6), we have

$$\partial(L_{\alpha(\Delta(\nu))}) < \partial(L_p') < \partial(L_{\alpha(\Delta(\nu))}) + \partial(X_{\Delta(\nu)}).$$

Consequently, there exist nonempty words B_1 and C_1 such that

$$X_\nu \doteq B_1C_1,$$
$$L_p' \doteq L_{\alpha(\Delta(\nu))}B_1,$$

whence, by (III.15), we obtain $\partial(L_p) = \partial(B_1)$ and $L_p \doteq L_{\alpha(\nu)}B_1$. Adding these equalities to (III.16), we see that the constructed table of words (3.15) satisfies the boundary connection (III.7).

Suppose Ω has a primitive boundary connection (III.10). Then in Ω_1 the boundaries l_h and l_g coincide, and hence $L_h' \doteq L_g$. In view of (III.15), we obtain

$$\partial(L_g) = \partial(L_h) + \partial(L_{\alpha(\Delta(\nu))}).$$

Since L_h is a proper beginning of X_ν, it follows that

$$L_g \doteq L_{\alpha(\Delta(\nu))}L_h,$$
$$\partial(L_g) < \partial(L_{\alpha(\Delta(\nu))}X_\nu).$$

Therefore, X_ν can be decomposed into nonempty words B_1 and C_1 such that

$$X_\nu \overset{\circ}{=} B_1C_1,$$

$$L_h \overset{\circ}{=} L_{\alpha(\nu)}B_1,$$

$$L_{\alpha(\Delta(\nu))}B_1 \overset{\circ}{=} L_g,$$

which means that the constructed table of words (3.15) satisfies the boundary connection (III.10).

The boundary connections whose original boundaries do not occur in the list (III.5) are not affected by the transformation $\Omega \to \Omega_1$; hence the constructed table of words (3.15) also satisfies these connections.

Thus, the table (3.15) is a solution of Ω. Assertion 3 is proved for Type III.

TYPE IV. *In Ω there is a carrier x_ν, there are no transfer bases, $w < \beta(\nu)$, where w is the subscript of the second essential boundary, and there are matched variables among the leading bases.*

The list of leading bases of Ω consists of the carrier x_ν with situation

$$l_1 x_\nu r_{\beta(\nu)} = t \tag{IV.1}$$

and a nonempty set of matched variables

$$x_{\theta_1}, \ \ldots, \ x_{\theta_d}, \ x_{\Delta(\theta_1)}, \ \ldots, \ x_{\Delta(\theta_d)} \tag{IV.2}$$

with situations

$$\begin{aligned} &l_1 x_{\theta_i} r_{\beta(\theta_i)} = t \quad (i = 1, \ \ldots, \ d), \\ &l_1 x_{\Delta(\theta_i)} r_{\beta(\theta_i)} = t \quad (i = 1, \ \ldots, \ d), \end{aligned} \tag{IV.3}$$

where the boundary comparison table contains the inequalities

$$l_{\beta(\nu)} < l_{\beta(\theta_i)} \quad (i = 1, \ \ldots, \ d). \tag{IV.4}$$

The transformation $\Omega \to \Omega_1$ of an equation of Type IV consists of replacing the situations (IV.3) of the variables (IV.2) by the new situations

$$\begin{aligned} &l_w x_{\theta_i} r_{\beta(\theta_i)} = t \quad (i = 1, \ \ldots, \ d), \\ &l_w x_{\Delta(\theta_i)} r_{\beta(\theta_i)} = t \quad (i = 1, \ \ldots, \ d). \end{aligned} \tag{IV.5}$$

We make no other changes in Ω.

The conditions (3.7) for the situations (IV.5) are guaranteed by the inequality $w < \beta(\nu)$ and (IV.4). The initial boundaries of Ω_1 are taken to be those boundaries which were initial in Ω.

Since Ω has no transfer bases, l_w cannot be the right boundary of any base. Consequently, the boundary l_w is initial, and condition D.3 is satisfied for Ω_1.

Exactly as in Type II, it can be proved that no boundary connection of Ω contains variables of the set (IV.2). Therefore, the changes of situations of variables of the set (IV.2) are not reflected in the boundary connections.

Thus, the resulting system Ω_1 fits the definition of a generalized equation.

Assertion 1 for Type IV is satisfied trivially, since the first three principal parameters are unchanged under the transformation $\Omega \to \Omega_1$.

Suppose Ω has a solution (3.15) with index I and exponent of periodicity s. In view of the conditions $w < \beta(\nu)$, (IV.3), and (IV.4), there exist nonempty words X'_{θ_i} and $X'_{\Delta(\theta_i)}$ such that

$$X_{\theta_i} \doteq L_w X'_{\theta_i} \quad (i=1, \ldots, d),$$
$$X_{\Delta(\theta_i)} \doteq L_w X'_{\Delta(\theta_i)} \quad (i=1, \ldots, d). \tag{IV.6}$$

We construct the table of words

$$X'_1, \ldots, X'_n, X'_{n+1}, \ldots, X'_{2n},$$
$$L_1, \ldots, L_\rho, \quad R_1, \ldots, R_\rho, \quad T, \tag{IV.7}$$

where $X'_j = X_j$ for all $j \neq \theta_1, \ldots, \theta_d, \Delta(\theta_1), \ldots, \Delta(\theta_d)$.

It is easy to see that (IV.7) satisfies the equalities of duals, the boundary equalities, the base situation equalities, and all conditions of the boundary comparison table of Ω_1.

The table (IV.7) also satisfies all boundary connections of Ω_1, since no variable of the set (IV.2) is contained in the paths of the boundary connections of Ω.

Thus, (IV.7) is a solution of Ω_1. Obviously, the exponent of periodicity of this solution does not exceed that of solution (3.15). The index of solution (IV.7) of Ω_1 is smaller than the index of solution (3.15) of Ω, since the distribution vector of (IV.7) is smaller (since $\partial(L_w) > 0$ and the set of variables (IV.2) is nonempty) than the distribution vector of (3.15). Assertion 2 is proved for Type IV.

Suppose Ω_1 has a solution (IV.7). We construct the table of words

$$X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n},$$
$$L_1, \ldots, L_\rho, \quad R_1, \ldots, R_\rho, \quad T, \tag{IV.8}$$

where we choose the X_j in accordance with (IV.6) for $j = \theta_1, \ldots, \theta_d, \Delta(\theta_1), \ldots, \Delta(\theta_d)$ and in accordance with $X_j \doteq X'_j$ for the remaining j. Obviously, (IV.8) is a solution of Ω. Assertion 3 is proved for Type IV.

Type V. *In Ω there is a carrier x_ν, there are no transfer bases, $w < \beta(\nu)$, where w is the subscript of the second essential boundary, and there are no matched variables among the leading bases.*

Since in Ω there are no transfer bases and no matched leading variables, the carrier x_ν with situation

$$l_1 x_\nu r_{\beta(\nu)} = t \tag{V.1}$$

is the only leading base. Since in Ω there are no transfer bases, the second essential boundary l_w cannot be the right boundary of any base. Consequently, the boundary l_w is initial.

By definition of l_w, the boundaries

$$l_2, \ldots, l_{w-1} \tag{V.2}$$

are inessential (recall that the list (V.2) can be empty). By condition D.4, each boundary l_p in (V.2) is the original boundary of some boundary connection.

We will prove that for any boundary connection (3.9) whose original boundary is a boundary l_p in the list

$$l_2, \ldots, l_{w-1}, l_w, \tag{V.3}$$

the variable x_{λ_1} is the carrier x_ν. Indeed, according to condition (3.10) for the given boundary connection we have $l_{\alpha(\lambda_1)} < l_p$. By condition D.3, the boundary $l_{\alpha(\lambda_1)}$ is initial and therefore essential. Consequently, $\alpha(\lambda_1) = 1$, from which it follows that x_{λ_1} is x_ν.

It follows from condition N.1 and Lemma 5.6 that if Ω contains two primitive boundary connections with the same original boundary in the list (V.3), then these connections have the same concluding boundary.

The transformation $\Omega \to \Omega_1$ of an equation of Type V consists of four transformations $\Omega \to \Omega_2$, $\Omega_2 \to \Omega_3$, $\Omega_3 \to \Omega_4$ and $\Omega_4 \to \Omega_1$, carried out in succession in the order indicated.

The transformation $\Omega \to \Omega_2$ consists of the following. Add $l_{\rho+1}$ and $r_{\rho+1}$ to the table of word variables. The boundary $l_{\rho+1}$ is taken to be initial. To the list of boundary equalities (3.4) add $l_{\rho+1}r_{\rho+1} = t$. To the boundary comparison table add the inequalities

$$l_{\alpha(\Delta(\nu))} < l_{\rho+1} < l_{\beta(\Delta(\nu))}. \tag{V.4}$$

To the table of word variables add x_u and x_{u+1}, where $u = 2n + m + 1$, satisfying the condition

$$x_u = x_{u+1}. \tag{V.5}$$

The variables x_u and x_{u+1} are called duals; and $\Delta(u) = u + 1$ and $\Delta(u + 1) = u$. To the base situation table add the equalities

$$l_w x_u r_{\beta(\nu)} = t, \tag{V.6}$$

$$l_{\rho+1} x_{u+1} r_{\beta(\Delta(\nu))} = t. \tag{V.7}$$

It is easy to see that the resulting system Ω_2, to within a renumbering of the variables, fits the definition of a generalized equation.

Assertions 1, 2 and 3 are only partially realized under the transformation $\Omega \to \Omega_2$. Let us indicate precisely to what extent they are realized and to what extent they are not.

The number of pairs of x-variables in Ω_2 is one larger than in Ω. The number of bases which are not variables in Ω_2 is the same as in Ω. The number of initial boundaries in Ω_2 is one larger than in Ω.

If Ω has a solution (3.15) with index I and exponent of periodicity s, then a solution of Ω_2 can be constructed by adding to (3.15) the missing components X_u, X_{u+1}, $L_{\rho+1}$ and $R_{\rho+1}$, which are uniquely determined from the following equalities:

$$\begin{aligned} X_\nu &\overset{\circ}{=} L_w X_u, \\ X_{u+1} &\overset{\circ}{=} X_u, \\ L_{\rho+1} X_{u+1} &\overset{\circ}{=} L_{\beta(\Delta(\nu))}, \\ L_{\rho+1} R_{\rho+1} &\overset{\circ}{=} T. \end{aligned} \tag{V.8}$$

The word X_u is an end of X_ν; hence the exponent of periodicity of the constructed solution of Ω_2 is equal to s. The index of the constructed solution of Ω_2 is greater than the index of solution (3.15) of Ω, but the first component of the index (the length of the solution) is the same in both cases.

If Ω_2 has a solution, then a solution of Ω is obtained by simply deleting the superfluous components.

The transformation $\Omega_2 \to \Omega_3$ consists of the following.

Delete from the boundary comparison table all inequalities involving $l_2, \ldots, l_{w-1}$, and add the inequalities

$$l_{\alpha(\Delta(\nu))} < l_2 < l_3 < \ldots < l_{w-1} < l_{\rho+1}. \tag{V.9}$$

Since the boundaries in (V.2) are inessential, they do not appear in the base situation table.

Replace each boundary connection of Ω_2

$$l_p, x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_k}, l_q, \tag{V.10}$$

where $k > 1$ and l_p is a boundary in (V.2), by the connection

$$l_p, x_{\lambda_2}, \ldots, x_{\lambda_k}, l_q \tag{V.11}$$

and add to the boundary comparison table the inequalities

$$l_{\alpha(\lambda_2)} < l_p < l_{\beta(\lambda_2)}, \tag{V.12}$$

which guarantee the fulfillment of the conditions (3.10) for the new connections.

Replace each boundary connection of Ω_2

$$l_w, x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_k}, l_q, \tag{V.13}$$

where $k > 1$ and l_w is the second essential boundary, by the connection

$$l_{\rho+1}, x_{\lambda_2}, \ldots, x_{\lambda_k}, l_q \tag{V.14}$$

and add to the boundary comparison table the inequalities

$$l_{\alpha(\lambda_2)} < l_{\rho+1} < l_{\beta(\lambda_2)}. \tag{V.15}$$

Delete each primitive boundary connection of Ω_2

$$l_h, x_{\lambda_1}, l_g, \tag{V.16}$$

where l_h is in (V.2), from the list of boundary connections, and identify l_h with l_g.

Delete each primitive boundary connection of Ω_2

$$l_w, x_{\lambda_1}, l_g \tag{V.17}$$

from the list of boundary connections, and identify l_g with $l_{\rho+1}$.

Since the boundary $l_{\alpha(\Delta(\nu))}$ is essential, it follows easily from (V.9) that in Ω_3 we have

$$l_w < l_i \tag{V.18}$$

for $i \neq 1$ and $i \neq w$. It is easy to see that the resulting system Ω_3, to within a renumbering of the variables, is a generalized equation. The first three principal parameters of Ω_3 agree with the corresponding parameters of Ω_2.

Suppose Ω_2 has a solution

$$\begin{aligned} &X_1, \ldots, X_n, X_u, X_{n+1}, \ldots, X_{2n}, X_{u+1}, \\ &L_1, L_2, \ldots, L_{w-1}, L_w, \ldots, L_\rho, L_{\rho+1}, \\ &R_1, R_2, \ldots, R_{w-1}, R_w, \ldots, R_\rho, R_{\rho+1}, \\ &T \end{aligned} \tag{V.19}$$

with index I and exponent of periodicity s.

Note that $\partial(L_w) < \partial(L_{\beta(\nu)}) = \partial(X_\nu)$, whence $\partial(L_i) + \partial(L_{\alpha(\Delta(\nu))}) < \partial(L_{\beta(\Delta(\nu))})$ for all $i = 1, 2, \ldots, w$.

We construct the table of words

$$\begin{aligned}&X_1, \ldots, X_n, X_u, X_{n+1}, \ldots, X_{2n}, X_{u+1},\\ &L_1, L_2', \ldots, L_{w-1}', L_w, \ldots, L_\rho, L_{\rho+1},\\ &R_1, R_2', \ldots, R_{w-1}', R_w, \ldots, R_\rho, R_{\rho+1},\\ &T,\end{aligned} \tag{V.20}$$

using the table (V.19) and the equalities

$$\begin{gathered}L_i'R_i' \doteq T \quad (i=2, \ldots, w-1),\\ \partial(L_i') = \partial(L_i) + \partial(L_{\alpha(\Delta(\nu))}) \quad (i=2, \ldots, w-1).\end{gathered} \tag{V.21}$$

We will prove that (V.20) is a solution of Ω_3.

It follows easily from (V.8) that

$$L_{\alpha(\Delta(\nu))}L_w \doteq L_{\rho+1}. \tag{V.22}$$

The equalities of duals, the boundary equalities, and the base situation equalities are satisfied in view of (V.21) and the fact that the transformation $\Omega_2 \to \Omega_3$ introduces no changes in the base situation table.

The inequalities (V.9) follow from (V.21) and (V.22).

Suppose Ω_2 contains a boundary connection (V.10). The table of words (V.19) satisfies this connection; hence there exist nonempty words

$$\begin{gathered}B_1, B_2, \ldots, B_k,\\ C_1, C_2, \ldots, C_k\end{gathered} \tag{V.23}$$

such that

$$\begin{gathered}X_{\lambda_i} \doteq B_iC_i \quad (i=1, \ldots, k),\\ L_p \doteq L_{\alpha(\lambda_1)}B_1,\\ L_{\alpha(\Delta(\lambda_1))}B_1 \doteq L_{\alpha(\lambda_2)}B_2,\\ L_{\alpha(\Delta(\lambda_i))}B_i \doteq L_{\alpha(\lambda_{i+1})}B_{i+1} \quad (i=2, \ldots, k-1),\\ L_{\alpha(\Delta(\lambda_k))}B_k \doteq L_q.\end{gathered} \tag{V.24}$$

Since $\lambda_1 = \nu$, we have $\partial(L_{\alpha(\nu)}) = 0$, and $L_p \doteq B_1$ in the conditions (V.24). In view of (V.21),

$$\partial(L_p') = \partial(B_1) + \partial(L_{\alpha(\Delta(\nu))}).$$

According to the third line of (V.24),

$$\partial(L_p') = \partial(L_{\alpha(\lambda_2)}) + \partial(B_2).$$

Since the words L_p' and $L_{\alpha(\lambda_2)}B_2$ are beginnings of T, and since the words B_2 and C_2 are nonempty, we have

$$\begin{gathered}L_p' \doteq L_{\alpha(\lambda_2)}B_2,\\ \partial(L_{\alpha(\lambda_2)}) < \partial(L_p') < \partial(L_{\beta(\lambda_2)}).\end{gathered}$$

Consequently, (V.20) satisfies the boundary connection (V.11) and the conditions (V.12).

Suppose Ω_2 contains a boundary connection (V.13). The table of words (V.19) satisfies this connection; hence there exist nonempty words (V.23) such that the conditions (V.24) are satisfied with the second line replaced by the equality $L_w \doteq L_{\alpha(\lambda_1)}B_1$. Since $\lambda_1 = \nu$, we have $L_w \doteq B_1$ and, in view of (V.22),

$$L_{\rho+1} \doteq L_{\alpha(\Delta(\nu))}B_1.$$

According to the third line of (V.24),

$$L_{\rho+1} \doteq L_{\alpha(\lambda_2)}B_2.$$

Since B_2 and C_2 are nonempty, the table (V.20) satisfies the boundary connection (V.14) and the conditions (V.15).

Suppose Ω_2 contains a primitive boundary connection (V.16). The table (V.19) satisfies this connection; hence there exist nonempty words B_1 and C_1 such that

$$L_h \doteq L_{\alpha(\nu)}B_1,$$
$$L_{\alpha(\Delta(\nu))}B_1 \doteq L_g.$$

Since $\partial(L_{\alpha(\nu)}) = 0$, we have $\partial(L_g) = \partial(L_{\alpha(\Delta(\nu))}) + \partial(L_h)$, from which, in view of (V.21), we obtain $L_g \doteq L'_h$, which is consistent with the identification of boundaries.

Suppose Ω_2 contains a primitive boundary connection (V.17). Then there exist words B_1 and C_1 such that

$$L_w \doteq L_{\alpha(\nu)}B_1,$$
$$L_{\alpha(\Delta(\nu))}B_1 \doteq L_g.$$

In this case, in view of (V.22), we have $L_g \doteq L_{\rho+1}$, which is consistent with the identification of boundaries.

Thus, we have proved that (V.20) is a solution of Ω_3. The exponent of periodicity of this solution is equal to s, and the length of this solution is the same as that of (V.19).

Suppose Ω_3 has a solution (V.20). We construct a table of words of the form (V.19), using (V.20) and the equalities

$$\begin{gathered} L_iR_i \doteq T \quad (i=2, \ldots, w-1), \\ \partial(L_i) = \partial(L'_i) - \partial(L_{\alpha(\Delta(\nu))}) \quad (i=2, \ldots, w-1). \end{gathered} \tag{V.25}$$

We will prove that this table is a solution of Ω_2.

The equalities of duals, the boundary equalities, and the base situation equalities are satisfied in view of (V.25) and the fact that the transformation $\Omega_2 \to \Omega_3$ introduces no changes in the base situation table. The conditions

$$\partial(L_1) < \partial(L_2) < \partial(L_3) < \ldots < \partial(L_{w-1}) < \partial(L_w)$$

follow from (V.9), (V.25), and (V.22).

Suppose Ω_2 has a boundary connection (V.10). Then Ω_3 has the connection (V.11) and the table of words (V.20) satisfies this connection. Consequently, there exist nonempty words

$$\begin{gathered} B_2, \ldots, B_k, \\ C_2, \ldots, C_k \end{gathered} \tag{V.26}$$

such that

$$\begin{gathered} X_{\lambda_i} \doteq B_i C_i \quad (i=2, \ldots, k), \\ L'_p \doteq L_{\alpha(\lambda_2)} B_2, \\ L_{\alpha(\Delta(\lambda_i))} B_i \doteq L_{\alpha(\lambda_{i+1})} B_{i+1} \quad (i=2, \ldots, k-1), \\ L_{\alpha(\Delta(\lambda_k))} B_k \doteq L_q. \end{gathered} \tag{V.27}$$

According to (V.9),

$$\partial(L_{\alpha(\Delta(\nu))}) < \partial(L'_p) < \partial(L_{\alpha(\Delta(\nu))} X_{\Delta(\nu)}).$$

Consequently, there exist nonempty words B_1 and C_1 such that

$$\begin{gathered} X_\nu \doteq B_1 C_1, \\ L'_p \doteq L_{\alpha(\Delta(\nu))} B_1; \end{gathered}$$

hence, in view of (V.25), we obtain $\partial(L_p) = \partial(B_1)$ and $L_p \doteq L_{\alpha(\nu)} B_1$. Adding these equalities to (V.27), we see that (V.20) satisfies the boundary connection (V.10).

Suppose Ω_2 has a boundary connection (V.13). Then Ω_3 has the connection (V.14) and there exist nonempty words (V.26) such that the conditions (V.27) are satisfied with the second line replaced by the equality

$$L_{p+1} \doteq L_{\alpha(\lambda_2)} B_2.$$

In view of $w < \beta(\nu)$ and (V.22), there exist nonempty words B_1 and C_1 such that $X_\nu \doteq B_1 C_1$ and

$$\begin{gathered} L_w \doteq L_{\alpha(\nu)} B_1, \\ L_{\alpha(\Delta(\nu))} B_1 \doteq L_{\alpha(\lambda_2)} B_2; \end{gathered}$$

hence (V.20) satisfies the boundary connection (V.13).

Suppose Ω_2 has a primitive boundary connection (V.16). Then in Ω_3 the boundaries l_h and l_g coincide; hence $L'_h \doteq L_g$. By (V.25), we obtain

$$\partial(L_g) = \partial(L_h) + \partial(L_{\alpha(\Delta(\nu))}).$$

Since L_h is a proper beginning of X_ν, it follows that

$$\begin{gathered} L_g \doteq L_{\alpha(\Delta(\nu))} L_h, \\ \partial(L_g) < \partial(L_{\alpha(\Delta(\nu))} X_\nu). \end{gathered}$$

Consequently, X_ν can be decomposed into nonempty words B_1 and C_1 such that

$$\begin{gathered} X_\nu \doteq B_1 C_1, \\ L_h \doteq L_{\alpha(\nu)} B_1, \\ L_{\alpha(\Delta(\nu))} B_1 \doteq L_g, \end{gathered}$$

which means that (V.20) satisfies the boundary connection (V.16).

Suppose Ω_2 has a primitive boundary connection (V.17). Then $L_{p+1} \doteq L_g$. By (V.22), we obtain $L_g \doteq L_{\alpha(\Delta(\nu))} L_w$. Since $w < \beta(\nu)$, the word X_ν can be decomposed into nonempty words B_1 and C_1 such that

$$X_\nu \doteq B_1 C_1,$$
$$L_w \doteq L_{\alpha(\nu)} B_1,$$
$$L_{\alpha(\Delta(\nu))} B_1 \doteq L_g,$$

which means that (V.20) satisfies the boundary connection (V.17).

Thus, we have proved that (V.19) is a solution of Ω_2.

The transformation $\Omega_3 \to \Omega_4$ consists of the following. Replace each occurrence of x_ν in the boundary connections of Ω_3 by the variable x_u. Replace each occurrence of $x_{\Delta(\nu)}$ in the boundary connections of Ω_3 by the variable x_{u+1}.

We proved above that Ω_3 has a solution if and only if Ω has. In exactly the same way, we can prove that Ω_3^0 has a solution if and only if Ω^0 has. It follows from this and the admissibility of Ω that Ω_3^0 has a solution.

Suppose Ω_3 contains a boundary connection

$$l_p,\ \pi_1,\ x_{\lambda_i},\ \pi_2,\ l_q, \tag{V.28}$$

where $\lambda_i = \nu$, x_ν is the carrier, and π_1 and π_2 are subpaths of the connection. Since Ω_3^0 has a solution, there exist nonempty words

$$\begin{gathered} B_1, \ldots, B_k, \\ C_1, \ldots, C_k \end{gathered} \tag{V.29}$$

in a one-letter alphabet such that

$$\begin{gathered} X_{\lambda_j} \doteq B_j C_j \quad (j=1, \ldots, k), \\ L_p \doteq L_{\alpha(\lambda_1)} B_1, \\ L_{\alpha(\Delta(\lambda_j))} B_j \doteq L_{\alpha(\lambda_{j+1})} B_{j+1} \quad (j=1, \ldots, k-1), \\ L_{\alpha(\Delta(\lambda_k))} B_k \doteq L_q. \end{gathered} \tag{V.30}$$

We will prove that

$$\partial(B_i) > \partial(L_w). \tag{V.31}$$

By Lemma 5.7, l_1 cannot be the original boundary of the connection (V.28). The boundary l_w cannot be the original boundary of (V.28), since we have replaced each connection of type (V.13) and have deleted each connection of type (V.17). Consequently, in view of (V.18),

$$l_w < l_p. \tag{V.32}$$

If $i = 1$ (the subpath π_1 is empty), then $L_p \doteq L_{\alpha(\nu)} B_i$, and since $\alpha(\nu) = 1$, it follows that $\partial(B_i) = \partial(L_p) > \partial(L_w)$ in view of (V.32).

If $i > 1$, then from the equality

$$L_{\alpha(\Delta(\lambda_{i-1}))} B_{i-1} \doteq L_{\alpha(\lambda_i)} B_i,$$

granting that $\alpha(\lambda_i) = 1$, we obtain

$$\partial(B_i) > \partial(L_{\alpha(\Delta(\lambda_{i-1}))}).$$

Assume that $\alpha(\Delta(\lambda_{i-1})) < w$. Then, by (V.18), we have $\alpha(\Delta(\lambda_{i-1})) = 1$; hence $x_{\Delta(\lambda_{i-1})}$ is

x_ν, i.e. the equation Ω_3, hence also Ω, contains the loop $x_{\Delta(\nu)}$, x_ν, which contradicts condition N.3. Thus, (V.31) is proved.

From (V.28) we construct the boundary connection

$$l_p,\ \pi_1,\ x_u,\ \pi_2,\ l_q. \tag{V.33}$$

We will prove that it satisfies conditions (3.10)–(3.12).

To prove (3.10) it suffices to prove that $l_{\alpha(u)} < l_p$ in the case where π_1 is empty. But then $L_p \doteq B_1$, and the desired condition follows from (V.31), since, according to (V.6), $L_w \doteq L_{\alpha(u)}$.

To prove (3.11) it suffices to prove the two inequalities

$$l_{\alpha(u)} \leqslant l_{\alpha(\Delta(\lambda_{i-1}))}, \tag{V.34}$$

$$l_{\alpha(\lambda_{i+1})} \leqslant l_{\alpha(\Delta(u))}. \tag{V.35}$$

If (V.34) does not hold, then, since $\alpha(u) = w$ by (V.6), we have

$$l_{\alpha(\Delta(\lambda_{i-1}))} < l_w,$$

from which, in view of (V.18), we obtain $\alpha(\Delta(\lambda_{i-1})) = 1$, and therefore $x_{\lambda_{i-1}}$ is $x_{\Delta(\nu)}$, contrary to condition N.3. Inequality (V.35) is obtained from condition (3.11) for the pair $x_{\lambda_{i-1}}$, x_{λ_i} of the boundary connection (V.28) and from the inequality $l_{\alpha(\Delta(\nu))} < l_{\alpha(\Delta(u))}$, which follows from (V.7) and (V.9).

To prove (3.12) it suffices to prove that $l_{\alpha(\Delta(u))} < l_q$ for $k = i$. According to (V.31), we have $\partial(B_k) > \partial(L_w)$. Consequently, for a one-letter solution of Ω_3^0 we have

$$\partial(L_q) > \partial(L_{\alpha(\Delta(\nu))}) + \partial(L_w),$$

from which, according to (V.22), $\partial(L_{p+1}) < \partial(L_q)$, and, according to (V.7),

$$\partial(L_{\alpha(\Delta(u))}) < \partial(L_q).$$

Suppose the table of words (V.20) satisfies the boundary connection (V.28). Then there exist nonempty words (V.29) in the alphabet of Ω_3 such that the equalities (V.30) are satisfied.

As in the one-letter alphabet, we have (V.31). Consequently, (V.20) also satisfies the boundary connection (V.33). To see this it is necessary to construct a decomposition of X_u into words B_i' and C_i, where $B_i \doteq L_w B_i'$, and to preserve the old decompositions for the remaining words.

Suppose (V.20) satisfies the boundary connection (V.33). Then it also satisfies the boundary connection (V.28). To see this it suffices to augment the word B_i corresponding to the decomposition $X_u \doteq B_i C_i$ given for (V.33) on the left by the word L_w.

Suppose Ω_3 contains a boundary connection l_p, π_1, x_{λ_i}, π_2, l_q, where $\lambda_i = \Delta(\nu)$, x_ν is the carrier, and π_1 and π_2 are subpaths of the connection. If $i < k$, then, according to condition (3.11) for the pair x_{λ_i}, $x_{\lambda_{i+1}}$ we have $\alpha(\lambda_{i+1}) = 1$. It follows that $x_{\lambda_{i+1}}$ is the carrier x_ν, contrary to condition N.3. Therefore, $i = k$ and our boundary connection has the form

$$l_p,\ \pi_1,\ x_{\Delta(\nu)},\ l_q. \tag{V.36}$$

We construct the boundary connection

$$l_p,\ \pi_1,\ x_{\Delta(u)},\ l_q. \tag{V.37}$$

Obviously, it suffices to verify condition (3.12):

$$l_{\alpha(u)} < l_q < l_{\beta(u)}. \tag{V.38}$$

Since condition (3.12) for the connection (V.36) implies $l_q < l_{\beta(\nu)}$ and since, by (V.6), $\beta(\nu) = \beta(u)$, the right-hand inequality in (V.38) holds. By Lemma 5.7, $q \neq 1$. According to condition D.5 for the boundary connection (V.36), l_q is the right boundary of some base. If $q = w$, this base is a leading base, which is impossible. Therefore, by (V.18),

$$l_w < l_q, \tag{V.39}$$

and since, by (V.5), $\alpha(u) = w$, it follows that the left-hand inequality in (V.38) holds.

Using (V.39), it is easy to prove that (V.20) satisfies the connection (V.36) if and only if it satisfies the connection (V.37).

Thus, any solution of Ω_3 is a solution of Ω_4, and conversely. The principal parameters and lengths of solutions of Ω_3 and Ω_4 agree.

The transformation $\Omega_4 \to \Omega_1$ consists of the following. From the table of word variables delete x_ν, $x_{\Delta(\nu)}$, l_1 and r_1. From the equalities of duals delete the equality $x_\nu = x_{\Delta(\nu)}$. From the boundary equalities delete the equality $l_1 r_1 = t$. From the base situation table delete the situations of x_ν and $x_{\Delta(\nu)}$. From the boundary comparison table delete all inequalities $l_1 < l_i$ and replace the initial equality $l_1 = 1$ by the new initial equality $l_w = 1$.

The boundary connections are not affected by this transformation, since the bases x_ν, $x_{\Delta(\nu)}$ and the boundary l_1 do not occur in them. Condition D.5 is preserved, since the variables x_ν and x_u, as well as the variables $x_{\Delta(\nu)}$ and $x_{\Delta(u)}$, have the same right boundary.

It is easy to see that the system Ω_1, to within a renumbering of the variables, is a generalized equation.

Under the transformation $\Omega_4 \to \Omega_1$ the number of pairs of x-variables decreases by one and the number of initial boundaries decreases by one. Consequently, under the complete transformation $\Omega \to \Omega_1$ the first three principal parameters do not increase, i.e. Assertion 1 is proved for Type V.

Suppose Ω_4 has a solution (V.20) with index I and exponent of periodicity s. A solution $\mathfrak{D}_1$ of Ω_1 can be obtained from (V.20) by deleting the components X_ν, $X_{\Delta(\nu)}$, L_1 and R_1 and shortening the remaining components L_i and the component T on the left by the word L_w.

Suppose Ω has a solution (3.15). Then Ω_2 has a corresponding solution (V.19), Ω_3 and Ω_4 have a corresponding solution (V.20), and Ω_1 has a corresponding solution $\mathfrak{D}_1$. The index of solution $\mathfrak{D}_1$ of Ω_1 is less than the index of solution (3.15) of Ω, since the length of a solution under the passages $\Omega \to \Omega_2$, $\Omega_2 \to \Omega_3$ and $\Omega_3 \to \Omega_4$ does not change, and under the passage $\Omega_4 \to \Omega_1$ decreases. The exponent of periodicity does not increase under each transformation. Assertion 2 is proved for Type V.

Suppose Ω_1 has a solution $\mathfrak{D}_1$. Then, in view of (V.9), there exists a nonempty word H such that

$$L_{\rho+1} \stackrel{\text{gr}}{=} L_{\alpha(\Delta(\nu))} H.$$

We construct the components of a solution (V.20) of Ω_4 from the components of $\mathfrak{D}_1$ as follows:

$$X_\nu \rightleftharpoons H X_u,$$

$$X_{\Delta(\nu)} \rightleftharpoons H X_{\Delta(u)},$$

$$L_1 \rightleftharpoons 1,$$

$$R_1 \rightleftharpoons HT.$$

All components L_i and the component T of $\mathfrak{D}_1$ are augmented on the left by the word H. The remaining components of $\mathfrak{D}_1$ are unchanged.

It is easy to see that the constructed table of words has the form (V.20) and is a solution of Ω_4 and Ω_3. From the solution (V.20) of Ω_3 we construct a solution (V.19) of Ω_2. From the solution (V.19) of Ω_2 we construct a solution (3.15) of Ω. Assertion 3 is proved for Type V.

Type VI. *In Ω there is a carrier x_ν and a transfer base w_π such that $\beta(\pi) < \beta(\nu)$.*

The equation Ω contains a carrier x_ν with situation

$$l_1 x_\nu r_{\beta(\nu)} = t \tag{VI.1}$$

and a transfer base w_π with situation

$$l_1 w_\pi r_{\beta(\pi)} = t, \tag{VI.2}$$

where the boundary comparison table contains the inequality

$$l_{\beta(\pi)} < l_{\beta(\nu)}. \tag{VI.3}$$

The transformation $\Omega \to \Omega_1$ of an equation of Type VI consists of the following.

To the table of word variables add $l_{\rho+1}$ and $r_{\rho+1}$, to the table of boundary equalities add the equality $l_{\rho+1} r_{\rho+1} = t$, and to the boundary comparison table add the inequalities

$$l_{\alpha(\Delta(\nu))} < l_{\rho+1} < l_{\beta(\Delta(\nu))}. \tag{VI.4}$$

In the base situation table replace the situation (VI.2) of the base w_π by the situation

$$l_{\alpha(\Delta(\nu))} w_\pi r_{\rho+1} = t. \tag{VI.5}$$

Condition (3.7) is guaranteed by condition (VI.4).

The initial boundaries of Ω_1 are taken to be those of Ω. Condition D.3 is satisfied in Ω_1, since the left boundary of the new situation of w_π is the initial boundary $l_{\alpha(\Delta(\nu))}$.

We make the following changes in the boundary connections.

Add the new connection

$$l_{\beta(\pi)},\ x_\nu,\ l_{\rho+1}. \tag{VI.6}$$

In view of conditions (VI.1)–(VI.3) and inequality (3.7) for the situation (VI.2), we have

$$l_{\alpha(\nu)} < l_{\beta(\pi)} < l_{\beta(\nu)}.$$

These inequalities and condition (VI.4) guarantee conditions (3.10)–(3.12) for the new boundary connection (VI.6) in Ω_1. We transform each boundary connection

$$l_p,\ x_{\lambda_1},\ \ldots,\ x_{\lambda_k},\ l_q \tag{VI.7}$$

into a connection

$$l_p,\ x_{\mu_1},\ \ldots,\ x_{\mu_t},\ l_h \tag{VI.8}$$

as follows.

Replace the concluding boundary l_q by the pair x_ν, $l_{\rho+1}$ if $q = \beta(\pi)$, but leave it unchanged if $q \neq \beta(\pi)$.

Note that if $q = \beta(\pi)$, then $\lambda_k \neq \Delta(\pi)$; otherwise, by condition (3.12) for the connection (VI.7), we would have $l_{\beta(\pi)} < l_{\beta(\pi)}$.

Suppose w_π is an x-variable. Then in front of each occurrence of x_π in the path $x_{\lambda_1}, \ldots, x_{\lambda_k}$ not immediately preceded by $x_{\Delta(\pi)}$ insert the variable x_ν, and after each occurrence of $x_{\Delta(\pi)}$ not immediately followed by x_π insert the variable $x_{\Delta(\nu)}$.

We will prove that the resulting boundary connection (VI.8) satisfies conditions (3.10)–(3.12) in Ω_1.

If x_{λ_1} is not x_π, then condition (3.10) for (VI.8) agrees with condition (3.10) for (VI.7).

If x_{λ_1} is x_π, then x_{μ_1} is x_ν and condition (3.10) for (VI.8) follows from condition (3.10) for (VI.7) and the conditions $\alpha(\pi) = \alpha(\nu) = 1$ and $\beta(\pi) < \beta(\nu)$.

Condition (3.11) for a pair x_ν, x_π of (VI.8), where x_ν is an inserted variable, and condition (3.11) for a pair $x_{\Delta(\pi)}$, $x_{\Delta(\nu)}$ of (VI.8), where $x_{\Delta(\nu)}$ is an inserted variable, follow from $\alpha(\pi) = \alpha(\Delta(\nu))$, which in turn follows from (VI.5).

Condition (3.11) for a pair x_{μ_i}, x_ν of (VI.8), where x_ν is the variable inserted in front of x_π, follows from condition N.1, since, by (VI.1), $\alpha(\nu) = 1$.

Condition (3.11) for a pair $x_{\Delta(\nu)}$, x_u of (VI.8), where $x_{\Delta(\nu)}$ is an inserted variable, is satisfied, since in (VI.7), by condition (3.11) for the corresponding pair $x_{\Delta(\pi)}$, x_u, we have $l_{\alpha(u)} \leqslant l_{\alpha(\pi)}$ and, by (VI.2), $\alpha(\pi) = 1$.

Condition (3.11) for the pair $x_{\mu_{t-1}}$, x_ν of (VI.8), where x_ν is the variable inserted along with the boundary $l_{\rho+1}$, follows from condition N.1, since, by (VI.1), $\alpha(\nu) = 1$.

Condition (3.11) for the remaining pairs x_{μ_i}, $x_{\mu_{i+1}}$ of (VI.8) agrees with condition (3.11) for (VI.7).

If in (VI.7) we have $\lambda_k = \Delta(\pi)$ and $q \neq \beta(\pi)$, then, by condition (3.12) for this boundary connection, $l_{\alpha(\pi)} < l_q < l_{\beta(\pi)}$; hence condition (3.12) for (V.I.8) follows from $\alpha(\nu) = \alpha(\pi)$ and $\beta(\pi) < \beta(\nu)$.

If in (VI.7) we have $\lambda_k \neq \Delta(\pi)$ and $q = \beta(\pi)$, then condition (3.12) for (VI.8) follows from (VI.4).

In the remaining cases, condition (3.12) for (VI.8) agrees with condition (3.12) for (VI.7).

It is easy to see that conditions D.4 and D.5 are satisfied.

Thus, the system Ω_1 is a generalized equation.

Assertion 1 for Type VI is satisfied trivially, since the first three principal parameters of Ω_1 agree with the corresponding parameters of Ω.

Suppose Ω has a solution (3.15) with index I and exponent of periodicity s. Let W_π denote the component X_π of (3.15) if $1 \leqslant \pi \leqslant 2n$ and the coefficient $a_{\psi(\pi)}$ if $2n + 1 \leqslant \pi \leqslant 2n + m$. We construct a table of words

$$\begin{gathered} X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}, \\ L_1, \ldots, L_\rho, L_{\rho+1}, \quad R_1, \ldots, R_\rho, R_{\rho+1}, \quad T, \end{gathered} \tag{VI.9}$$

using the solution (3.15) and the equalities

$$\begin{gathered} L_{\rho+1} \overset{\circ}{=} L_{\alpha(\Delta(\nu))} W_\pi, \\ L_{\rho+1} R_{\rho+1} \overset{\circ}{=} T. \end{gathered} \tag{VI.10}$$

We will prove that (VI.9) is a solution of Ω_1.

The equalities of duals, the boundary equalities, the conditions of the boundary comparison table, and the base situation equalities of Ω_1 follow from the corresponding equalities and conditions of Ω, the equalities (VI.10), and the conditions (3.7) and (VI.3).

We will prove that (VI.9) satisfies all boundary connections of Ω_1.

According to (VI.1) and (VI.2), we have $L_{\alpha(\nu)} \doteq 1$ and $L_{\beta(\pi)} \doteq W_\pi$. Let B_1 denote the word W_π, and define the word C_1 by means of the equality $X_\nu \doteq B_1C_1$. In view of (3.7) and (VI.3), B_1 and C_1 are nonempty. Since, according to (VI.10), we have

$$L_{\beta(\pi)} \doteq L_{\alpha(\nu)}B_1,$$

$$L_{\alpha(\Delta(\nu))}B_1 \doteq L_{\rho+1},$$

it follows that (VI.9) satisfies the boundary connection (VI.6).

We will prove that if (3.15) satisfies the connection (VI.7), then (VI.9) satisfies the connection (VI.8).

By hypothesis, there exist nonempty words $B_1, \dots, B_k$ and $C_1, \dots, C_k$ such that

$$X_{\lambda_i} \doteq B_iC_i \quad (i = 1, \dots, k), \tag{VI.11}$$

$$L_p \doteq L_{\alpha(\lambda_1)}B_1, \tag{VI.12}$$

$$L_{\alpha(\Delta(\lambda_i))}B_i \doteq L_{\alpha(\lambda_{i+1})}B_{i+1} \quad (i = 1, \dots, k-1), \tag{VI.13}$$

$$L_{\alpha(\Delta(\lambda_k))}B_k \doteq L_q. \tag{VI.14}$$

We construct decompositions of $X_{\mu_1}, \dots, X_{\mu_t}$ into nonempty words $B'_1, \dots, B'_t$ and $C'_1, \dots, C'_t$ such that $X_{\mu_i} \doteq B'_iC'_i$ $(i = 1, \dots, t)$ as follows. (Recall that the list of words $X_{\mu_1}, \dots, X_{\mu_t}$ is obtained from the list $X_{\lambda_1}, \dots, X_{\lambda_k}$ by insertions of X_ν and $X_{\Delta(\nu)}$.)

For the words $X_{\lambda_1}, \dots, X_{\lambda_k}$ passed on into the sequence $X_{\mu_1}, \dots, X_{\mu_t}$ we preserve the old decompositions (VI.11). For the word X_ν inserted in front of X_π with decomposition $X_\pi \doteq BC$ we construct the decomposition $X_\nu \doteq BC^*$, where C^* is nonempty, since, in view of (VI.1)–(VI.3), $X_\nu \doteq X_\pi K$ for some K. For the word $X_{\Delta(\nu)}$ inserted after $X_{\Delta(\pi)}$ with decomposition $X_{\Delta(\pi)} \doteq BC$ we construct the decomposition $X_{\Delta(\nu)} \doteq BC^*$, where C^* is nonempty. For the word X_ν inserted at the end of the list $X_{\mu_1}, \dots, X_{\mu_t}$ (in the case $q = \beta(\pi)$) we construct the decomposition $X_\nu \doteq BC$, where $B \doteq X_\pi$ and C is nonempty by (VI.3).

We will prove that the constructed decomposition satisfies the boundary connection (VI.8), i.e. the following conditions are satisfied:

$$L_p \doteq L_{\alpha(\mu_1)}B'_1, \tag{VI.15}$$

$$L_{\alpha(\Delta(\mu_i))}B'_i \doteq L_{\alpha(\mu_{i+1})}B'_{i+1} \quad (i = 1, \dots, t-1), \tag{VI.16}$$

$$L_{\alpha(\Delta(\mu_t))}B'_t \doteq L_h. \tag{VI.17}$$

If X_{λ_1} is not X_π, (VI.15) agrees with (VI.12).

If X_{λ_1} is X_π with $X_{\lambda_1} \doteq B_1C_1$, then X_{μ_1} is X_ν with $X_{\mu_1} \doteq B_1C_1^*$, and (VI.15) follows from the equality $\alpha(\pi) = 1$ in Ω, the equality $\alpha(\nu) = 1$, and (VI.12).

Condition (VI.16) for the pair X_ν, X_π, where X_ν is the value of the inserted variable, and condition (VI.16) for the pair $X_{\Delta(\pi)}$, $X_{\Delta(\nu)}$, where $X_{\Delta(\nu)}$ is the value of the inserted variable, follow from the conditions on the decompositions of the words X_{μ_i} and the equality $\alpha(\Delta(\nu)) = \alpha(\pi)$ in Ω_1.

To prove (VI.16) for the pair X_u, X_ν, where X_ν is the value of the variable inserted in front of X_π, we must prove that

$$L_{\alpha(\Delta(u))}B'_i \doteq L_{\alpha(v)}B'_{i+1}.$$

In view of our decomposition and (VI.13), we have $X_\pi \doteq B'_{i+1}C^*$ and

$$L_{\alpha(\Delta(u))}B'_i \doteq L_{\alpha(\pi)}B'_{i+1}.$$

The desired equality follows from the equality $\alpha(\pi) = 1$ in Ω and the equality $\alpha(\nu) = 1$.

To prove (VI.16) for the pair $X_{\Delta(\nu)}, X_u$, where $X_{\Delta(\nu)}$ is the value of the inserted variable, we must prove that

$$L_{\alpha(v)}B'_i \doteq L_{\alpha(u)}B'_{i+1}$$

In view of our decomposition and (VI.13), we have $X_{\Delta(\pi)} \doteq B'_i C^*$ and

$$L_{\alpha(\pi)}B'_i \doteq L_{\alpha(u)}B'_{i+1}.$$

The desired equality follows from the equality $\alpha(\pi) = 1$ in Ω and the equality $\alpha(\nu) = 1$.

To prove (VI.16) for the pair $X_{\mu_{t-1}}, X_\nu$, where X_ν is the value of the variable inserted along with the boundary $l_{\rho+1}$, we must prove that

$$L_{\alpha(\Delta(\mu_{t-1}))}B'_{t-1} \doteq L_{\alpha(v)}B'_t.$$

In view of our decomposition, $B'_t \doteq X_\pi$, and, by (VI.1), $\alpha(\nu) = 1$. Hence the desired equality follows from (VI.14), the equality

$$L_{\alpha(\Delta(\mu_{t-1}))}B'_{t-1} \doteq L_{\beta(\pi)}$$

and the equality $\alpha(\pi) = 1$ in Ω.

In the remaining cases, (VI.16) agrees with (VI.13) for the corresponding pair.

To prove (VI.17) in the case when $\lambda_k = \Delta(\pi)$ and $q \neq \beta(\pi)$ we must prove that

$$L_{\alpha(v)}B'_t \doteq L_q.$$

In view of our decomposition, we have $B'_t \doteq B'_{t-1}$, $X_{\Delta(\pi)} \doteq B'_{t-1}C^*$, and, by (VI.14), $L_{\alpha(\pi)}B'_{t-1} \doteq L_q$. Consequently, the desired equality follows from the equality $\alpha(\pi) = 1$ in Ω and the equality $\alpha(\nu) = 1$.

To prove (VI.17) in the case when $\lambda_k \neq \Delta(\pi)$ and $q = \beta(\pi)$ we must prove that

$$L_{\alpha(\Delta(v))}B'_t \doteq L_{\rho+1}.$$

In view of our decomposition, in this case we have $B'_t \doteq X_\pi$; hence the desired equality follows from the first equality in (VI.10).

In the remaining cases, (VI.17) agrees with (VI.14) for the corresponding pair.

Thus, we have proved that (VI.9) is a solution of Ω_1. The exponent of periodicity of solution (3.15) of Ω and that solution (VI.9) of Ω_1 are obviously the same. The index of (VI.9) is smaller than the index of (3.15), since the fact that the carrier x_ν is not matched with its dual $x_{\Delta(\nu)}$ implies that in the vector of the base w_π of Ω_1 relative to solution (VI.9) the block of ones is shifted to the right in comparison with the vector of the base w_π of Ω relative to solution (3.15). Assertion 2 is proved for Type VI.

Suppose Ω_1 has a solution (VI.9). We will prove that the table of words

$$\begin{gathered} X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}, \\ L_1, \ldots, L_\rho, \quad R_1, \ldots, R_\rho, \quad T \end{gathered} \tag{VI.18}$$

is a solution of Ω.

The equalities of duals, the boundary equalities, and the conditions of the boundary comparison table of Ω follow from the corresponding equalities and conditions of Ω_1.

The base situation equalities of Ω, except for (VI.2), follow from the corresponding equalities of Ω_1.

Since (VI.9) satisfies the new boundary connection (VI.6), there exist nonempty words B_1 and C_1 such that

$$X_\nu \doteq B_1 C_1,$$

$$L_{\beta(\pi)} \doteq L_{\alpha(\nu)} B_1,$$

$$L_{\alpha(\Delta(\nu))} B_1 \doteq L_{\rho+1}.$$

According to (VI.5), we have

$$L_{\alpha(\Delta(\nu))} W_\pi R_{\rho+1} \doteq T;$$

hence $W_\pi \doteq B_1$ and, since $\alpha(\nu) = 1$, $L_{\beta(\pi)} \doteq W_\pi$. It follows from this equality that (VI.18) satisfies the base situation equality (VI.2) of w_π.

We will prove that (VI.18) satisfies all boundary connections of Ω.

Suppose some boundary connection of Ω has the form (VI.7). In Ω_1 there corresponds to it the connection (VI.8), and the table of words (VI.9) satisfies (VI.8).

Consequently, there exist nonempty words $B'_1, \ldots, B'_t$ and $C'_1, \ldots, C'_t$ such that

$$X_{\mu_i} \doteq B'_i C'_i \quad (i = 1, \ldots, t) \tag{VI.19}$$

and conditions (VI.15)–(VI.17) are satisfied.

We construct decompositions of $X_{\lambda_1}, \ldots, X_{\lambda_k}$ into nonempty words $B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ such that $X_{\lambda_i} \doteq B_i C_i$ $(i = 1, \ldots, k)$, preserving in these words the decompositions (VI.19).

We will prove that with these decompositions the table of words (VI.18) satisfies the boundary connection (VI.7), i.e. conditions (VI.12)–(VI.14) are satisfied.

If X_{λ_1} is not X_π, then condition (VI.12) agrees with (VI.15).

If X_{λ_1} is X_π, then, by (VI.15),

$$L_p \doteq L_{\alpha(\nu)} B'_1,$$

$$L_{\alpha(\Delta(\nu))} B'_1 \doteq L_{\alpha(\pi)} B'_2,$$

where $L_{\alpha(\nu)} \doteq 1$ and $L_{\alpha(\Delta(\nu))} \doteq L_{\alpha(\pi)}$. In view of our decomposition, $B'_2 \doteq B_1$, and in Ω we have $\alpha(\pi) = 1$. Consequently, $L_p \doteq L_{\alpha(\pi)} B_1$ and we have proved that in this case condition (VI.12) is satisfied.

We prove condition (VI.13) for the pair $X_{\lambda_{i-1}}, X_{\lambda_i}$, where $X_{\lambda_{i-1}}$ is not $X_{\Delta(\pi)}$, and X_{λ_i} is X_π. Then, for some j, X_{μ_j} is $X_{\lambda_{i-1}}$ and $X_{\mu_{j+2}}$ is X_{λ_i}, where $X_{\mu_{j+1}}$ is X_ν. According to (VI.16),

$$L_{\alpha(\Delta(\mu_j))} B'_j \doteq L_{\alpha(\nu)} B'_{j+1},$$

$$L_{\alpha(\Delta(\nu))} B'_{j+1} \doteq L_{\alpha(\pi)} B'_{j+2}.$$

In view of our decomposition $B'_j \doteq B_{i-1}$ and $B'_{j+2} \doteq B_{i+1}$. Since $\alpha(\nu) = 1$ and $\alpha(\Delta(\nu)) = \alpha(\pi)$ in Ω_1 and $\alpha(\pi) = 1$ in Ω, it follows that

$$L_{\alpha(\Delta(\lambda_{i-1}))} B_{i-1} \doteq L_{\alpha(\pi)} B_{i+1},$$

as required.

We prove condition (VI.13) for the pair $X_{\lambda_{i-1}}, X_{\lambda_i}$, where $X_{\lambda_{i-1}}$ is $X_{\Delta(\pi)}$, and X_{λ_i} is not

X_π. Then, for some j, X_{μ_j} is $X_{\lambda_{i-1}}$, $X_{\mu_{j+1}}$ is $X_{\Delta(\nu)}$, and $X_{\mu_{j+2}}$ is X_{λ_i}. According to (VI.16),

$$L_{\alpha(\pi)}B'_j \mathrel{\overset{\circ}{=}} L_{\alpha(\Delta(\nu))}B'_{j+1},$$

$$L_{\alpha(\nu)}B'_{j+1} \mathrel{\overset{\circ}{=}} L_{\alpha(\mu_{j+2})}B'_{j+2}.$$

In view of our decomposition, $B'_j \mathrel{\overset{\circ}{=}} B_{i-1}$ and $B'_{j+2} \mathrel{\overset{\circ}{=}} B_i$. Since $\alpha(\nu) = 1$ and $\alpha(\Delta(\nu)) = \alpha(\pi)$ in Ω_1 and $\alpha(\pi) = 1$ in Ω, it follows that

$$L_{\alpha(\Delta(\lambda_{i-1}))}B_{i-1} \mathrel{\overset{\circ}{=}} L_{\alpha(\lambda_i)}B_i,$$

as required.

Condition (VI.13) for the remaining pairs $X_{\lambda_{i-1}}$, X_{λ_i} agrees with (VI.16) for the corresponding pairs.

We prove condition (VI.14) for the case when $\lambda_k = \Delta(\pi)$ and $q \neq \beta(\pi)$. Then $X_{\mu_{t-1}}$ is $X_{\Delta(\pi)}$ and X_{μ_t} is $X_{\Delta(\nu)}$. According to (VI.16) and (VI.17), we have

$$L_{\alpha(\pi)}B'_{t-1} \mathrel{\overset{\circ}{=}} L_{\alpha(\Delta(\nu))}B'_t,$$

$$L_{\alpha(\nu)}B'_t \mathrel{\overset{\circ}{=}} L_q.$$

In view of our decomposition, $B'_{t-1} \mathrel{\overset{\circ}{=}} B_k$. Since $\alpha(\nu) = 1$ and $\alpha(\Delta(\nu)) = \alpha(\pi)$ in Ω_1 and $\alpha(\pi) = 1$ in Ω, it follows that $L_{\alpha(\pi)}B_k \mathrel{\overset{\circ}{=}} L_q$, as required.

We prove condition (VI.14) for the case when $\lambda_k \neq \Delta(\pi)$ and $q = \beta(\pi)$. Then $X_{\mu_{t-1}}$ is X_{λ_k} and X_{μ_t} is X_ν. According to (VI.16) and (VI.17), we have

$$L_{\alpha(\Delta(\mu_{t-1}))}B'_{t-1} \mathrel{\overset{\circ}{=}} L_{\alpha(\nu)}B'_t,$$

$$L_{\alpha(\Delta(\nu))}B'_t \mathrel{\overset{\circ}{=}} L_{\rho+1}.$$

In view of our decomposition, $B'_{t-1} \mathrel{\overset{\circ}{=}} B_k$. According to (VI.10), $B'_t \mathrel{\overset{\circ}{=}} W_\pi$. Since $\alpha(\pi) = 1$ in Ω and $\alpha(\nu) = 1$, it follows that

$$L_{\alpha(\Delta(\lambda_k))}B_k \mathrel{\overset{\circ}{=}} L_{\beta(\pi)}.$$

Condition (VI.14) for the remaining cases agrees with condition (VI.17).

Thus, the table of words (VI.18) is a solution of Ω. Assertion 3 is proved for Type VI.

TYPE VII. *In Ω there is a carrier x_ν and a transfer base w_π such that $\beta(\pi) = \beta(\nu)$.*

The equation Ω contains a carrier x_ν with situation

$$l_1 x_\nu r_{\beta(\nu)} = t \tag{VII.1}$$

and a transfer base w_π with situation

$$l_1 w_\pi r_{\beta(\pi)} = t, \tag{VII.2}$$

where

$$\beta(\pi) = \beta(\nu). \tag{VII.3}$$

The transformation $\Omega \to \Omega_1$ of an equation of Type VII consists of the following. In the base situation table replace the situation (VII.2) of the base w_π by the situation

$$l_{\alpha(\Delta(\nu))} w_\pi r_{\beta(\Delta(\nu))} = t. \tag{VII.4}$$

The initial boundaries of Ω_1 are taken to be those of Ω.

We transform each boundary connection

$$l_p,\ x_{\lambda_1},\ \ldots,\ x_{\lambda_k},\ l_q \tag{VII.5}$$

into a connection

$$l_p,\ x_{\mu_1},\ \ldots,\ x_{\mu_t},\ l_q \tag{VII.6}$$

as follows. If w_π is an x-variable, then in front of each occurrence of x_π in the path $x_{\lambda_1}, \ldots, x_{\lambda_k}$ not immediately preceded by $x_{\Delta(\pi)}$ we insert the variable x_ν, and after each occurrence of $x_{\Delta(\pi)}$ not immediately followed by x_π we insert the variable $x_{\Delta(\nu)}$.

Exactly as in Type VI, we prove that the resulting boundary connection satisfies conditions (3.10)–(3.12) in Ω_1.

It is easy to see that the system Ω_1 is a generalized equation and that Assertion 1 holds for Type VII.

Suppose Ω has a solution (3.15) with index I and exponent of periodicity s. The proof that this table of words is a solution of Ω_1 is the same as in Type VI. The index of solution (3.15) of Ω_1 is smaller than the index of solution (3.15) of Ω for the same reason as in Type VI. Assertion 2 is proved for Type VII.

Suppose Ω_1 has a solution (3.15). The proof that this table of words is a solution of Ω is the same as in Type VI. Assertion 3 is proved for Type VII.

§8. An algorithm recognizing the solvability of equations in a free semigroup

A generalized equation Ω with principal parameters n, m, τ and δ will be denoted by $\Omega(n, m, \tau, \delta)$.

Suppose n_0, m_0, τ_0 and s_0 are positive integers and $\delta_0 \rightleftharpoons 4n_0^3(n_0 + 1)(s_0 + 2)$. We define an algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$.

The algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ is applied to any nonempty list of normalized admissible generalized equations

$$\Omega_1(n_1, m_1, \tau_1, \delta_1),\ \ldots,\ \Omega_p(n_p, m_p, \tau_p, \delta_p), \tag{8.1}$$

in which no two equations are identical and where, for each $i = 1, \ldots, p$, we have the inequalities

$$\begin{aligned} n_i &\leqslant n_0, \\ m_i &\leqslant m_0, \\ \tau_i &\leqslant \tau_0, \\ \delta_i &\leqslant \delta_0. \end{aligned} \tag{8.2}$$

The result of applying $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ to the list (8.1) is the answer **yes**, the answer **no**, or a nonempty list of normalized admissible generalized equations

$$\Omega_1'(n_1', m_1', \tau_1', \delta_1'),\ \ldots,\ \Omega_r'(n_r', m_r', \tau_r', \delta_r'), \tag{8.3}$$

in which no two equations are identical and where, for each $i = 1, \ldots, r$, we have the inequalities

$$\begin{aligned} n_i' &\leqslant n_0, \\ m_i' &\leqslant m_0, \\ \tau_i' &\leqslant \tau_0, \\ \delta_i' &\leqslant \delta_0. \end{aligned} \tag{8.4}$$

The algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ consists of three steps.

Step 1. Using the algorithm of the Main Lemma (see §7), for each normalized admissible generalized equation $\Omega_i(n_i, m_i, \tau_i, \delta_i)$ in (8.1) construct a generalized equation $\Omega_i^{(1)}(n_i^{(1)}, m_i^{(1)}, \tau_i^{(1)}, \delta_i^{(1)})$ related to it by means of Assertions 1, 2 and 3. As a result, we obtain a list of generalized equations

$$\Omega_1^{(1)}(n_1^{(1)}, m_1^{(1)}, \tau_1^{(1)}, \delta_1^{(1)}), \ \dots, \ \Omega_p^{(1)}(n_p^{(1)}, m_p^{(1)}, \tau_p^{(1)}, \delta_p^{(1)}), \tag{8.5}$$

where, in view of Assertion 1 of the Main Lemma and the inequalities (8.2), for each $i = 1, \dots, p$ we have

$$\begin{aligned} n_i^{(1)} &\leqslant n_0, \\ m_i^{(1)} &\leqslant m_0, \\ \tau_i^{(1)} &\leqslant \tau_0. \end{aligned} \tag{8.6}$$

Using the algorithm of Lemma 5.4, determine whether each equation in (8.5) is true, false, or admissible. If at least one equation in (8.5) is true, then the algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ yields the answer **yes**. If every equation in (8.5) is false, then $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ yields the answer **no**. If the list (8.5) contains no true equations but does contain admissible equations, then the algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ produces all admissible equations in this list:

$$\Omega_1^{(1)}(n_1^{(1)}, m_1^{(1)}, \tau_1^{(1)}, \delta_1^{(1)}), \ \dots, \ \Omega_q^{(1)}(n_q^{(1)}, m_q^{(1)}, \tau_q^{(1)}, \delta_q^{(1)}). \tag{8.7}$$

(We have assumed here that all false equations in (8.5) occur at the end of the list.)

Step 2. Using the algorithm of the Normalization Lemma (see §6), we construct for each equation in (8.7) a list of generalized equations, each of which is either normalized admissible or false. Combine these lists. If in the combined list each equation is false, then the algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ yields the answer **no**. Otherwise, $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ produces all normalized admissible equations of the combined list in a new list

$$\Omega_1^{(2)}(n_1^{(2)}, m_1^{(2)}, \tau_1^{(2)}, \delta_1^{(2)}), \ \dots, \ \Omega_v^{(2)}(n_v^{(2)}, m_v^{(2)}, \tau_v^{(2)}, \delta_v^{(2)}), \tag{8.8}$$

where, in view of the Normalization Lemma and (8.6), for each $i = 1, \dots, v$ we have

$$\begin{aligned} n_i^{(2)} &\leqslant n_0, \\ m_i^{(2)} &\leqslant m_0, \\ \tau_i^{(2)} &\leqslant \tau_0. \end{aligned} \tag{8.9}$$

Step 3. From the list (8.8) delete each equation $\Omega_i^{(2)}(n_i^{(2)}, m_i^{(2)}, \tau_i^{(2)}, \delta_i^{(2)})$ for which $\delta_i^{(2)} > \delta_0$ (recall that $\delta_0 = 4n_0^3(n_0 + 1)(s_0 + 2)$). If in so doing we delete the entire list (8.8), then the algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ yields the answer **no**. Otherwise, $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ produces the remaining equations in a list of normalized admissible generalized equations

$$\Omega_1^{(2)}(n_1^{(2)}, m_1^{(2)}, \tau_1^{(2)}, \delta_1^{(2)}), \ \dots, \ \Omega_u^{(2)}(n_u^{(2)}, m_u^{(2)}, \tau_u^{(2)}, \delta_u^{(2)}), \tag{8.10}$$

where for each $i = 1, \dots, u$ we have the inequalities

$$\begin{aligned} n_i^{(2)} &\leqslant n_0, \\ m_i^{(2)} &\leqslant m_0, \\ \tau_i^{(2)} &\leqslant \tau_0, \\ \delta_i^{(2)} &\leqslant \delta_0. \end{aligned} \tag{8.11}$$

If the list (8.10) contains identical equations, retain exactly one representative from each class of identical equations. As a result, we obtain the desired list of equations (8.3) whose principal parameters satisfy (8.4).

Suppose s_0 is a positive integer. We say that a list of generalized equations $\Omega_1, \ldots, \Omega_r$ has *limiter* s_0 if either none of these equations has a solution or one of these equations has a solution with exponent of periodicity at most s_0.

LEMMA 8.1. *Suppose n_0, m_0, τ_0 and δ_0 are positive integers. Suppose also that a list of normalized admissible generalized equations* (8.1) *satisfying the estimates* (8.2) *for the principal parameters has limiter s_0. Apply to* (8.1) *the algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$. If this algorithm yields the answer* **yes**, *then some equation in* (8.1) *has a solution. If this algorithm yields the answer* **no**, *then no equation in* (8.1) *has a solution. If this algorithm produces the list* (8.3), *then* (8.3) *has limiter s_0 and, moreover: if some equation in* (8.1) *has a solution with index I, then some equation in* (8.3) *has a solution with index I', where $I' < I$; if some equation in* (8.3) *has a solution, then some equation in* (8.1) *has a solution.*

PROOF. If some equation in (8.5) has a solution, then, by Assertion 3 of the Main Lemma, some equation in (8.1) has a solution. Since the list (8.1) has limiter s_0, some equation in (8.1) has a solution with exponent of periodicity at most s_0. By Assertion 2 of the Main Lemma, some equation in (8.5) has a solution with exponent of periodicity at most s_0. Consequently, the list (8.5) has limiter s_0.

Furthermore, according to the Main Lemma: if some equation in (8.1) has a solution with index I, then some equation in (8.5) has a solution with index I_1, where $I_1 < I$; if some equation in (8.5) has a solution, then some equation in (8.1) has a solution.

Suppose the algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ yields the answer **yes** at the first step. Then some equation in (8.5) is true and therefore has a solution. By Assertion 3 of the Main Lemma, the corresponding equation in (8.1) has a solution.

Suppose $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ yields the answer **no** at the first step. Then each equation in (8.5) is false; hence (Lemma 5.3) none of them has a solution. By Assertion 2 of the Main Lemma, no equation in (8.1) has a solution.

Suppose $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ produces the list of admissible equations (8.7) at the first step. It is easy to see that (8.7) has limiter s_0 and is related to (8.1) via solutions as is the list (8.5).

Suppose $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ yields the answer **no** at the second step. Then, by Lemma 5.3, no equation in the combined list has a solution. By assertion 3) of the Normalization Lemma, no equation in (8.1) has a solution.

Suppose $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ produces the list (8.8) at the second step. By assertion 3) of the Normalization Lemma, (8.8) has limiter s_0 and is related to (8.1) via solutions as is the list (8.5).

According to Lemma 4.2, if for some equation $\Omega_i^{(2)}(n_i^{(2)}, m_i^{(2)}, \tau_i^{(2)}, \delta_i^{(2)})$ in (8.8) we have $\delta_i^{(2)} > \delta_0$, then the exponent of periodicity of any of its solutions is greater than s_0.

Suppose $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ yields the answer **no** at the third step. Then, since (8.8) has limiter s_0, no equation in this list has a solution. Consequently, no equation in (8.1) has a solution.

Suppose $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ produces the list (8.10) at the third step. If some equation in (8.10) has a solution, then some equation in (8.8) has a solution. Since (8.8) has limiter s_0, some equation in this list has a solution with exponent of periodicity at most s_0. Since (8.10) is obtained from (8.8) by deleting the equations whose solutions all have exponent of periodicity greater than s_0, some equation in (8.10) has a solution with exponent of periodicity at most s_0. Consequently, the list (8.10) has limiter s_0.

It is easy to see that (8.10) is related to (8.1) via solutions as is the list (8.5).

Since each equation in (8.10) is contained in (8.3), the lemma follows immediately from the assertions just proved.

We say that *a list of equations has a solution* if at least one equation in this list has a solution, and has no solution if no equation in this list has a solution.

LEMMA 8.2. *There exists an algorithm which, for any list of normalized admissible generalized equations and integer s which is a limiter of this list, recognizes whether or not this list has a solution.*

PROOF. Suppose

$$\Omega_1^{(1)}(n_1, m_1, \tau_1, \delta_1), \ \ldots, \ \Omega_{p_1}^{(1)}(n_{p_1}, m_{p_1}, \tau_{p_1}, \delta_{p_1}) \tag{8.12}$$

is a list of normalized admissible generalized equations with limiter s.

To determine if this list has a solution we choose positive integers n_0, m_0, τ_0 and s_0 such that $s \leqslant s_0$ and such that for each $i = 1, \ldots, p_1$ we have $n_i \leqslant n_0$, $m_i \leqslant n_0$, $\tau_i \leqslant \tau_0$ and $\delta_i \leqslant 4n_0^3(n_0 + 1)(s_0 + 2)$. Apply to (8.12) the algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$. By Lemma 8.1, this either establishes whether or not the list (8.12) has a solution or else produces a second list of equations, to which we can again apply $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$. Apply $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ to the second list. This either establishes whether or not the list (8.12) has a solution or else produces a third list of equations, and so on. If after $\lambda - 1$ steps we have not established whether or not the list (8.12) has a solution, we have produced a sequence of lists of normalized admissible generalized equations

$$\begin{array}{l} \Omega_1^{(1)}, \ \ldots, \ \Omega_{p_1}^{(1)}, \\ \Omega_1^{(2)}, \ \ldots, \ \Omega_{p_2}^{(2)}, \\ \ldots\ldots\ldots\ldots \\ \Omega_1^{(\lambda)}, \ \ldots, \ \Omega_{p_\lambda}^{(\lambda)}. \end{array} \tag{8.13}$$

Each of these lists is nonempty, and no list contains two identical equations. In each equation in each list the principal parameters do not exceed n_0, m_0, τ_0 and δ_0, where $\delta_0 = 4n_0^3(n_0 + 1)(s_0 + 2)$. By Lemma 5.8, the number of distinct equations in the table (8.13) is bounded by a number $F(n_0, m_0, \tau_0, \delta_0)$. Consequently, if $\lambda \geqslant 2^{F(n_0,m_0,\tau_0,\delta_0)}$, then (8.13) contains at least two identical lists.

If in (8.13) the ith list agrees with the jth, then obviously the $(i + 1)$th list agrees with the $(j + 1)$th, and, for any $q > 0$, the $(i + q)$th list agrees with the $(j + q)$th. Consequently, if (8.13) already contains two identical lists, then subsequent application of the algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ cannot yield the answer **yes** or the answer **no**.

On the other hand, if some equation in the first list has a solution with index I_1, then some equation in the second list has a solution with index I_2, where $I_2 < I_1$. Furthermore, some equation in the third list has a solution with index I_3, where $I_3 < I_2 < I_1$, and so on. Consequently, if some equation in the first list has a solution, then after a finite number of steps the algorithm $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ yields the answer **yes**. Therefore, if (8.13) contains two identical lists, then the list of equations $\Omega_1^{(1)}, \ldots, \Omega_{p_1}^{(1)}$ has no solution. Thus, on applying $\mathfrak{A}_{n_0,m_0,\tau_0,s_0}$ at most $2^{F(n_0,m_0,\tau_0,\delta_0)}$ times we can establish whether or not the list $\Omega_1^{(1)}, \ldots, \Omega_{p_1}^{(1)}$ has a solution.

LEMMA 8.3. *There exists an algorithm which, for any list of generalized equations without* *boundary connections and an integer s which is a limiter of this list, recognizes whether or not this list has a solution.*

PROOF. Suppose

$$\Omega_1, \ldots, \Omega_r \tag{8.14}$$

is a list of generalized equations without boundary connections which has limiter s. By Lemma 5.4, there exists an algorithm which, for any generalized equation, recognizes whether it is true, false, or admissible. If some equation in (8.14) is true, then it has a solution. If the equations in (8.14) are all false, then, by Lemma 5.3, they have no solution. If (8.14) contains no true equations but does contain admissible ones, we list all admissible equations:

$$\Omega_1, \ldots, \Omega_p. \tag{8.15}$$

(We have assumed here that all false equations in (8.14) occur at the end of the list.) The list (8.15) consists of equations without boundary connections, has the same limiter s as the list (8.14), and has a solution if and only if (8.14) has.

According to Lemma 6.1, for each equation in (8.15) we can construct a list of equations, each of which either is false or else is admissible and satisfies condition N.1. Combine these lists and delete all false equations from the combined list. If in so doing we delete the entire combined list, then, by Lemma 6.1, the list (8.15), hence also the list (8.14), has no solution. Otherwise, write down all admissible equations of the combined list:

$$\Omega_1', \ldots, \Omega_q'. \tag{8.16}$$

By Lemma 6.1, the path index of each equation in (8.16) does not exceed the largest path index among all equations in (8.15). Therefore, the equations in (8.16) have no boundary connections, and, since they satisfy condition N.1, they are all normalized. By Lemma 6.1, the list (8.16) has the same limiter s as the list (8.15) and has a solution if and only if (8.15) has. Thus, the desired algorithm reduces to that of Lemma 8.2.

THEOREM. *There exists an algorithm recognizing whether or not any equation in a free semigroup has a solution.*

PROOF. Suppose Σ is an arbitrary equation in a free semigroup. If Σ is coefficient-free, then it has a solution (for example, the solution whose components are all empty words).

Suppose Σ is a coefficient equation. In accordance with Lemma 3.1, construct a list of generalized equations

$$\Omega_1, \ldots, \Omega_r \tag{8.17}$$

without boundary connections such that Σ has a solution with exponent of periodicity s if and only if at least one equation in (8.17) has such a solution.

If an equation Ω_i in (8.17) has a solution, so does Σ. According to Lemma 1.3, if Σ has a solution, then it has a solution with exponent of periodicity at most

$$s_1 = (6d)^{2^{(2d^4)}} + 2,$$

where d is the notational length of Σ. Then some equation Ω_j in (8.17) has a solution with exponent of periodicity at most s_1.

Thus, we have proved that the list (8.17) has limiter s_1. Consequently, the desired algorithm reduces to that of Lemma 8.3.

In conclusion, the author would like to express his sincere gratitude to S. I. Adjan for many useful remarks concerning the exposition of this paper, and also to Ju. I. Hmelevskiĭ for his participation in the discussions.

Received 13/FEB/77

BIBLIOGRAPHY

1. Ju. I. Hmelevskiĭ, *Equations in free semigroups*, Trudy Mat. Inst. Steklov. **107** (1971); English transl., Proc. Steklov Inst. Math. **107** (1971) (1976).

2. V. K. Bulitko, *Equations and inequalities in a free group and a free semigroup*, Tul. Gos. Ped. Inst. Učen. Zap. Mat. Kafedr Vyp. 2 Geometr. i Algebra (1970), 242–252. (Russian) MR **52** #14045.

3. S. I. Adjan, *The Burnside problem and identities in groups*, "Nauka", Moscow, 1975. (Russian)

4. Ju. L. Eršov et al., *Elementary theories*, Uspehi Mat. Nauk **20** (1965), no. 4 (124), 37–108; English transl. in Russian Math. Surveys **20** (1965).

Translated by G. A. KANDALL