Can Nondeterminism Help Complementation?

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Abstract

Complementation and determinization are two fundamental notions in automata theory. The close relationship between the two has been well observed in the literature. In the case of nondeterministic finite automata on finite words (NFA), complementation and determinization have the same state complexity, namely $\Theta(2^n)$ where n is the state size. The same similarity between determinization and complementation was found for Büchi automata, where both operations were shown to have $2^{\Theta(n \lg n)}$ state complexity. An intriguing question is whether there exists a type of ω-automata whose determinization is considerably harder than its complementation. In this paper, we show that for all common type ω-automata, the determinization problem has the same state complexity as the corresponding complementation problem at the granularity of $2^{\Theta(\cdot)}$.

1 Introduction

Automata on infinite words (ω-automata) have wide applications in synthesis and verification of reactive concurrent systems. Complementation and determinization are two fundamental notions in automata theory. The complementation problem is to construct, given an ω-automaton $\mathcal{A}$, an ω-automaton $\mathcal{B}$ that recognizes the complementary language of $\mathcal{A}$. Provided that a given $\mathcal{A}$ is nondeterministic, the determinization problem is to construct a deterministic ω-automaton $\mathcal{B}$ that recognizes the same language as $\mathcal{A}$ does.

The close relationship between determinization and complementation has been well observed in the literature (see [Cho74, Var07] for discussions). A deterministic ω-automaton can be trivially complemented by dualizing its acceptance condition. As a result, a lower bound on complementation applies to determinization while an upper bound on determinization applies to complementation. Besides easily rendering complementation, determinization is crucial in decision problems for tree temporal logics, logic games and system synthesis. For example, using game-theoretical semantics [GH82], complementation of ω-tree automata (ω-tree complementation) not only requires complementation of ω-automata, but also requires the complementary ω-automata be deterministic (called co-determinization). Therefore, lowering the cost of ω-determinization also improves the performance of ω-tree complementation.

In the case of nondeterministic finite automata on finite words (NFA), complementation and determinization have the same state complexity, namely $\Theta(2^n)$ where n is the state size. The same similarity between determinization and complementation was found for Büchi automata, where both operations were shown to have $2^{\Theta(n \lg n)}$ state complexity. An intriguing question is whether there exists a type of ω-automata whose determinization is considerably harder than its complementation. In this paper, we show that for all common type ω-automata, the determinization problem has the same state complexity as the corresponding complementation problem at the granularity of $2^{\Theta(\cdot)}$ (see Figure 1).

Related Work. Büchi started the theory of ω-automata. The original ω-automata used to establish the decidability of S1S [Büc66] are now referred to as Büchi automata. Shortly after Büchi's work, McNaughton proved a fundamental theorem in the theory of ω-automata, that is, ω-regular languages (the languages that are recognized nondeterministic Büchi automata) are exactly those recognizable by the deterministic version of a type of ω-automata, now referred to as Rabin automata [McN66].

The complexity of McNaughton's Büchi determinization is double exponential. In 1988, Safra proposed a type of tree structures (now referred to as Safra trees) to obtain a Büchi determinization that converts a nondeterministic Büchi automaton of state size n to an equivalent deterministic Rabin automaton of state size $2^{O(n \lg n)}$ and index size $O(n)$ [Saf88]. Safra's construction is essentially optimal as the lower bound on state size for Büchi complementation is $2^{\Omega(n \lg n)}$ [Mic88, Löd99]. Later, Safra generalized the Büchi determinization to a Streett determinization which, given a nondeterministic Streett automaton of state size n and index size k, produces an equivalent deterministic Rabin automaton of state size $2^{O(nk \lg nk)}$ and index size $O(nk)$ [Saf92]. A variant Büchi determinization using a similar tree structure was proposed by Muller and Schupp [MS95]. Recently, Piterman improved both of Safra's determinization procedures with a node renaming scheme [Pit06]. Piterman's constructions are more efficient than Safra's, though the asymptotical bounds in terms of $2^{O(\cdot)}$ are the same. A big advantage of Piterman's constructions, however, is to output deterministic parity automata, which is easier to manipulate than deterministic Rabin automata. For example, there exists no efficient procedure to complement deterministic Rabin automata to Büchi automata [SV89], while such complementation is straightforward and efficient for deterministic parity automata.

In [CZL09, CZ11a, CZ11b] we established tight bounds for the complementation of ω-automata with rich acceptance conditions, namely Rabin, Streett and parity automata (see Figure 1). The state complexities of the corresponding determinization problems, however, are yet to be settled. In particular, a large gap exists between the lower and upper bounds for Streett determinization. A Streett automaton can be viewed as a Büchi automaton $\langle Q, \Sigma, Q_0, \Delta, \mathcal{F} \rangle$ except that the acceptance condition $\mathcal{F} = \langle G, B \rangle_I$, where $I = [1..k]$ for some k and $G, B : I \to 2^Q$, comprises k pairs of *enabling sets* $G(i)$ and *fulfilling sets* $B(i)$. A run is accepting if for every $i \in I$, if the run visits $G(i)$ infinitely often, then so does it to $B(i)$. Therefore, Streett automata naturally express *strong fairness* conditions that characterize meaningful computations [FK84, Fra86]. Moreover, Streett automata are exponentially more succinct than Büchi automata in encoding infinite behaviors of systems [SV89]. As a results, Streett automata have an advantage in modeling the behaviors of concurrent and reactive systems. For Streett determinization, the gap between current lower and upper bounds is huge: the lower bound is $2^{\Omega(n^2 \lg n)}$ [CZ11a] and the upper bound is $2^{O(nk \lg nk)}$ [Saf92, Pit06] when k is large (say $k = \omega(n)$).

In this paper, we show a Streett determinization whose state complexity matches the one for Streett complementation. More precisely, our construction has state complexity $2^{O(n \lg n + nk \lg k)}$ for $k = O(n)$ and $2^{O(n^2 \lg n)}$ for $k = \omega(n)$. We note that this improvement is not only meant for large k; when $k = O(\log n)$, the difference between $2^{O(n \lg n + nk \lg k)}$ and $2^{O(nk \lg nk)}$ is already substantial. The phenomenon that determinization and complementation have the same state complexity does not stop at Streett automata; we also show that this phenomenon holds for generalized Büchi automata, parity automata and Rabin automata. This raises a very interesting question: do determinization and complementation always "walk hand in hand"? Although the exact complexities

Type	Bound		Lower	Upper
Büchi	$2^{\Theta(n \lg n)}$		[Mic88]	[Saf88]
Generalized Büchi	$2^{\Theta(n \lg nk)}$	$k = O(2^n)$	[Yan06]	[KV05]
Streett	$2^{\Theta(n \lg n + nk \lg k)}$ $2^{\Theta(n^2 \lg n)}$	$k = O(n)$ $k = \omega(n)$	[CZ11a]	[CZ11b]
Rabin	$2^{\Theta(nk \lg n)}$	$k = O(2^n)$	[CZL09]	[KV05]
Parity	$2^{\Theta(n \lg n)}$	$k = O(n)$	[Mic88]	[CZ11b]

Figure 1: Complementation and determinization complexities for ω-automata of common types. The listed citations are meant for complementation only.

of complementation and determinization for some or all types of ω-automata could be different[1], the "coincidence" at the granularity of $2^{\Theta(\cdot)}$ is already intriguing.

Our Approaches. Our improved construction bases on two ideas. The first one is what we have exploited in obtaining tight upper bounds for Streett complementation [CZ11b], namely, the larger the size of Streett index size, the higher correlations in the runs of Streett automata. We used two tree structures: *ITS* (*Increasing Tree of Sets*) and *TOP* (*Tree of Ordered Partitions*) to characterize those correlations. We observed that there is a similarity between *TOP* and Safra trees for Büchi determinization [Saf88]. As Safra trees for Streett determinization are generalization of those for Büchi determinization, we conjectured that *ITS* should have a role in improving Streett determinization. Our study confirmed this expectation; using *ITS* we can significantly reduce the size of Safra trees for Streett determinization.

The second idea is a new naming scheme. Bounding the size of Safra trees alone cannot bring down the state complexity when the Streett index is small (i.e., $k = O(n)$), because the naming cost becomes a dominating factor in this case. Naming is an integral part of Safra trees. Every node in a Safra tree is associated with a name, which is used to track changes of the node between the tree (state) and its successors. The current name allocation is a retail-style strategy; when a new node is created, a name from the pool of unused names is selected arbitrarily and assigned, and when a node is removed, its name is recycled to the pool. In contrast, our naming scheme is more like wholesale; the name space is divided into even blocks and every block is allocated at the same time. When a branch is created, an unused block is assigned to it, and when a branch is changed, the corresponding block is recycled.

Paper Organization. Section 2 presents notations and basic terminology in automata theory. Section 3 introduces Safra's construction for Büchi and Streett determinization. Section 4 presents our construction for Streett determinization. Section 5 establishes tight upper bounds for the determinization of Streett, generalized Büchi, parity, and Rabin automata. Section 6 concludes with some discussion on future work. All proofs are omitted from the main text, but they can be found in the appendix.

[1] Recent work by Colcombet and Zdanowski, and by Schewe showed that the state complexity of determinization of Büchi automata on alphabets of *unbounded* size is between $\Omega((1.64n)^n)$ [CZ09] and $O((1.65n)^n)$ [Sch09b], which is strictly higher than the state complexity of Büchi complementation which is between $\Omega(L(n))$ [Yan06] and $O(n^2 L(n))$ [Sch09a] (where $L(n) \approx (0.76n)^n$)). However, Schewe's determinization construction produces Rabin automata with exponentially large index size, and it is not known weather Colcombet and Zdanowski's lower bound result can be generalized to Büchi automata on conventional alphabets of fixed size.

2 Preliminaries

Basic Notations. Let $\mathbb{N}$ denote the set of natural numbers. We write $[i..j]$ for $\{k \in \mathbb{N} \mid i \leq k \leq j\}$, $[i..j)$ for $[i..j-1]$, and $[n]$ for $[0..n)$. For an infinite sequence ϱ, we use $\varrho(i)$ to denote the i-th component for $i \in \mathbb{N}$. For a finite sequence α, we use $|\alpha|$ to denote the length of α, $\alpha[i]$ ($i \in [1..|\alpha|]$) to denote the object at the i-th position, and $\alpha[i..j]$ (resp. $\alpha[1..j)$) to denote the subsequence of α from position i to position j (resp. $j-1$). We reserve n and k as parameters of a determinization instance (n for state size and k for index size), and define $\mu = \min(n,k)$ and $I = [1..k]$.

Automata and Runs. A finite automaton on infinite words (ω-automaton) is a tuple $\mathcal{A} = (\Sigma, Q, Q_0, \Delta, \mathcal{F})$ where Σ is an alphabet, Q is a finite set of states, $Q_0 \subseteq Q$ is a set of initial states, $\Delta \subseteq Q \times \Sigma \times Q$ is a set of transition relations, and $\mathcal{F}$ is an acceptance condition.

A *finite run* of $\mathcal{A}$ from state q to state q' over a finite word w is a sequence of states $\varrho = \varrho(0)\varrho(1)\cdots\varrho(|w|)$ such that $\varrho(0) = q$, $\varrho(|w|) = q'$ and $\langle \varrho(i), w(i), \varrho(i+1) \rangle \in \Delta$ for all $i \in [|w|]$. An infinite word (ω-words) over Σ is an infinite sequence of letters in Σ. A *run* ϱ of $\mathcal{A}$ over an ω-word w is an infinite sequence of states in Q such that $\varrho(0) \in Q_0$ and, $\langle \varrho(i), w(i), \varrho(i+1) \rangle \in \Delta$ for $i \in \mathbb{N}$. Let $\mathit{Inf}(\varrho)$ be the set of states that occur infinitely many times in ϱ. An automaton accepts w if ϱ satisfies $\mathcal{F}$, which usually is defined as a predicate on $\mathit{Inf}(\varrho)$. By $\mathscr{L}(\mathcal{A})$ we denote the set of ω-words accepted by $\mathcal{A}$.

Acceptance Conditions and Types. ω-automata are classified according their acceptance conditions. Below we list automata of common types. Let G, B be two functions $I \to 2^Q$.

- *Generalized Büchi*: $\langle B \rangle_I$: $\forall i \in I, \mathit{Inf}(\varrho) \cap B(i) \neq \emptyset$.
- *Büchi*: $\langle B \rangle_I$ with $I = \{1\}$ (i.e., $k = 1$).
- *Streett*, $\langle G, B \rangle_I$: $\forall i \in I$, $\mathit{Inf}(\varrho) \cap G(i) \neq \emptyset \to \mathit{Inf}(\varrho) \cap B(i) \neq \emptyset$.
- *Parity*, $\langle G, B \rangle_I$ with $B(1) \subset G(1) \subset \cdots \subset B(k) \subset G(k)$.
- *Rabin*, $[G, B]_I$: $\exists i \in I$, $\mathit{Inf}(\varrho) \cap G(i) \neq \emptyset \wedge \mathit{Inf}(\varrho) \cap B(i) = \emptyset$.

For simplicity, we denote a Büchi condition by F (i.e., $F = B(1)$) and call it the final set of $\mathcal{A}$. Note that Streett and Rabin conditions are dual to each other, and Büchi, generalized Büchi and parity automata are all subclasses of Streett automata. For a Streett condition $\langle G, B \rangle_I$, if there exist $i, i' \in I$, $B(i) = B(i')$, then we can simplify the condition by replacing both $\langle G(i), B(i) \rangle$ and $\langle G(i'), B(i') \rangle$ by $\langle G(i) \cup G(i'), B(i) \rangle$. For this reason, for every Streett condition $\langle G, B \rangle_I$, we assume that B is injective and $k = |I| \leq 2^n$.

Trees. A tree is a set $V \subseteq \mathbb{N}^*$ such that if $v \cdot i \in V$ then $v \in V$ and $v \cdot j \in V$ for $j \leq i$. In this paper we only consider finite trees. Elements in V are called *nodes* and ϵ is called the *root*. Nodes $v \cdot i$ are *children* of v and they are *siblings* to one another. A node $v \cdot i$ is said to be *older* then $v \cdot j$ if $i < j$. The set of v's children is denoted by $ch(v)$. A node is a leaf if it has no children. Given an alphabet Σ, a Σ-*labeled tree* is a pair $\langle V, L \rangle$, where V is a tree and $L : V \to \Sigma$ assigns a letter to each node in V. We refer to $v \in V$ as V-*value* (*structural value*) and $L(v)$ as L-*value* (*label value*) of v. An *ordered* and labeled tree is a tuple $\langle V, L, \prec \rangle$ where $\langle V, L \rangle$ is a labeled tree and $\prec$ is a partial order on V.

3 Safra's Determinization

In this section we introduce Safra's constructions for Büchi and Streett determinization [Saf88, Saf92].

By the nature of determinization, it is not surprising that all constructions rely on certain types of subset construction. For Büchi automata, this subset construction takes a particular form. Let $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, F \rangle$ be a nondeterministic Büchi automaton and $\mathcal{B}$ a purported deterministic ω-automaton such that $\mathscr{L}(\mathcal{A}) = \mathscr{L}(\mathcal{B})$. It is not hard to build $\mathcal{B}$ such that if a run $\varrho = q_0, q_1, \ldots \in Q^\omega$ of $\mathcal{A}$ over an infinite word w is accepting, then so is the run $\tilde{\varrho} = \tilde{q}_0, \tilde{q}_1, \ldots \in (2^Q)^\omega$ of $\mathcal{B}$ over w, obtained by the standard subset construction [RS59], and $q_i \in \tilde{q}_i$ for every $i \geq 0$. This gives us $\mathscr{L}(\mathcal{A}) \subseteq \mathscr{L}(\mathcal{B})$. The harder part is to also guarantee $\mathscr{L}(\mathcal{B}) \subseteq \mathscr{L}(\mathcal{A})$, that is, an accepting run $\tilde{\varrho}$ of $\mathcal{B}$ over w induces an accepting run ϱ of $\mathcal{A}$ over w. Let $S \overset{F}{\hookrightarrow} S'$ denote that for every state $q' \in S'$, there exists a state $q \in S$ and a finite run ρ of $\mathcal{A}$ such that ρ goes from q to q' and visits F. The key idea in many determinization constructions [McN66, Saf88, Saf92, MS95, Pit06] relies on the following lemma, which itself is a consequence of König's lemma.

Lemma 1 ([McN66, Saf88]). *Let $\tilde{\varrho} = \tilde{q}_0, \tilde{q}_1, \ldots \in (2^Q)^\omega$ and $(S_i)_\omega = S_0, S_1, \ldots$ an infinite subsequence of $\tilde{\varrho}$ such that for every $i \geq 0$, $S_i \overset{F}{\hookrightarrow} S_{i+1}$, then there is an accepting run $\varrho = q_0, q_1, \ldots \in Q^\omega$ of $\mathcal{A}$ such that $q_i \in \tilde{q}_i$ for every $i \geq 0$.*

Safra proposed an essentially optimal construction for Büchi determinization [Saf88]. The ingenuity of his construction is to efficiently organize subsets of states using a type of tree structures, now referred to as Safra trees. In a Safra tree, each node is equipped with a set label, a subset of Q. We simply say that a node *contains* the states in its set label. The standard subset construction is carried out on all nodes in parallel, and each node v in the tree gives birth to a new child which contains the states both in F and in v. We say that a node v turns *green* if every state in v appears in a child of v, and vice versa. We have $S \overset{F}{\hookrightarrow} S'$ where S and S' are the set labels of v at two consecutive moments of v being green. If v turns green infinitely often, then we have an desired $(S_i)_\omega$ as stated in Lemma 1.

A key ingredient in Safra's construction is to assign names and colors to nodes, in order to track changes on nodes to identify those that turn green infinitely many times. For the sake of presentation clarity, we separate the naming and coloring convention from the core of the construction, the transition function (or the tree transformation rule). In this paper, use three colors: *green*, *red* and *yellow*.

Procedure 1 (Naming and Coloring Convention). *The following convention of naming and coloring is applied in the transition function (see Procedure 2).*

1. *Newly created nodes are marked red.*
2. *Nodes whose all descendants are removed are marked green.*
3. *Nodes are marked yellow unless they have been marked red or green.*
4. *Nodes with the empty set label are removed.*
5. *When a node is created, a name from the pool of unused name is selected and assigned to the node; when a node is removed, its name is recycled back to the pool.*

Procedure 2 (Safra Determinization Scheme). *Given $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, \mathcal{F} \rangle$ as a nondeterministic automaton, this procedure outputs a deterministic Rabin automaton $\mathcal{D} = \langle \tilde{Q}, \tilde{q}_0, \Sigma, \tilde{\Delta}, [\tilde{G}, \tilde{B}]_{\tilde{I}} \rangle$ such that*

2.1 $\tilde{Q}$ *is the set of Safra trees.*

2.2 $\tilde{q}_0 \in \tilde{Q}$ *is the tree with just the root.*

2.3 $\tilde{q}' = \tilde{\Delta}(\tilde{q})$ *is defined as follows.*

2.3.1 Expansion*: Spawn new children.*

2.3.2 Subset Construction*. Update the set label of each node using the subset construction.*

2.3.3 Horizontal Merge*. Make sure that set labels of any two siblings are disjoint.*

2.3.4 Vertical Merge*. Remove all descendants of a node if the node turns green.*

2.4 $[\tilde{G}, \tilde{B}]_{\tilde{I}}$ *is such that for every* $i \in \tilde{I}$

$$\tilde{G}(i) = \{\tilde{q} \in \tilde{Q} \mid \tilde{q} \text{ contains a red node with name } i \text{ or does not contains a node with name } i\}$$
$$\tilde{B}(i) = \{\tilde{q} \in \tilde{Q} \mid \tilde{q} \text{ contains a green node with name } i\}$$

From now on, we concentrate on defining transition functions; this amounts to customizing Step (2.3) for Büchi and Streett automata, respectively.

Büchi Determinization.

Definition 1 (Safra Trees for Büchi Determinization (STB))**.** *A* Safra tree for Büchi determinization *(STB) is an ordered and labeled tree* $\langle V, L, \prec_{str}\rangle$ *with* $L = \langle L_n, L_s, L_c\rangle$ *where*

1.1 $L_n : V \to [1..n]$ *assigns each node a unique name.*

1.2 $L_s : V \to 2^Q$ *assigns each node a subset of* Q *such that for every node* v, $\bigcup_{v' \in ch(v)} L_s(v') \subset L_s(v)$ *and* $L_s(v') \cap L_s(v'') = \emptyset$ *for every two distinct* $v', v'' \in ch(v)$.

1.3 $L_c : V \to \{green, red, yellow\}$ *assigns each node a color.*

1.4 $\prec_{str}: V \times V$ *linearly orders siblings such that* $v \prec_{str} v'$ *iff* v *is to the left of* v'.

We note that there are several subtleties in this definition. First, nodes are identified with their names instead of their V-values (structural values); that is exactly the purpose of naming. When we say that a node v in a tree t and its successor t' (i.e., $t' = \tilde{\Delta}(t)$), we actually mean that two nodes with the same name, one in t and the other in t', and their corresponding V-values do not matter. But for simplicity, we keep using V-values to refer to nodes. We shall address an exception, however, in our construction for Streett determinization where due to renaming, the "same" node v can have different names in t and t' (Section 4). Second, we require that newly created nodes be added to the *right* of all existing siblings. That is, $v \prec_{str} v'$ iff v is *older than* v'. Formally, this would require defining the structural ordering $\prec_{str}$ between nodes in a tree t and its successor t'. But we avoid this complication as the meaning is clear from the context. Third, $\prec_{str}$ is in fact of no importance in Safra's original construction; it is totally unnecessary to require that *younger* siblings be put to the right of *older* ones. What matters is that this *older-than* relation determines the priority of state ownership between sibling nodes at the stage of *horizontal merge* (Step (2.3.3)). But we shall explain in Section 4 why $\prec_{str}$ is needed to get the desired complexity result for our construction. We choose to introduce the relation here for a uniform presentation of all determinization constructions in this paper.

Procedure 3 (Büchi Determinization [Saf88]). *Given a nondeterministic Büchi automaton* $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, F\rangle$*, this procedure outputs a deterministic Rabin automaton* $\mathcal{D} = \langle \tilde{Q}, \tilde{q}_0, \Sigma, \tilde{\Delta}, [\tilde{G}, \tilde{B}]_{\tilde{I}}\rangle$*, where* $\tilde{Q}$ *is the set of STB,* $\tilde{I} = [1..n]$*,* $\tilde{q}_0$ *and* $[\tilde{G}, \tilde{B}]_{\tilde{I}}$ *are as defined in Procedure 2 (for the root* v *in* $\tilde{q}_0$ *we have* $L_n(v) = 1$*,* $L_s(v) = Q_0$*, and* $L_c = red$*), and* $\tilde{\Delta}$ *is defined such that* $\tilde{q}' = \tilde{\Delta}(\tilde{q})$ *if and only if* $\tilde{q}'$ *is the STB obtained by applying the following transformation rule to* $\tilde{q}$*.*

3.1 Expansion. *For each node* v *in* $\tilde{q}$*, add a new child* v' *with label* $L_s(v) \cap F$*.*

3.2 Subset Construction. *For each node* v *in* $\tilde{Q}$*, update set label* $L_s(v)$ *to* $\Delta(L_s(v))$*.*

3.3 Horizontal Merge. *For any state* q *and any two siblings* v *and* v' *such that* $q \in L_s(v) \cap L_s(v')$*, if* $v \prec_{str} v'$*, then remove* q *from* v' *and all its descendants.*

3.4 Vertical Merge. *For every node* v*, if* $L_s(v) = \cup_{v' \in ch(v)} L_s(v')$*, then remove all descendants of* v*.*

Streett Determinization. Safra generalized the construction to Streett determinization [Saf92]. Let $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, \langle G, B\rangle_I\rangle$ be a Streett automaton. A $J \subseteq I$ serves as a *witness set* for an accepting run ϱ of $\mathcal{A}$ in the following sense: for every j in J, ϱ visits $B(j)$ infinitely often while for every $j \in I \setminus J$, ϱ visits $G(j)$ only finitely many times. We call indices in $I \setminus J$ *negative obligations* and indices in J *positive obligations*. It is easily seen that a run ϱ is accepting if and only if ϱ admits a witness set $J \subseteq I$. We say that a finite run ρ *fulfills* J if for every j in J, ρ visits $B(j)$, but for every $j \in I \setminus J$, ρ does not visit $G(j)$. By $S \overset{J}{\hookrightarrow} S'$ we mean that every state in S' is reachable from a state in S via a finite run ρ that fulfills J.

Lemma 2 ([Saf92]). *Let* $\tilde{\varrho} = \tilde{q}_0, \tilde{q}_1, \ldots \in (2^Q)^\omega$ *and* $(S_i)_\omega = S_0, S_1, \ldots$ *an infinite subsequence of* $\tilde{\varrho}$ *such that for every* $i \geq 0$*,* $S_i \overset{J}{\hookrightarrow} S_{i+1}$*, then there is an accepting run* $\varrho = q_0, q_1, \ldots \in Q^\omega$ *of* $\mathcal{A}$ *such that* $q_i \in \tilde{q}_i$ *for every* $i \geq 0$*.*

However, there are 2^k potential witness sets and a naive implementation would give a double exponential construction. Safra's idea is to organize both witness sets and state sets in a tree where each node is labeled by a witness set as well as by a state set. We illustrate the idea using an example. Let $k = 3$ and $I = [1..3]$.

The initial state in the deterministic automaton $\mathcal{B}$ is a root v labeled with I and Q_0. The obligation of v is to detect runs that fulfills I, i.e., visit $B(1)$, $B(2)$, and $B(3)$ infinitely often. Once a run visits $B(3)$ (fulfilling a positive obligation), the run moves to a new child, waiting to visit $B(2)$, and once the run visits $B(2)$, the run moves to anther new child, waiting to visit $B(1)$, and so on. Technically speaking, states are moved around, which induces moving of runs.

This *sequential sweeping* can be stalled because some runs in v, from some point on, could never visit $B(3)$. So v should spawn a child v_3 with the witness set $I \setminus \{3\} = \{1, 2\}$, to detect runs that fulfill $\{1, 2\}$ (see Figure 2a). A run in v_3 should never visit $G(3)$, its negative obligation. Or the run will be reset (the exact meaning of reset is shown in Step (4.2.3.2)). If a run visits $B(3)$, then the run should be moved into a child v_2 with the witness set $I \setminus \{2\} = \{1, 3\}$. Similarly, we may have a child v_1 associated with the witness set $I \setminus \{1\} = \{2, 3\}$. Note that this spawning should be recursively applied until we arrive at leaves whose witness sets are singletons. For example, runs staying in v_3 could never visit $B(2)$, and hence v_3 should also spawn a child v_{32} with the witness set $I \setminus \{3, 2\} = \{1\}$. In general, two nodes, even if they are siblings, may have the same witness set as they are created at different times.

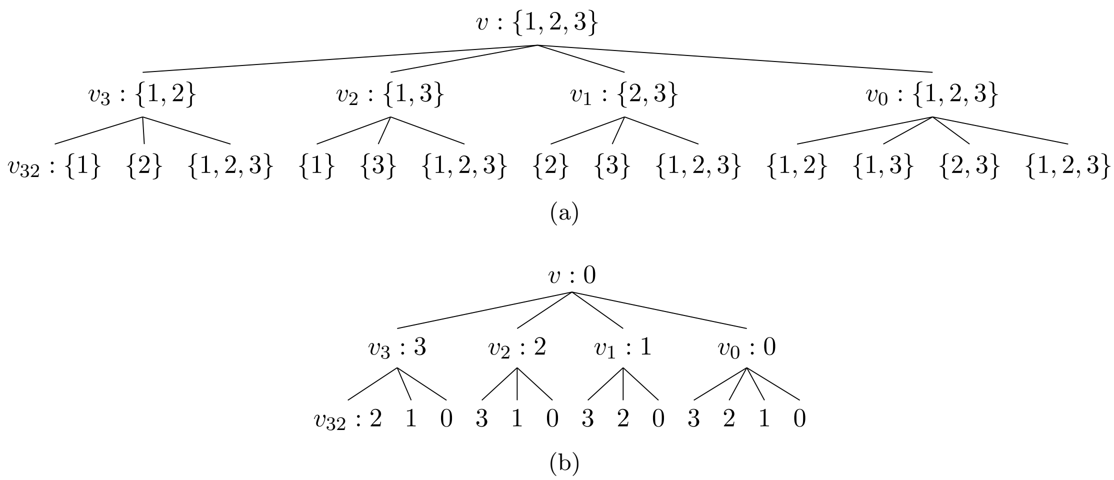

Figure 2: Two equivalent representations of witness sets. In (a) nodes are explicitly labeled with witness sets while in (b) witness sets are implicitly represented by index labels.

Let v' be a node with the witness set J. If a run fulfills J, then the run moves into a special child that also has I as its witness set. If all children of v' are special, then we say v' turns *green*. It is not hard to verify that if S and S' are the state sets of v' at any two consecutive moments when v' turns green, then we have $S \overset{J}{\hookrightarrow} S'$. But v' may never turn green because not all runs fulfill J. That is why these special children of v' also have I as their witness sets, for they also spawn children, behaving just as the root. We refer the reader to [Sch02] for a detailed exposition of this spawning and sweeping process.

Figure 2 shows part of a Safra tree for Streett determinization in two equivalent representations (with respect to witness sets). The representation shown in Figure 2b is used in Definition 2. Note that, in general, not all witness sets necessarily appear in a Safra tree.

Definition 2 (Safra Trees for Streett Determinization (*STS*))**.** *A* Safra tree for Streett determinization *(STS) is an ordered and labeled tree $\langle V, L, \prec_{str}\rangle$ with $L = \langle L_n, L_s, L_c, L_h\rangle$ where*

2.1 $L_n : V \to [1..nk]$ assigns each node a unique name.

2.2 $L_s : V \to 2^Q$ assigns each node a subset of Q such that for every node v, $L_s(v) = \cup_{v' \in ch(v)} L_s(v')$ and $L_s(v') \cap L_s(v'') = \emptyset$ for every two distinct $v', v'' \in ch(v)$.

2.3 $L_c : V \to \{green, red, yellow\}$ assigns each node a color.

2.4 $L_h : V \to I \cup \{0\}$ assigns each node an index in $I \cup \{0\}$. For a node v, let $L_h^{\to}(v)$ denote the sequence of I-elements from the root to v with 0 excluded and $L_h^{set}(v)$ the set of I-elements occurring in $L_h^{\to}(v)$. We require that for every node v, there is no repeated index in the sequence $L_h^{\to}(v)$.

2.5 $\prec_{str}$: $V \times V$ linearly orders siblings such that $v \prec_{str} v'$ iff v is to the left of v', and if $L_h(v) > L_h(v')$, then $v \prec_{str} v'$.

We require that a newly created node be added to the right of all existing siblings of the same L_h-value. That is, for two sibling v and v', $v \prec_{str} v'$ iff $L_h(v) > L_h(v')$, or $L_h(v) = L_h(v')$ and v is *older than* v'. We define a variant sibling ordering $\prec_{pri}$ such that $v \prec_{pri} v'$ iff $L_h(v) < L_h(v')$, or $L_h(v) = L_h(v')$ and v is *older than* v'. As mentioned before, the ordering $\prec_{str}$ is of no importance in Safra's determinization constructions, while both $\prec_{str}$ and $\prec_{pri}$ are needed for the correctness of our construction (Section 4).

Procedure 4 (Streett Determinization [Saf92])**.** *Given a nondeterministic Streett automaton* $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, \langle G, B\rangle_I\rangle$*, this procedure outputs a deterministic Rabin automaton* $\mathcal{B} = \langle \tilde{Q}, \tilde{q}_0, \Sigma, \tilde{\Delta}, [\tilde{G}, \tilde{B}]_{\tilde{I}}\rangle$*, where* $\tilde{Q}$ *is the set of STS,* $\tilde{I} = [1..nk]$*,* $\tilde{q}_0$ *and* $[\tilde{G}, \tilde{B}]_{\tilde{I}}$ *are as defined in Procedure 2 (for the root* v *in* $\tilde{q}_0$ *we have* $L_n(v) = 1$*,* $L_s(v) = Q_0$*,* $L_c(v) = red$ *and* $L_h(v) = 0$*), and* $\tilde{\Delta}$ *is defined such that* $\tilde{q}' = \tilde{\Delta}(\tilde{q})$ *if and only if* $\tilde{q}'$ *is the STS obtained by applying the following transformation rule to* $\tilde{q}$*.*

4.1 Subset Construction*: for each node* v *in* $\tilde{Q}$*, update set label* $L_s(v)$ *to* $\Delta(L_s(v))$*.*

4.2 Expansion*. Apply the following transformations* downwards *from the root:*

4.2.1 If v *is a leaf with* $L_h^{set}(v) = I$*, stop.*

4.2.2 If v *is a leaf with* $L_h^{set}(v) \neq I$*, add a child* v' *to* v *such that* $L_s(v') = L_s(v)$ *and* $L_h(v') = \max(I \setminus L_h^{set}(v))$*.*

4.2.3 If v *is a node with* j *children* $v_1, \ldots, v_j$*. Let* $i_1, \ldots, i_j$ *be the corresponding index labels. Consider the following cases for each* $j' \in [1..j]$*.*

4.2.3.1 If $L_s(v_{j'}) \cap B(i_{j'}) \neq \emptyset$*. Add a child* v' *to* v *with* $L_s(v') = L_s(v_{j'}) \cap B(i_{j'})$ *and* $L_h(v') = \max([0..i_{j'}) \cap ((I \cup \{0\}) \setminus L_h^{set}(v)))$*, and remove the states in* $L_s(v_{j'}) \cap B(i_{j'})$ *from* $v_{j'}$ *and all the descendants of* $v_{j'}$*.*

4.2.3.2 If $L_s(v_{j'}) \cap B(i_{j'}) = \emptyset$ *and* $L_s(v_{j'}) \cap G(i_{j'}) \neq \emptyset$*. Add a child* v' *to* v *with* $L_s(v') = L_s(v_{j'}) \cap G(i_{j'})$ *and* $L_h(v') = L_h(v)$*, and remove the states in* $L_s(v_{j'}) \cap G(i_{j'})$ *from* $v_{j'}$ *and all the descendants of* $v_{j'}$*.*

4.3 Horizontal Merge*. For any state* q *and any two siblings* v *and* v' *such that* $q \in L_s(v) \cap L_s(v')$*, if* $v \prec_{pri} v'$*, then remove* q *from* v' *and all its descendants.*

4.4 Vertical Merge*. For each* v*, if all children of* v *have index label* 0*, then remove all descendants of* v*.*

Step (4.2.3.1) says that if a run in $v_{j'}$ fulfills the positive obligation $i_{j'}$ by visiting $B(i_{j'})$ ($L_s(v_{j'}) \cap B(i_{j'}) \neq \emptyset$), then the run moves into a new node v' ($L_s(v') = L_s(v_{j'}) \cap B(i_{j'})$), and v' continues to monitor if the run hits the next largest positive obligation in the witness set of its parent ($L_h(v') = \max([0..i_{j'}) \cap ((I \cup \{0\}) \setminus L_h^{set}(v)))$). Step (4.2.3.2) says that if this is not the case and the run in $v_{j'}$ also violates the negative obligation $i_{j'}$ by visiting $G(i_{j'})$ ($L_s(v_{j'}) \cap G(i_{j'}) \neq \emptyset$), then this run is *reset*, in the sense that the states in $L_s(v_{j'}) \cap G(i_{j'})$ moved into a new child of v.

4 Improved Streett Determinization

In this section, we show our improved construction for Streett determinization.

Improvement I. The first idea is what we have applied to Streett complementation [CZ11b], namely, the larger the k, the more overlaps between $B(i)$'s and between $G(i)$'s ($i \in I$). Let us revisit the previous example, illustrated in Figure 2. Assume that $G(2) \subseteq G(3)$ (we say that $G(3)$ covers $G(2)$). If a run stays at v_3, then the run is not supposed to visit $G(3)$, or otherwise the run should have been reset by Step (4.2.3.2). Since the run cannot visit $G(2)$ either, there is no point to check if it is to visit $B(2)$, and hence v_3 does not need to have a child with index label 2 (in this case the node v_{32}). This simple idea already puts a cap on the size of STS. But it turns out that we can save the most if we exploit the redundancy on B instead of on G.

Reduction of Tree Size. Step (4.2.3.1) at a non-leaf node v is to check, for every child v' of v, if a run visits $B(L_h(v'))$, and in the positive case, move the run into a new node. Let v' be a non-root node and v the parent of v'. Let $I_v = L_h^{set}(v)$. As Step (4.2) is executed recursively from top to bottom, it can be assured that at the moment of its arriving at v, for every $i \in I_v$, we have $L_s(v') \cap B(i) = \emptyset$. By an abuse of notation, we write $B(v)$ for $\cup_{j \in I_v} B(j)$, and hence we have $L_s(v') \cap B(v) = \emptyset$. Thus, there is no chance of missing a positive obligation even if we restrict L_h to be such that $B(L_h(v')) \not\subseteq B(v)$. It follows that each node "watches" at least one more state that has not been watched by its ancestors, and therefore along any path of an STS, there are at most μ nodes with non-zero index labels (recall that $\mu = \min(n, k)$ and note that the root is excluded as its index label is 0). Also, it can be shown by induction on tree height that an STS contains at most n nodes with index label 0, using the fact that set labels of sibling nodes are pairwise disjoint and the fact that if v is the parent of v' and $L_h(v') = 0$, then $L_s(v') \subset L_s(v)$. Therefore, the number of nodes in an STS is bounded by $n(\mu + 1)$.

Reduction of Index Labels. Let $I'_v = \{i \in I \mid B(i) \subseteq B(v)\}$. The above analysis tells us that $L_h(v') \in (I \setminus I'_v)$. However, we can further improve L_h such that there is no $j \in (I \setminus I'_v)$, $(B(j) \setminus B(v)) \subset (B(L_h(v')) \setminus B(v))$ and for any $j \in (I \setminus I'_v)$, $(B(j) \setminus B(v)) = (B(L_h(v')) \setminus B(v))$ implies $L_h(v') < j$. We say that $L_h(v')$ minimally extends $L_h^{\rightarrow}(v)$ if this condition holds.

To formalize the intuition of *minimal extension*, we introduce two functions $Cover : I^* \to 2^I$ and $Mini : I^* \to 2^I$ as in [CZ11b]. $Cover$ maps finite sequences of I-elements to subsets of I such that

$$Cover(\alpha) = \{\, j \in I \ \mid \ B(j) \subseteq \bigcup_{i=1}^{|\alpha|} B(\alpha[i]) \,\}.$$

Note that $Cover(\epsilon) = \emptyset$. $Mini$ also maps finite sequences of I-elements to subsets of I such that $j \in Mini(\alpha)$ if and only if $j \in I \setminus Cover(\alpha)$ and

$$\forall j' \in I \setminus Cover(\alpha) \left(j' \neq j \to B(j') \cup \bigcup_{i=1}^{|\alpha|} B(\alpha[i]) \not\subset B(j) \cup \bigcup_{i=1}^{|\alpha|} B(\alpha[i]) \right), \tag{1}$$

$$\forall j' \in I \setminus Cover(\alpha) \left(j' < j \to B(j') \cup \bigcup_{i=1}^{|\alpha|} B(\alpha[i]) \neq B(j) \cup \bigcup_{i=1}^{|\alpha|} B(\alpha[i]) \right). \tag{2}$$

$Mini(\alpha)$ consists of index candidates to *minimally* enlarge $Cover(\alpha)$; ties (with respect to set inclusion) are broken by numeric minimality (Condition (2)).

Improvement II. The second idea is a batch-mode naming scheme to reduce name combinations. As shown before, an STS can have $n(\mu{+}1)$ nodes, which translates to $(n(\mu{+}1))!$ name combinations according to the current "first-come-first-serve" naming scheme, that is, picking an unused name when a new node is created and recycling the name when a node is removed. When $k = O(n)$, the naming cost is higher than all other complexity factors combined, as $(n(\mu + 1))! = 2^{O(nk \lg nk)}$. However, this can be overcome by dividing the name space into even buckets and "wholesaling" buckets to specific paths in an STS.

Reduction of Names. Let t be an STS. A *left spine* (LS) is a maximal path $l = v_1 \cdots v_m$ such that v_m is a leaf, for any $i \in [2..m]$, v_i is the left-most child of v_{i-1}, and v_1 is not a left-most child of its parent. We call v_1 the head of l. Let $head : V \to V$ be such that $head(v) = v'$ if v' is the head of the LS where v belongs to. We say that l is the i-th LS of t if v_m is the i-th leaf of t, counting from left to right. It is clear that a tree with m leaves has m LS, and every node is on exactly one LS. Thus, an STS has at most n LS. Now the reason of using the structural ordering $\prec_{str}$ is clear. If a non-head node on an LS has index label 0, then so should all of its siblings. But then all of them should have been removed. Therefore, only the head of an LS can have index label 0, which means that an LS can have at most $\mu + 1$ nodes.

We use $n(\mu + 1)$ names and divide them evenly into n buckets. Let b_i ($i \in [1..n]$) denote the i-th bucket, $b_{ij} = (\mu + 1)(i - 1) + j$ ($i \in [1..n], j \in [1..\mu + 1]$) the j-th name in the i-th bucket. We say that $b_{i1} = (\mu + 1)(i - 1) + 1$ is the *initial value* of b_i. Our naming strategy is as follows. Every LS $l = v_1 \cdots v_m$ in an STS is associated with a name bucket b and $v_1, \cdots, v_m$ are assigned names continuously from the initial value of b. For example, if l is associated with bucket b_t, then $L_n(v_i) = (\mu + 1)(t - 1) + i$ for $i \in [1..m]$. The bucket association for each LS in an STS t can be viewed as selection function $bucket : V \to [1..n]$ such that $bucket(v) = i$ if node v is assigned a name in the i-th bucket.

This naming strategy, however, comes with a complication; what if a leftmost sibling v is removed (due to Step (4.2.3)), and the second leftmost sibling v' (if exists) and all nodes belonging to the LS of which v' is the head, "graft into" the LS that v belongs to? The answer is that at the end of tree transformation, we need to rename those nodes that have moved into another LS. If a tree transformation turns t into t', and during the process a node v joins another LS l in t' (which is also in t before the transformation), then we rename v to a name in the bucket that l uses in t, and recycle the bucket with which v was associated in t.

Example 1 (New Naming Scheme)**.** *Figure 3 illustrates the changes of names in a sequence of tree transformations. We assume that there are* 10 *buckets* $b_1 - b_{10}$*, each of which is of size* 4*. Nodes in the graphs are denoted in the form* $v : L_n(v)$*; all other types of labels are omitted for simplicity.*

Definition 3 (Reduced Safra Trees for Streett Determinization (μSTS))**.** *A* reduced Safra tree for Streett determinization *(μSTS) is an STS* $\langle V, L, \prec_{str}\rangle$ *with* $L = \langle L_n, L_s, L_c, L_h\rangle$ *that satisfies the following additional conditions:*

3.1 Condition on L_h. *For each node* v*, if* $Mini(L_h^{\rightarrow}(v)) \neq \emptyset$*, then* v *is not a leaf node and for any child* v' *of* v*,* $L_h(v') \in Mini(L_h^{\rightarrow}(v))$*.*

3.2 Condition on L_n. *There exists a function* $bucket : V \to [1..n]$ *such that for every LS* $v_1 \cdots v_m$*, we have* $bucket(v_i) = bucket(v_j)$ *for* $i, j \in [1..m]$ *and* $L_n(v_i) = (\mu + 1)(bucket(v_i) - 1) + i$ *for* $i \in [1..m]$*.*

Condition (3.2), together with the requirement that L_n is injective (Condition (2.1)), guarantees that no two distinct LS in a tree share a bucket. Condition (3.1) says that every μSTS has fully grown left spines, that is, no leaf v can be further extended, as $Mini(L_h^{\rightarrow}(v)) = \emptyset$. To achieve this, we need the following procedure applied as the last step of each tree transformation.

Procedure 5 (Grow Left Spine)**.** *Repeat the following procedure until no new nodes can be added: if* v *is a leaf and* $Mini(L_h^{\rightarrow}(v)) \neq \emptyset$*, add a new child* v' *to* v *with* $L_s(v') = L_s(v)$*,* $L_h(v') = \max(Mini(L_h^{\rightarrow}(v)))$*,* $L_c(v') = red$*, and* $L_n(v') = L_n(v) + 1$*.*

We note that the requirement that a μSTS has fully grown left spines is not essential; we can "grow" a μSTS "on-the-fly" as in [Saf92, Pit06]. But this requirement simplifies the analysis on the number of combinations of index labels (see the proof of Theorem 2).

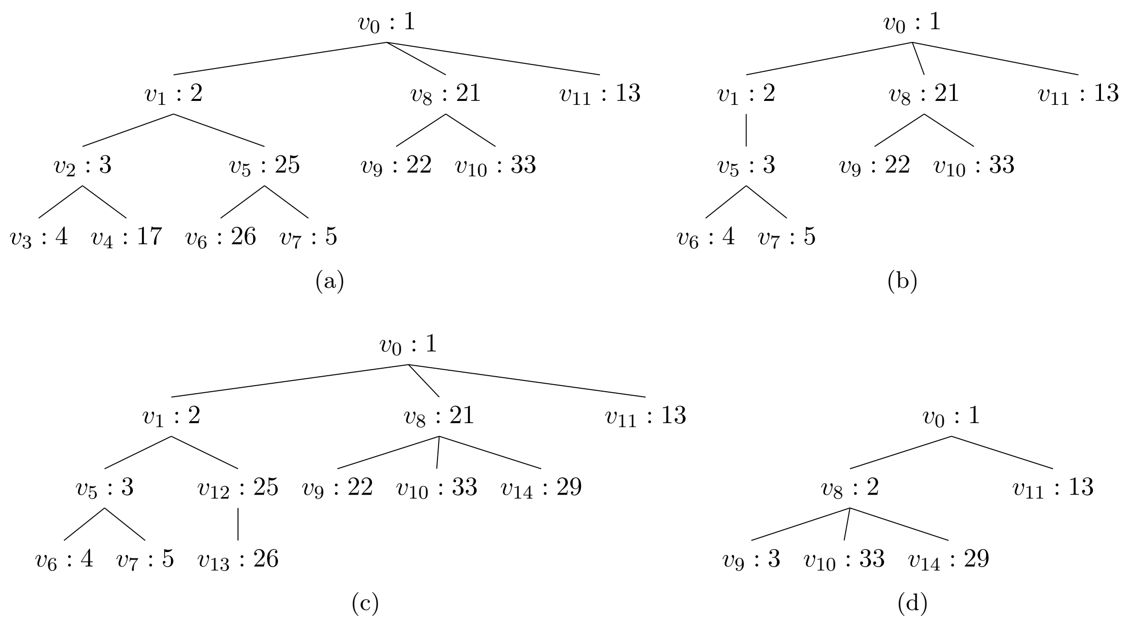

Figure 3: (a) shows an *STS* with 7 *LS*: $l_1 : v_0v_1v_2v_3$, $l_2 : v_4$, $l_3 : v_5v_6$, $l_4 : v_7$, $l_5 : v_8v_9$, $l_6 : v_{10}$ and $l_7 : v_{11}$, associated with buckets b_1, b_5, b_7, b_2, b_6, b_9 and b_4, respectively. (b) shows the resulting *STS* after removing v_2 and its descendants. Nodes v_5 and v_6 migrate into l_1, and accordingly their name bucket b_7 is recycled. Also recycled is bucket b_5 due to the deletion of v_4. (c) shows the resulting *STS* after adding nodes v_{12}, v_{13} and v_{14} and forming two new *LS*: $v_{12}v_{13}$ and v_{14}. Here v_{12} and v_{13} reuse the previously recycled bucket b_7, while node v_{14} takes an unused bucket b_8. (d) shows the resulting *STS* after removing v_1 and its descendants. Nodes v_8 and v_9 migrate into l_1 and take names 2 and 3, respectively. Buckets b_2, b_7 and b_6 are recycled accordingly.

Procedure 6 (Naming and Coloring on μSTS)**.** *Naming and Coloring convention for μSTS is the one for STS plus the following.*

6.1 Nodes in an LS, from the head downwards, are assigned continuously increasing names, starting from the initial value of a bucket.

6.2 When an LS is created, nodes in the LS are assigned names from an unused name bucket; when an LS is removed (which only happens when its head is removed), the name bucket of the LS is recycled.

6.3 When an LS l is grafted into another LS l', the name bucket of l is recycled and nodes on l are renamed according to Rule (6.1)*, as if they were on l' originally.*

6.4 Renamed nodes are marked red.

Procedure 7 (Improved Streett Determinization)**.** *Given a nondeterministic Streett automaton $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, \langle G, B\rangle_I\rangle$, this procedure outputs a deterministic Rabin automaton $\mathcal{B} = \langle \tilde{Q}, \tilde{q}_0, \Sigma, \tilde{\Delta}, [\tilde{G}, \tilde{B}]_{\tilde{I}}\rangle$, where $\tilde{Q}$ is the set of μSTS, $\tilde{I} = [1..n(\mu+1)]$, $[\tilde{G}, \tilde{B}]_{\tilde{I}}$ is as defined in Procedure 4, $\tilde{q}_0$ is a tree that is just a fully grown left spine obtained by growing the single root tree (which is defined in Procedure 4) according to Procedure* (5)*, and $\tilde{\Delta}$ is defined such that $\tilde{q}' = \tilde{\Delta}(\tilde{q})$ if and only if $\tilde{q}'$ is the μSTS obtained by applying the following transformation rule to $\tilde{q}$.*

7.1 Subset Construction. *For each node v in $\tilde{Q}$, update set label $L_s(v)$ to $\Delta(L_s(v))$.*

7.2 Expansion. *Apply the following transformations to non-leaf nodes recursively from the root. Let v be a node with j children $v_1, \ldots, v_j$ with $i_1, \ldots, i_j$ as the corresponding index labels. Consider the following cases for each $j' \in [1..j]$.*

7.2.1 If $L_s(v_{j'}) \cap B(i_{j'}) \neq \emptyset$. Add a child v' to v with $L_s(v') = L_s(v_{j'}) \cap B(i_{j'})$ and $L_h(v') = \max([0..i_{j'}) \cap (\{0\} \cup Mini(L_h^{\rightarrow}(v))))$, and remove the states in $L_s(v_{j'}) \cap B(i_{j'})$ from $v_{j'}$ and all the descendants of $v_{j'}$.

7.2.2 If $L_s(v_{j'}) \cap B(i_{j'}) = \emptyset$ and $L_s(v_{j'}) \cap G(i_{j'}) \neq \emptyset$. Add a child v' to v with $L_s(v') = L_s(v_{j'}) \cap G(i_{j'})$ and $L_h(v') = L_h(v)$, and remove the states in $L_s(v_{j'}) \cap G(i_{j'})$ from $v_{j'}$ and all the descendants of $v_{j'}$.

7.3 Horizontal Merge. *For any state q and any two siblings v and v' such that $q \in L_s(v) \cap L_s(v')$, if $v \prec_{pri} v'$, then remove q from v' and all its descendants.*

7.4 Vertical Merge. *For each v, if all children of v have index label* 0*, then remove all descendants of v.*

7.5 Grow the tree fully according to Procedure (5).

Note that besides naming and renaming, the only major difference between Procedures 7 and 4 is the way of selecting the next positive obligation. In Step (7.2.1), $Mini(L_h^{\rightarrow}(v))$ is used in the calculation, while in Step (4.2.3.1), $L_h^{set}(v)$ is used.

Theorem 1 (Streett Determinization: Correctness)**.** *Let $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, \langle G, B \rangle_I \rangle$ be a Streett automaton with $|Q| = n$ and $I = [1..k]$, and $\mathcal{B} = \langle \tilde{Q}, \tilde{q}_0, \Sigma, \tilde{\Delta}, [\tilde{G}, \tilde{B}]_{\tilde{I}} \rangle$ the deterministic Rabin automaton obtained by Procedure 7. We have $\mathscr{L}(\mathcal{A}) = \mathscr{L}(\mathcal{B})$.*

5 Complexity

In this section we state the complexity results for the determinization of Streett, generalized Büchi, parity and Rabin automata. The corresponding proofs are given in the appendix.

Theorem 2 (Streett Determinization: Complexity)**.** *Let $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, \langle G, B \rangle_I \rangle$ be a Streett automaton with $|Q| = n$ and $I = [1..k]$, and $\mathcal{B} = \langle \tilde{Q}, \tilde{q}_0, \Sigma, \tilde{\Delta}, [\tilde{G}, \tilde{B}]_{\tilde{I}} \rangle$ the deterministic Rabin automaton obtained by Procedure 7. If $k = O(n)$, then $|\tilde{Q}| = 2^{O(n \lg n + nk \lg k)}$ and $|\tilde{I}| = O(nk)$. If $k = \omega(n)$, then $|\tilde{Q}| = 2^{O(n^2 \lg n)}$ and $|\tilde{I}| = O(n^2)$.*

Figure 4 breaks down the total cost into five categories and compares our construction with previous ones in each category. It turns out that the complexity analysis can be easily adapted for generalized Büchi and parity automata as they are subclasses of Streett automata.

Corollary 1 (Generalized Büchi Determinization: Complexity)**.** *Let $\mathcal{A}$ be a generalized Büchi automaton with state size n and index size k. There is an equivalent deterministic Rabin automaton $\mathcal{B}$ with state size $2^{O(n \lg nk)}$. The index size is $O(nk)$ if $k = O(n)$ and $O(n^2)$ if $k = \omega(n)$.*

† In Piterman's construction, the number of ordered trees times the number of name combinations is bounded by $(nk)^{nk}$. However, the second factor is still the dominating one, costing $2^{\Omega(nk \lg nk)}$. Also, there is no notion of color in Piterman's construction. Instead, each tree is associated with two special names both ranging from 1 to nk, resulting in the cost $O((nk)^2)$.

Cost	Safra's	Piterman's	Ours	
Ordered Tree	$2^{O(nk)}$	$2^{O(nk)}$	$2^{O(n \lg n)}$	$k = O(2^n)$
Name	$2^{O(nk \lg nk)}$	$2^{O(nk \lg nk)\dagger}$	$2^{O(n \lg n)}$	$k = O(2^n)$
Color	$2^{O(nk)}$	$O((nk)^2)^{\dagger}$	$2^{O(n \lg k)}$	$k = O(n)$
			$2^{O(n \lg n)}$	$k = \omega(n)$
Set Label	$2^{O(n \lg n)}$	$2^{O(n \lg n)}$	$2^{O(n \lg n)}$	$k = O(2^n)$
Index Label	$2^{O(nk \lg nk)}$	$2^{O(nk \lg nk)}$	$2^{O(n \lg n + nk \lg k)}$	$k = O(n)$
			$2^{O(n^2 \lg n)}$	$k = \omega(n)$
Total	$2^{O(nk \lg nk)}$	$2^{O(nk \lg nk)}$	$2^{O(n \lg n + nk \lg k)}$	$k = O(n)$
			$2^{O(n^2 \lg n)}$	$k = \omega(n)$

Figure 4: Cost breakdown of Streett determinization

Corollary 2 (Parity Determinization: Complexity)**.** *Let $\mathcal{A}$ be a parity automaton with state size n and index size k. There is an equivalent deterministic Rabin automaton $\mathcal{B}$ with state size $2^{O(n \lg n)}$ and index size $O(nk)$.*

The determinization of a Rabin automaton with acceptance condition $[G, B]_I$ (the dual of $\langle G, B \rangle_I$ for $I = [1..k]$) can be straightforwardly obtained by running, in parallel, k modified Safra trees, each of which monitors runs for an individual Rabin condition $[G(i), B(i)]$ ($i \in I$). Figure 1 summarizes the determinization complexities for ω-automata of common types.

Theorem 3 (Rabin Determinization: Complexity)**.** *Let $\mathcal{A}$ be a Rabin automaton with state size n and index size k. There is an equivalent deterministic Rabin automaton $\mathcal{B}$ with state size $2^{O(nk \lg n)}$ and index size $O(nk)$.*

We note that it is unlikely that there exists a Safra-tree style determinization for Rabin automata, because an analogue of Lemma 2 fails due to the existential nature of Rabin acceptance conditions.

6 Concluding Remarks

In this paper we improved Safra's construction and obtained tight upper bounds on state complexity for the determinization of Streett, generalized Büchi, parity and Rabin automata. Combining these with the lower bound results in [CZ11a] and previous findings in the literature, we now have a complete characterization of determinization complexity of ω-automata of common types.

Our results show an interesting phenomenon that in the asymptotic notation $2^{\Theta(\cdot)}$, complementation complexity is identical to determinization complexity. The same phenomenon happens to finite automata on finite words. We believe it is worth investigating the reason behind this phenomenon.

As mentioned earlier, determinization procedures that output parity automata, like Piterman's constructions, are preferable to the classic ones that output Rabin automata. We plan to investigate how to combine Piterman's node renaming scheme with ours to obtain determinization procedures that output parity automata with optimal state complexity.

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A Proof for Theorem 1

Theorem 1 (Streett Determinization: Correctness). Let $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, \langle G, B\rangle_I\rangle$ be a Streett automaton with $|Q| = n$ and $I = [1..k]$, and $\mathcal{B} = \langle \tilde{Q}, \tilde{q}_0, \Sigma, \tilde{\Delta}, [\tilde{G}, \tilde{B}]_{\tilde{I}}\rangle$ the deterministic Rabin automaton obtained by Procedure 7. We have $\mathscr{L}(\mathcal{A}) = \mathscr{L}(\mathcal{B})$.

Proof. ($\mathscr{L}(\mathcal{A}) \subseteq \mathscr{L}(\mathcal{B})$). This amounts to showing that if $\varrho = \tilde{q}_0\tilde{q}_1\cdots$ is an run of $\mathcal{B}$ over an infinite word $w = w_0w_1\ldots \in \mathscr{L}(\mathcal{A})$, then there exist $i \in \tilde{I}$ and $j \geq 0$ such that a node v with name i exists in all ϱ_j and v turns green infinitely often. This part of proof is almost the same as the one in [Saf92]. The only complication comes from renaming. As mentioned before, nodes are identified with their names, not their positions (V-values). So we have the situation that v with name i exists in $\tilde{q}_j$, but it is renamed to i' in the following state $\tilde{q}_{j+1}$. This happens when v is a second left-most sibling in $\tilde{q}_j$ and the leftmost sibling of v is removed in $\tilde{q}_{j+1}$, resulting in v joining another LS in $\tilde{q}_{j+1}$. However, such "grafting" is one directional, in the sense that the V-value of v after a grafting is lexicographically smaller than the V-value before the grafting. Due to well-foundedness of lexicographical order, if v stays in each state of a suffix of ϱ, then v eventually has a fixed name i. This means that ϱ eventually never visits $\tilde{G}(i)$, but visits $\tilde{B}(i)$ infinitely often, and hence $w \in \mathscr{L}(\mathcal{B})$.

($\mathscr{L}(\mathcal{B}) \subseteq \mathscr{L}(\mathcal{A})$). We ought to show that if $w = w_0w_1\ldots \in \mathscr{L}(\mathcal{B})$, then the run $\varrho = \tilde{q}_0\tilde{q}_1\cdots$ of $\mathcal{B}$ over w induces an accepting run of $\mathcal{A}$ over w. The assumption that ϱ is accepting means that there exists an $i \in \tilde{I}$ such that ϱ eventually never visits $\tilde{G}(i)$, but visits $\tilde{B}(i)$ infinitely often, or equivalently, i names a green or yellow node in every state in a suffix of ϱ and there are infinitely many occurrences when the nodes named by i are green. Since renamed nodes are marked red, all nodes named by i in the suffix have to be the same one. It follows that a node v eventually stays in every state in a suffix of ϱ and v turns green infinitely often. The rest of the proof is the same as the one in [Saf92], with the help of Lemma 2 and the fact that using *Mini* to select the index labels of the children of v is sound and complete. □

B Formal Analysis of Complexity

To formally analyze the complexity of our Streett determinization, we first introduce a notion called *increasing tree of sets* (*ITS*) [CZ11b]. Here we switch to an informal notation of labeled trees and we identify a node with the sequence of labels from the root to the node.

Definition 4 (Increasing Tree of Sets (*ITS*) [CZ11b]). *An ITS $\mathcal{T}(n,k,B)$ is an unordered I-labeled tree such that a node α exists in $\mathcal{T}(n,k,B)$ iff $\forall i \in [1..|\alpha|]$, $\alpha[i] \in Mini(\alpha[1..i))$.*

By the definition an *ITS* is uniquely determined by parameter n, k and B. Beside that, several properties are easily seen. First, the length of the longest path in $\mathcal{T}(n,k,B)$ is bound by μ. Second, if β is a direct child of α, then β must contribute at least one new element that has not been seen from the root to α. Third, the new contributions made by β cannot be covered by contributions made by another sibling β', with ties broken by selecting the one with smallest index. As $B : I \to 2^Q$ is one to one, we also view *ITS* as 2^Q-labeled trees.

Example 2 (*ITS*). *Consider $n = 3$, $k = 4$, $Q = \{q_0, q_1, q_2\}$, and $B : [1..4] \to 2^Q$ such that*

$$B(1) = \{q_0, q_1\}, \qquad B(2) = \{q_0\}, \qquad B(3) = \{q_1, q_2\}, \qquad B(4) = \{q_2\}\,.$$

Figure 5 shows the corresponding $\mathcal{T}(n,k,B)$.

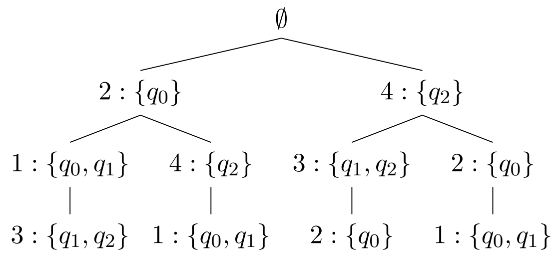


Figure 5: The *ITS* $\mathcal{T}(n,k,B)$ in Example 2. Note that $\{q_0,q_1\}$ and $\{q_1,q_2\}$ cannot appear in the first level because $\{q_0,q_1\}$ covers $\{q_0\}$ and $\{q_1,q_2\}$ covers $\{q_2\}$. The leftmost node at the bottom level is labeled by $\{q_1,q_2\}$ instead of by $\{q_2\}$ due to the index minimality requirement.

Now let $\mathbb{T}(n,k)$ denote the class of all *ITS* with fixed n and k. Let $T_h(n,k)$ be the maximum number of nodes an *ITS* in $\mathbb{T}(n,k)$ can have.

Lemma 3 ([CZ11b])**.** *$T_h(n,k) = 2^{O(k\lg k)}$ for $k = O(n)$ and $T_h(n,k) = 2^{O(n\lg n)}$ for $k = \omega(n)$.*

Proof. Note that for fixed n, k and $G : I \to 2^Q$, $\mathcal{T}(n,k,G)$ is uniquely determined. Also note that the height of $\mathcal{T}(n,k,G)$ is bound by $\mu = \min(n,k)$ and the maximum branching is bound by k. Let $|\mathcal{T}(n,k,G)|$ denote the number of non-root nodes in $\mathcal{T}(n,k,G)$

Case 1: $k = O(n)$. In this case we have $\mu = O(k)$. Therefore, we have

$$|\mathcal{T}(n,k,G)| \le \sum_{i=1}^{\mu} k^i \le \mu k^\mu = 2^{\lg\mu + \mu\lg k} = 2^{O(k\lg k)}$$

Since G is chosen arbitrarily, we have $T_h(n,k) = 2^{O(k\lg k)}$.

Case 2: $k = \omega(n)$. Let $G' : I' \to 2^Q$ where $I' = [1..k']$ be an extension of G such that *range*(G') contains all singletons from Q, i.e.,

$$\begin{aligned} &\forall i \in [1..k]\ G'(i) = G(i)\,, \\ &\{\{q\} \mid q \in Q\} \subseteq \mathit{range}(G')\,. \end{aligned}$$

By Lemma 4, we assume without loss of generality that $[1..n]$ name the n singletons such that $G'(i) = q_{i-1}$ for $i \in [1..n]$. Because of existence of all singletons, the minimal extension at each node is always done by adding singletons. Formally, for any index sequence α, $\mathit{Mini}(\alpha) \subsetneq [1..n]$. Therefore, each nonempty path in $\mathcal{T}(n,k',G')$ corresponds to a nonempty prefix of a permutation of $[1..n]$ and vice versa. Note that each leave is of height n, $\mathcal{T}(n,k',G')$ has exactly $n!$ nodes at both the second last level and the last level. For $n \ge 2$, each node above the second last level has at leat two children, which implies the number of nodes above the second last level is bound by $n!$. All in all, we have

$$|\mathcal{T}(n,k',G')| \le 3n! = 2^{O(n\lg n)}.$$

By Lemma 5, $|\mathcal{T}(n,k,G)| \le |\mathcal{T}(n,k',G')|$. As G is chosen arbitrarily, we have $T_h(n,k) = 2^{O(n\lg n)}$. ∎

Lemma 4. *Let $B : I \to 2^Q$, $B' : I \to 2^Q$ be two injective functions such that $\mathit{range}(B) = \mathit{range}(B')$. Then $|T(n,k,B)| = |T(n,k,B')|$.*

Proof. The condition $range(B) = range(B')$ means that B and B' just name subsets of Q differently. We extend B, B' to functions from I^* to 2^Q such that for $\alpha \in I^*$,

$$B(\alpha) = \bigcup_{i=1}^{|\alpha|} B(\alpha[i]) , \qquad\qquad B'(\alpha) = \bigcup_{i=1}^{|\alpha|} B'(\alpha[i]) .$$

By Definition 4, children of a node α in an *ITS* is completely determined by $B(\alpha)$. So a node α in $T(n, k, B)$ has the same number of children as a node α' in $T(n, k, B')$ if $B(\alpha) = B'(\alpha')$. Moreover, if $\alpha_1, \ldots, \alpha_j$ are children of α and $\alpha'_1, \ldots, \alpha'_j$ are children of α', Then $B(\alpha_1), \ldots, B(\alpha_j)$ are just a permutation of $B'(\alpha'_1), \ldots, B'(\alpha'_j)$. By induction on tree height, there is a one-to-one correspondence between nodes in $T(n, k, B)$ and nodes in $T(n, k, B')$. Thus $|\mathcal{T}(n, k, B)| = |\mathcal{T}(n, k, B')|$. □

Lemma 5. *Let $\mathcal{T}(n, k, G)$ and $\mathcal{T}(n, k', G')$ be two ITS such that G' extends G by naming a singleton that G does not name. Then $|\mathcal{T}(n, k, G)| \le |\mathcal{T}(n, k', G')|$.*

Proof. It suffices to show that $|\mathcal{T}(n, k, G)| \le |\mathcal{T}(n, k', G')|$ when $k' = k+1$, $I' = [1..k']$, $G'(i) = G(i)$ for $i \in I$ and $G'(k+1)$ names a new singleton. Without loss of generality we assume $G'(k+1) = \{q_0\}$. We show a tree transformation θ that turns $\mathcal{T}(n, k, G)$ into $\mathcal{T}(n, k+1, G')$.

Recall that we identify a node with the path from root to the node. By a node α, we mean $\alpha[1], \ldots, \alpha[|\alpha|]$ are labels on the path and $\alpha[|\alpha|]$ is the label of α. Let T_α denote the subtree rooted at α and $G(\alpha)$ be

$$\bigcup_{i=1}^{|\alpha|} G(\alpha[i]) .$$

We say that a state $q \in Q$ is named by α if $q \in G(\alpha)$. We define θ on a subtree T_α as follows.

Case 1. The new state q_0 is named by α. In this case, $\theta(T_\alpha) = T_\alpha$.

Case 2. The new state q_0 has not been named by α. Let us assume that $\{q_0\}$ is named by α's children $\alpha_1, \ldots, \alpha_j$ ($j = 0$ means no children names $\{q_0\}$). Let $\alpha_{j+1}, \ldots, \alpha_{j'}$ be the rest children of α. Formally, we have

$$\begin{aligned} &q_0 \notin G(\alpha) , \\ &q_0 \in G(\alpha_i)(i \in [1..j], j \ge 0) , \\ &q_0 \notin G(\alpha_i)(i \in [j+1..j'], j \ge 0) . \end{aligned}$$

We have two sub-cases to consider.

Case 2.1. For some $i \in [1..j]$, $G(\alpha_i) \cup G(\alpha) = \{q_0\} \cup G(\alpha)$. In this case we must have $j = 1$. Let $\theta(T_\alpha)$ be a tree obtained from T_α by replacing the label of α_1 by $k+1$.

Case 2.2. For all $i \in [1..j]$, $G(\alpha_i) \cup G(\alpha) \supset \{q_0\} \cup G(\alpha)$. In this case let $\theta(T_\alpha)$ be a tree obtained from T_α by the following procedure.

1. Add to α a new leaf α' labeled with $k+1$,
2. Remove subtrees $\alpha_1, \ldots, \alpha_j$ from α and make them children of α'
3. Add to α' $j' - j$ new leaves $\alpha'_1, \ldots, \alpha'_{j'-j}$, respectively, labeled with

$$\alpha_{j+1}[|\alpha_{j+1}|], \ldots, \alpha_{j'}[|\alpha_{j'}|].$$

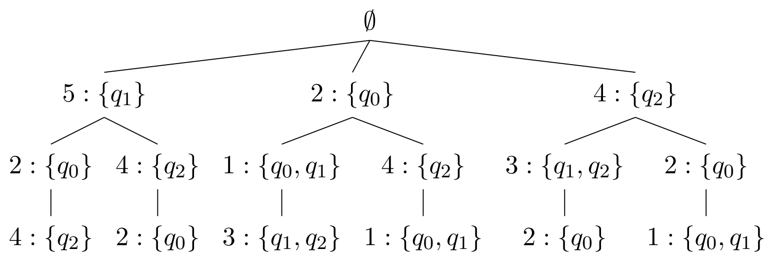
4. Grow the new $j' - j$ leaves to full *ITS* according to *Mini* defined with respect to n, k' and G'.

In any case, $\theta(T_\alpha)$ is an *ITS* with root labeled with $\alpha[|\alpha|]$. It is clear from the above procedure that $|\theta(T_\alpha)| \geq |T_\alpha|$. Now let $\Theta(T)$ a tree obtained from T by applying θ on T level by level, from top to bottom. Example 3 shows such a transformation. It is not hard to verify that $\mathcal{T}(n, k', G') = \Theta(\mathcal{T}(n, k, G))$. Therefore, we have $|\mathcal{T}(n, k, G)| \leq |\mathcal{T}(n, k', G')|$. $\square$

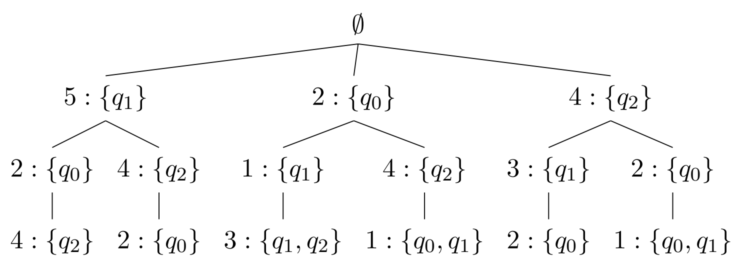
Example 3. *Consider the same setup: $n = 3$, $k = 4$, $Q = \{q_0, q_1, q_2\}$, and $B : [1..4] \to 2^Q$ such that*

$$B(1) = \{q_0, q_1\} \qquad B(2) = \{q_0\} \qquad B(3) = \{q_1, q_2\} \qquad B(4) = \{q_2\}$$

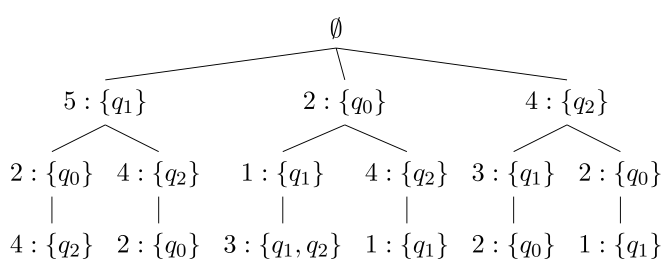
as in Example 2. Let $T_0 = \mathcal{T}(n, k, B)$. Let $k' = 5$, $B' = B \cup \{\langle 5, \{q_1\}\rangle$. Applying θ to the root of T_0 we have T_1:



Applying θ to nodes $\langle 2\rangle$ and $\langle 4\rangle$ in T_1, we have T_2:



Applying θ to nodes $\langle 2, 4\rangle$ and $\langle 4, 2\rangle$ in T_2, we have T_3:



And no more application of θ is possible. It is not hard to verify that $T_3 = \mathcal{T}(n, k + 1, B')$.

Theorem 2 (Streett Determinization: Complexity). Let $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, \langle G, B\rangle_I\rangle$ be a Streett automaton with $|Q| = n$ and $I = [1..k]$, and $\mathcal{B} = \langle \tilde{Q}, \tilde{q}_0, \Sigma, \tilde{\Delta}, [\tilde{G}, \tilde{B}]_{\tilde{I}}\rangle$ the deterministic Rabin automaton obtained by Procedure 7. If $k = O(n)$, then $|\tilde{Q}| = 2^{O(n \lg n + nk \lg k)}$ and $|\tilde{I}| = O(nk)$. If $k = \omega(n)$, then $|\tilde{Q}| = 2^{O(n^2 \lg n)}$ and $|\tilde{I}| = O(n^2)$.

Proof. As mentioned before, a μSTS has at most $n(\mu+1)$ nodes. Let $T(e,l)$ denote the number of ordered trees with e edges and l leaves. $T(e,l)$ are called *Narayana numbers*, which for $e,l \geq 1$, assume the following closed form [Nar59, DZ80]:

$$T(e,l) = \frac{1}{e}\binom{e}{l}\binom{e}{l-1}.$$

Let $OT(n,\mu)$ be the set of ordered trees with at most n leaves and $n(\mu+1)$ nodes. We have

$$|OT(n,\mu)| \leq \sum_{e=1}^{n\cdot(\mu+1)} \sum_{l=1}^{\min(n,e)} T(e,l) \leq n^3 \cdot T(n^2,n) \leq n\binom{n^2}{n}\binom{n^2}{n-1} = 2^{O(n\lg n)}.$$

A μSTS is a tree in $OT(n,\mu)$ augmented with four components: name, color, set label, and index label. By Condition (2.2), the set labels of nodes are completely determined by the set labels of leaves. There are at most n^n ways to label leaves, each of which corresponds to a function from $[1..n] \to Q$.

Let us consider index labels on each LS. As stated before, only the head of an LS can be labeled by 0. As each LS is fully grown, the sequence of index labels from the leaf of an LS to the head of the LS corresponds to a path in the ITS from a leaf going upward. So for each LS, there are at most $T_h(n,k)$ ways to do index labeling. Therefore, by Lemma 3, the number of ways of index labeling a μSTS is bounded by $(T_h(n,k))^n$, which is $2^{O(nk\lg k)}$ if $k = O(n)$ and $2^{O(n^2\lg n)}$ if $k = \omega(n)$.

The names along an LS from the head downwards is continuously increasing from the initial value of a bucket. So they are completely determined once a bucket is chosen. There are at most n buckets, and therefore the number of ways to name a μSTS is bounded by $n! = 2^{O(n\lg n)}$.

The colors on an LS also have some regularity. It is not hard to see that there is at most one green node in an LS and all red nodes must be descendants of yellow nodes. So there are at most $(\mu+1)^2$ choices to color an LS and hence the number of color combinations on a μSTS is bounded by $(\mu+1)^{2n}$, which is $2^{O(n\lg k)}$ if $k = O(n)$ and $2^{O(n\lg n)}$ if $k = \omega(n)$.

Put all together, we have state complexity $2^{O(n\lg n + nk\lg k)}$ if $k = O(n)$ and $2^{O(n\lg n)}$ if $k = \omega(n)$. Figure 4 compares the costs attributed by each component of STS between the previous constructions and ours. □

Corollary 1 (Generalized Büchi Determinization: Complexity). Let $\mathcal{A}$ be a generalized Büchi automaton with state size n and index size k. There is an equivalent deterministic Rabin automaton $\mathcal{B}$ with state size $2^{O(n\lg nk)}$. The index size is $O(nk)$ if $k = O(n)$ and $O(n^2)$ if $k = \omega(n)$.

Proof. Let $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, \langle B\rangle_I\rangle$. The acceptance condition $\langle B\rangle_I$ can be viewed as a Streett condition $\langle G, B\rangle_I$ where $G(i) = Q$ for every $i \in I$. In this case, the ITS is just a one-level tree (not counting the root). $T_h(n,k)$ is then bounded by k and the number of ways of putting index labels is bounded by $(T_h(n,k))^n = k^n$. Other complexity factors are all bounded by $n^{O(n)}$. So the total number of μSTS is bounded by $k^n \cdot n^{O(n)} = 2^{O(n\lg nk)}$. □

Corollary 2 (Parity Determinization: Complexity). Let $\mathcal{A}$ be a parity automaton with state size n and index size k. There is an equivalent deterministic Rabin automaton $\mathcal{B}$ with state size $2^{O(n\lg n)}$ and index size $O(nk)$.

Proof. Let $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, \langle G, B\rangle_I\rangle$. Because $B_1 \subset \cdots \subset B_k$, the ITS is just a directed line. As a result, there is no index combinations and other complexity factors are all bounded by $n^{O(n)}$. So we have $2^{O(n\lg n)}$. □

Theorem 3 (Rabin Determinization: Complexity). Let $\mathcal{A}$ be a Rabin automaton with state size n and index size k. There is an equivalent deterministic Rabin automaton $\mathcal{B}$ with state size $2^{O(nk \lg n)}$ and index size $O(nk)$.

Proof. This theorem is a folklore, but here we sketch a proof just for the sake of completeness. Let $\mathcal{A} = \langle Q, Q_0, \Sigma, \Delta, [G, B]_I \rangle$. For each fixed i, $\mathcal{A}_i = \langle Q, Q_0, \Sigma, \Delta, [G(i), B(i)] \rangle$ is a Rabin automaton. $\mathcal{A}_i$ can be transformed into a Büchi automaton in $O(n)$ state, which is equivalent to a deterministic Rabin automaton $\mathcal{B}_i$ with $2^{O(n \lg n)}$ states. The desired $\mathcal{B}$ is just a product of $(\mathcal{B}_i)_{i \in I}$. Let $\mathcal{B}_i = \langle Q^{(i)}, q_0^{(i)}, \Sigma, \Delta_i, [G^{(i)}, B^{(i)}]_{I^{(i)}} \rangle$. We define $\mathcal{B} = \langle \tilde{Q}, \tilde{q}_0, \Sigma, \tilde{\Delta}, [\tilde{G}, \tilde{B}]_{\tilde{I}} \rangle$ where

$$\begin{aligned}
\tilde{Q} &= Q^{(1)} \times \cdots \times Q^{(k)}, \\
\tilde{q}_0 &= \langle q_0^{(1)}, \ldots, q_0^{(k)} \rangle, \\
\tilde{\Delta}(\langle q^{(1)}, \ldots, q^{(k)} \rangle, a) &= \langle \Delta_0(q^{(1)}, a), \ldots, \Delta_k(q^{(k)}, a) \rangle, \\
\tilde{I} &= I^{(1)} \times \{1\} \cup \cdots \cup I^{(k)} \times \{k\}, \\
\tilde{G}(\langle i_j, i \rangle) &= Q^{(1)} \times \cdots \times G^{(i)}(i_j) \times \cdots \times Q^{(k)}, \\
\tilde{B}(\langle i_j, i \rangle) &= Q^{(1)} \times \cdots \times B^{(i)}(i_j) \times \cdots \times Q^{(k)}.
\end{aligned}$$

It is easily to verify that $\mathcal{B}$ is deterministic, $\mathscr{L}(\mathcal{B}) = \mathscr{L}(\mathcal{A})$, the state size of $\mathcal{B}$ is $2^{O(nk \lg n)}$ and the index size of $\mathcal{B}$ is $O(nk)$. □