Decision Procedures for Recursive Data Structures with Integer Constraints

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Abstract. This paper is concerned with the integration of recursive data structures with Presburger arithmetic. The integrated theory includes a length function on data structures, thus providing a tight coupling between the two theories, and hence the general Nelson-Oppen combination method for decision procedures is not applicable to this theory, even for the quantifier-free case. We present four decision procedures for the integrated theory depending on whether the language has infinitely many constants and whether the theory has quantifiers. Our decision procedures for quantifier-free theories are based on Oppen's algorithm for acyclic recursive data structures with infinite atom domain.

1 Introduction

Recursively defined data structures are essential constructs in programming languages. Intuitively, a data structure is *recursively defined* if it is partially composed of smaller or simpler instances of the same structure. Examples include lists, stacks, counters, trees, records and queues. To verify programs containing recursively defined data structures we must be able to reason about these data structures. Decision procedures for several data structures exist. However, in program verification decision procedures for a single theory are usually not applicable as programming languages often involve multiple data domains, resulting in verification conditions that span multiple theories. Common examples of such "mixed" constraints are combinations of data structures with integer constraints on the size of those structures.

In this paper we consider the integration of Presburger arithmetic with an important subclass of recursively defined data structures known as *recursive data structures*. This class of structures satisfies the following properties of term algebras: (i) the data domain is the set of data objects generated exclusively by applying constructors, and (ii) each data object is uniquely generated. Examples

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of such structures include lists, stacks, counters, trees and records; queues do not belong to this class as they are not uniquely generated: they can grow at both ends.

Our language of the integrated theory has two sorts; the integer sort $\mathbb{Z}$ and the data (term) sort λ. Intuitively, the language is the set-theoretic union of the language of recursive data structures and the language of Presburger arithmetic plus the additional length function $|.| : \lambda \to \mathbb{Z}$. Formulae are formed from data literals and integer literals using logical connectives and quantifications. Data literals are exactly those literals in the theory of recursive data structures. Integer literals are those that can be built up from integer variables (including the length function applied to data terms), addition and the usual arithmetic functions and relations.

We present four decision procedures for different variants of the theory depending on whether the language has infinitely many atoms and whether the theory is quantifier-free. Our decision procedures for quantifier-free theories are based on Oppen's algorithm for acyclic recursive data structures with infinite atom domain [17]. When integer constraints in the input are absent, our decision procedures can be viewed as an extension of Oppen's original algorithm to cyclic structures and to the structures with finite atom domain.

Related Work Our component theories are both decidable. Presburger arithmetic was first shown to be decidable in 1929 by the quantifier elimination method [6]. Efficient algorithms were later discovered by Cooper et al [4, 18]. It is well-known that recursive data structures can be modeled as term algebras which were shown to be decidable by the quantifier elimination method [13, 11, 8]. Decision procedures for the quantifier-free theory of recursive data structures were discovered by Nelson, Oppen et al [15, 17, 5]. In [17] Oppen gave a linear algorithm for acyclic structures and a quadratic algorithm was given in [15] for cyclic structures. If the values of the selector functions on atoms are specified, then the problem is NP-complete [17].

A general combination method for decision procedures for quantifier-free theories was developed by Nelson and Oppen in 1979 [14]. However, this method is not applicable to the combination of our component theories. The method requires that component theories be loosely coupled, that is, have disjoint signatures, and are stably infinite[1]. The method is not applicable to tightly coupled theories such as single-sorted theories with shared signatures or, as in our case, multisorted theories with functions mapping elements in one sort to another.

The integration of Presburger arithmetic with recursive data structures was discussed by Bjørner [1] and an incomplete procedure was implemented in STeP (Stanford Temporal Prover) [2]. Zarba constructed decision procedures for the combined theory of sets and integers [22] and multisets and integers [21] by extending the Nelson-Oppen combination method.

[1] A theory is stably infinite if a quantifier-free formula in the theory is satisfiable if and only if it is satisfiable in an infinite model.

Paper Organization Section 2 provides the preliminaries: it introduces the notation and terminology. Section 3 defines recursive data structures and explains Oppen's algorithm, the basis for our decision algorithm for quantifier-free theories. Section 4 introduces the combined theory of recursive data structures and Presburger arithmetic and outlines our approach for constructing the decision procedures. Sections 5-8 describe the four classes of decision procedures for the different variants of the theory in detail. Section 9 discusses complexity issues and Section 10 concludes with some ideas for future work. Because of space limitations most proofs have been omitted. They are available for reference in the extended version of this paper at http://theory.stanford.edu/~tingz/papers/ijcar04_extended.pdf.

2 Preliminaries

We assume the first-order syntactic notions of variables, parameters and quantifiers, and semantic notions of structures, satisfiability and validity as in [6].

A signature Σ is a set of parameters (function symbols and predicate symbols) each of which is associated with an arity. The function symbols with arity 0 are also called constants. The set of Σ-terms $\mathcal{T}(\Sigma, \mathcal{X})$ is recursively defined by: (i) every constant $c \in \Sigma$ or variable $x \in \mathcal{X}$ is a term, and (ii) if $f \in \Sigma$ is an n-place function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

An atomic formula (atom) is a formula of the form $P(t_1, \ldots, t_n)$ where P is an n-place predicate symbol and $t_1, \ldots, t_n$ are terms (equality is treated as a binary predicate symbol). A literal is an atomic formula or its negation. A ground formula is a formula with no variables. A variable occurs free in a formula if it is not in the scope of a quantifier. A sentence is a formula in which no variable occurs free. A formula without quantifiers is called quantifier-free. Every quantifier-free formula can be put into disjunctive normal form, that is, a disjunction of conjunctions of literals.

A Σ-structure (or Σ-interpretation) $\mathfrak{A}$ is a tuple $\langle A, I \rangle$ where A is a non-empty domain and I is a function that associates each n-place function symbol f (resp. predicate symbol P) with an n-place function $f^{\mathfrak{A}}$ (resp. relation $P^{\mathfrak{A}}$) on A. We usually denote $\mathfrak{A}$ by $\langle A; \Sigma \rangle$ which is called the signature of $\mathfrak{A}$. We use Gothic letters (like $\mathfrak{A}$) for structures and Roman letters (like A) for the underlying domain. A variable valuation (or variable assignment) ν (w.r.t. $\mathfrak{A}$) is a function that assigns each variable an element of A. The truth value of a formula is determined when an interpretation and a variable assignment is given.

A formula φ is satisfiable (or consistent) if it is true under some variable valuation; it is unsatisfiable (or inconsistent) otherwise. A formula φ is valid if it is true under every variable valuation. A formula φ is valid if and only if $\neg\varphi$ is unsatisfiable. We say that $\mathfrak{A}$ is a model of a set T of sentences if every sentence in T is true in $\mathfrak{A}$. A sentence φ is (logically) implied by T (or T-valid), written $T \models \varphi$, if φ is true in every model of T. Similarly we say that φ is T-satisfiable if φ is true in some model of T and it is T-unsatisfiable otherwise. The notions of (T-)validity and (T-)satisfiability naturally extend to a set of formulae. A theory

T is a set of sentences that is closed under logical implication, that is, if $T \models \varphi$, then $\varphi \in T$. By a theory of structure $\mathfrak{A}$, written $\mathsf{Th}(\mathfrak{A})$, we shall mean the set of all valid sentences in $\mathfrak{A}$. We write $\mathsf{Th}^{\forall}(\mathfrak{A})$ for the quantifier-free fragment of $\mathsf{Th}(\mathfrak{A})$.

A term algebra (TA) of Σ with basis $\mathcal{X}$ is the structure $\mathfrak{A}$ whose domain is $\mathcal{T}(\Sigma,\mathcal{X})$ and for any n-place function symbol $f \in \Sigma$ and $t_1,\ldots,t_n \in \mathcal{T}(\Sigma,\mathcal{X})$, $f^{\mathfrak{A}}(t_1,\ldots,t_n) = f(t_1,\ldots,t_n)$. We assume that Σ does not contain any predicate symbols except equality.

Presburger arithmetic (PA) is the first-order theory of addition in the arithmetic of integers. The corresponding structure is denoted by $\mathfrak{A}_{\mathbb{Z}} = \langle \mathbb{Z}; 0, +, < \rangle$.

In this paper all decision procedures for quantifier-free theories are refutation-based; to determine the validity of a formula φ it suffices to determine the unsatisfiability of $\neg\varphi$, which further reduces to determining the unsatisfiability of each disjunct in the disjunctive normal form of $\neg\varphi$. Henceforth, in discussions related to quantifier-free theories, an input formula always refers to a conjunction of literals.

3 Recursive Data Structures and Oppen's Algorithm

We present a general language of recursive data structures. For simplicity, we do not distinguish syntactic terms in the language from semantic terms in the corresponding interpretation. The meaning should be clear from the context.

Definition 1. *A recursive data structure $\mathfrak{A}_\lambda : \langle \lambda; \mathcal{A}, \mathcal{C}, \mathcal{S}, \mathcal{T} \rangle$ consists of*

1. *λ: The data domain, which consists of all terms built up from constants by applying constructors. Elements in λ are called λ-terms (or data terms).*
2. *$\mathcal{A}$: A set of atoms (constants): a, b, c, ...*
3. *$\mathcal{C}$: A finite set of constructors: α, β, γ, ... The arity of α is denoted by $\mathsf{ar}(\alpha)$. We say that an object is α-typed (or an α-term) if its outmost constructor is α.*
4. *$\mathcal{S}$: A finite set of selectors. For each constructor α with arity $k > 0$, there are k selectors $\mathsf{s}_1^\alpha, \ldots, \mathsf{s}_k^\alpha$ in $\mathcal{S}$. We call s_i^α $(1 \le i \le k)$ the i^{th} α-selector. For a term x, $\mathsf{s}_i^\alpha(x)$ returns the i^{th} component of x if x is an α-term and x itself otherwise.*
5. *$\mathcal{T}$: A finite set of testers. For each constructor α there is a corresponding tester Is_α. For a term x, $\mathsf{Is}_\alpha(x)$ is true if and only if x is an α-term. In addition there is a special tester Is_A such that $\mathsf{Is}_A(x)$ is true if and only if x is an atom. Note that there is no need for individual atom testers as $x = a$ serves as $\mathsf{Is}_a(x)$.*

The theory of recursive data structures is essentially the theory of term algebras (with the empty variable basis) which is axiomatizable as follows.

Proposition 1 (Axiomatization of Recursive Data Structures [8]). *Let $\bar{z}_\alpha$ abbreviate $z_1, \ldots, z_{\mathsf{ar}(\alpha)}$. The following formula schemes, in which variables are implicitly universally quantified over λ, axiomatize $\mathsf{Th}(\mathfrak{A}_\lambda)$.*

A. $t(x) \neq x$, *if* t *is built solely by constructors and* t *properly contains* x.
B. $a \neq b$, $a \neq \alpha(x_1 \ldots, x_{\mathsf{ar}(\alpha)})$, *and* $\alpha(x_1 \ldots, x_{\mathsf{ar}(\alpha)}) \neq \beta(y_1, \ldots, y_{\mathsf{ar}(\beta)})$, *if* a *and* b *are distinct atoms and if* α *and* β *are distinct constructors.*
C. $\alpha(x_1, \ldots, x_{\mathsf{ar}(\alpha)}) = \alpha(y_1, \ldots, y_{\mathsf{ar}(\alpha)}) \rightarrow \bigwedge_{1 \leq i \leq \mathsf{ar}(\alpha)} x_i = y_i$.
D. $\mathsf{Is}_\alpha(x) \leftrightarrow \exists\ \bar{z}_\alpha \alpha(\bar{z}_\alpha) = x$, $\mathsf{Is}_A(x) \leftrightarrow \bigwedge_{\alpha \in \mathcal{C}} \neg \mathsf{Is}_\alpha(x)$.
E. $\mathsf{s}_i^\alpha(x) = y \leftrightarrow \exists \bar{z}_\alpha \big(\alpha(\bar{z}_\alpha) = x \wedge y = z_i)\big) \vee \big(\forall \bar{z}_\alpha (\alpha(\bar{z}_\alpha) \neq x) \wedge x = y\big)$.

In general, selectors and testers can be defined by constructors and vice versa. One direction has been shown by (D) and (E), which are pure definitional axioms.

Example 1. Consider the LISP list structure $\mathfrak{A}_{\mathsf{List}} = \langle \mathsf{List};\ \mathsf{cons}, \mathsf{car}, \mathsf{cdr} \rangle$ where List denotes the domain, cons is the 2-place constructor (pairing function) and car and cdr are the corresponding left and right selectors (projectors) respectively. Let $\{\mathsf{car}, \mathsf{cdr}\}^+$ denote any nonempty sequence of car and cdr. The axiom schemas in Proposition 1 reduce to the following.

$$(i)\ \mathsf{Is}_A(x) \leftrightarrow \neg \mathsf{Is}_{\mathsf{cons}}(x), \quad (ii)\ \mathsf{car}(\mathsf{cons}(x, y)) = x, \quad (iii)\ \mathsf{cdr}(\mathsf{cons}(x, y)) = y,$$
$$(iv)\ \mathsf{Is}_A(x) \leftrightarrow \{\mathsf{car}, \mathsf{cdr}\}^+(x) = x, \quad (v)\ \mathsf{Is}_{\mathsf{cons}}(x) \leftrightarrow \mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x)) = x.$$

Decision Procedures for Recursive Data Structures

In [17] Oppen presented a decision procedure for acyclic data structures $\mathfrak{A}_\lambda$. The basic idea of the decision procedure is to generate all equalities between terms implied by asserted equalities in the formula and check for inconsistencies with the asserted disequalities in the formula.

The decision procedure relies on the fact that $\mathsf{Th}(\mathfrak{A}_\lambda)$ is convex.

Definition 2 (Convexity). *A theory is* convex *if whenever a conjunction of literals implies a disjunction of atoms, it also implies one of the disjuncts.*

Let Φ be a conjunction of literals and Ψ a disjunction of equalities. The convexity of $\mathsf{Th}(\mathfrak{A}_\lambda)$ can be rephrased as follows: if none of the disjuncts in Ψ is implied by Φ, then $\neg\Psi$ is Φ-satisfiable. Hence Φ is satisfiable if and only if for any terms s and t, whenever $s \neq t \in \Phi$, $\Phi \not\models s = t$. The idea of Oppen's algorithm is to discover all logically implied equalities (between terms in Φ) using the DAG representation and the bidirectional closure algorithm, which we introduce below.

Definition 3 (DAG Representation). *A term* t *can be represented by a tree* T_t *such that (i) if* t *is a constant or variable, then* T_t *is a leaf vertex labeled by* t*, and (ii) if* t *is in the form* $\alpha(t_1, \ldots, t_k)$*, then* T_t *is the tree having the root labeled by* t *and having* $T_{t_1}, \ldots, T_{t_k}$ *as its subtrees. A* directed acyclic graph *(DAG)* G_t *of* t *is obtained from* T_t *by "factoring out" the common subtrees (subterms).*

For a vertex u, let $\pi(u)$ denote the label, $\delta(u)$ the outgoing degree and $u[i]$ $(1 \leq i \leq \delta(u))$ the i^{th} successor of u. The DAG of a formula is the DAG representing all terms in the formula. For example, Figure 1 shows the DAG for $\mathsf{cons}(y, z) = \mathsf{cons}(\mathsf{cdr}(x), z) \wedge \mathsf{cons}(\mathsf{car}(x), y) \neq x$ under the assumption that x is not an atom.

Definition 4 (Bidirectional Closure). *Let R be a binary relation on a DAG and let u, v be any two vertices such that $\delta(u) = \delta(v)$. We say that R' is the* unification closure *of R (denoted by $R\downarrow$) if R' is the smallest equivalence relation extending R such that $\pi(u) = \pi(v)$ implies $(u[i], v[i]) \in R'$ for $1 \leq i \leq \delta(u)$. We say that R' is the* congruence closure *of R (denoted by $R\uparrow$) if R' is the smallest equivalence relation extending R such that $(u[i], v[i]) \in R'$ $(1 \leq i \leq \delta(u))$ implies $\pi(u) = \pi(v)$. If R' is both unification and congruence closed (w.r.t. R), we call it the* bidirectional closure*, denoted by $R\Updownarrow$.*

Let R be the set of all pairs asserted equal in Φ. It has been shown that $R\Updownarrow$ represents all equalities logically implied by Φ [17]. Therefore Φ is unsatisfiable if and only if there exists t and s such that $t \neq s \in \Phi$ and $(t, s) \in R\Updownarrow$.

Algorithm 1 (Oppen's Decision Procedure for Acyclic $\mathfrak{A}_\lambda$ [17]).
Input: $\Phi : q_1 = r_1 \wedge \ldots \wedge q_k = r_k \ \wedge \ s_1 \neq t_1 \wedge \ldots \wedge s_l \neq t_l$

1. *Construct the DAG G of Φ.*
2. *Compute the bidirectional closure $R\Updownarrow$ of $R = \{(q_i, r_i) \mid 1 \leq i \leq k\}$.*
3. *Return* FAIL *if $\exists i (s_i, t_i) \in R\Updownarrow$; return* SUCCESS *otherwise.*

In our setting $\mathfrak{A}_\lambda$ is cyclic and values of α-selectors on non α-terms are specified, e.g., $s_i^\alpha(x) = x$ if x is not an α-term. It was shown that for such structures the decision problem is NP-complete [15]. The complication is that it is not known a priori whether $\mathsf{s}(x)$ is a proper subterm of x and hence it is not possible to use the DAG representation directly. A solution to this problem is to guess the type information of terms occurring immediately inside selector functions before applying Algorithm 1.

Definition 5 (Type Completion). *Φ' is a* type completion *of Φ if Φ' is obtained from Φ by adding tester predicates such that for any term $\mathsf{s}(t)$ either $\mathsf{Is}_\alpha(t)$ (for some constructor α) or $\mathsf{Is}_A(t)$ is present in Φ'.*

Example 2. A possible type completion for $y = \mathsf{car}(\mathsf{cdr}(x))$ is $y = \mathsf{car}(\mathsf{cdr}(x)) \wedge \mathsf{Is}_{\mathsf{cons}}(x) \wedge \mathsf{Is}_A(\mathsf{cdr}(x))$.

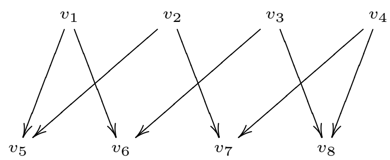
A type completion Φ' is compatible with Φ if the satisfiability of Φ implies that Φ' is satisfiable and if any solution of Φ' is a solution of Φ. Obviously Φ is satisfiable if and only if it has a satisfiable compatible completion. This leads to the following nondeterministic algorithm that relies on the successful guess of a satisfiable compatible completion if such completion exists.

Algorithm 2 (The Decision Procedure for Cyclic $\mathfrak{A}_\lambda$). *Input: Φ.*

1. *Guess a type completion Φ' of Φ and simplify selector terms accordingly.*
2. *Call Algorithm 1 on Φ'.*

Example 3. Figure 1 shows the DAG representation of

$$\mathsf{cons}(y, z) = \mathsf{cons}(\mathsf{cdr}(x), z) \wedge \mathsf{cons}(\mathsf{car}(x), y) \neq x \tag{1}$$



$v_1 : x$	$v_5 : \mathsf{car}(x)$
$v_2 : \mathsf{cons}(\mathsf{car}(x), y)$	$v_6 : \mathsf{cdr}(x)$
$v_3 : \mathsf{cons}(\mathsf{cdr}(x), z)$	$v_7 : y$
$v_4 : \mathsf{cons}(y, z)$	$v_8 : z$

Fig. 1. The DAG of (1) under the assumption $\mathsf{Is}_{\mathsf{cons}}(x)$.

under the guess $\mathsf{Is}_{\mathsf{cons}}(x)$. Initially $R = \{(v_3, v_4)\}$ as v_3 and v_4 are asserted equal in (1). (For simplicity reflexive pairs are not listed.) By the unification algorithm (v_6, v_7) are merged, which gives $R{\downarrow} = \{(v_3, v_4), (v_6, v_7)\}$. Then by the congruence algorithm (v_1, v_2) are merged, resulting in $R{\Updownarrow} = \{(v_1, v_2), (v_3, v_4), (v_6, v_7)\}$. Obviously this branch fails as $v_1 \neq v_2$ is asserted by (1). The remaining branch (with presence of $\mathsf{Is}_A(x)$) simplifies to $\mathsf{Is}_A(x) \wedge x = y$ which is clearly satisfiable, and therefore so is (1).

Note that the correctness of both Algorithm 1 and 2 relies on the (implicit) assumption that the atom domain is infinite, since otherwise the theory is not convex. As a counter example, for the structure $\mathfrak{A}_\lambda$ with only two atoms a and b, we have $\mathsf{Is}_A(x) \models x = a \vee x = b$, but neither $\mathsf{Is}_A(x) \models x = a$ nor $\mathsf{Is}_A(x) \models x = b$. We shall see (in Section 6) that our algorithm extends Oppen's original algorithm to structures with finite atom domain.

4 The Approach for the Integrated Theory

In this section we describe the different variants of the integrated theory and outline our approach for constructing the decision procedures for each of these variants. The details of each of these four decision procedures are presented in the next four sections.

Definition 6. *The structure of the integrated theory is* $\mathfrak{B}_\lambda = (\mathfrak{A}_\lambda; \mathfrak{A}_\mathbb{Z}; |.| : \lambda \to \mathbb{Z})$ *where* $\mathfrak{A}_\lambda$ *is a recursive data structure,* $\mathfrak{A}_\mathbb{Z}$ *is Presburger arithmetic, and* $|.|$ *denotes the length function defined recursively by: (i) for any atom* a, $|a| = 1$, *and (ii) for a term* $\alpha(t_1, \ldots, t_k)$, $|\alpha(t_1, \ldots, t_k)| = \sum_{i=1}^{k} |t_i|$.

Notice that we have chosen a length function that does not count the intermediate vertices. However, the algorithms can easily be modified for an alternative length function. We view terms of the form $|t|$ (where t is a non-ground λ-term) as ($\mathsf{generalized}$) integer variables. If ν is an assignment of data variables, let $|\nu|$ denote the corresponding assignment of generalized integer variables. From now on, we use $\Phi_\mathbb{Z}$ (resp. Φ_λ) to denote a conjunction of integer literals (resp. a conjunction of data literals). Whenever it is clear from context, we write $\mathfrak{B}$ for $\mathfrak{B}_\lambda$. By $\mathfrak{B}^\omega$, $\mathfrak{B}^{<\omega}$ and $\mathfrak{B}^{=k}$ $(k > 0)$ we denote the structures with infinitely many atoms, with finitely many atoms and with exactly k atoms, respectively.

As was mentioned before, the general purpose combination method in [14] is not directly applicable due to the presence of the length function. The following example illustrates the problem.

Example 4. Consider $\mathfrak{B}^{\omega}_{\mathsf{List}}$. The constraints $\Phi_\lambda : x = \mathsf{cons}(\mathsf{car}(y), y)$ and $\Phi_\mathbb{Z} : |x| < 2|\mathsf{car}(x)|$ are clearly satisfiable, respectively, in $\mathfrak{A}_{\mathsf{List}}$ and $\mathfrak{A}_\mathbb{Z}$. However, since Φ_λ implies that $\mathsf{car}(x) = \mathsf{car}(y)$, x contains two copies of $\mathsf{car}(y)$ and so its length should be at least two times the length of $\mathsf{car}(x)$. Therefore, $\Phi_\mathbb{Z} \wedge \Phi_\lambda$ is unsatisfiable.

A simple but crucial observation is that constraints of data structures impose "hidden" constraints on the lengths of those structures. Here we distinguish two types of integer constraints; "external integer constraints" that occur explicitly in input formulae, and "internal integer constraints" that are "induced" by satisfying assignments for the pure data structure constraints. In the following we will show that if we can express sound and complete length constraints (in the sense defined below) in Presburger arithmetic, we can derive decision procedures by utilizing the decision procedures for Presburger arithmetic and for recursive data structures.

Definition 7 (Induced Length Constraint). *A* length constraint *Φ_Δ with respect to Φ_λ is a Presburger formula in which only generalized integer variables occur free. Φ_Δ is* sound *if for any satisfying assignment ν_λ of Φ_λ, $|\nu_\lambda|$ is a satisfying assignment for Φ_Δ. Φ_Δ is* complete *if, whenever Φ_λ is satisfiable, for any satisfying assignment ν_Δ of Φ_Δ there exists a satisfying assignment ν_λ of Φ_λ such that $|\nu_\lambda| = \nu_\Delta$. We also say that Φ_Δ is* realizable *w.r.t. Φ_λ provided that Φ_λ is satisfiable. We say that Φ_Δ is* induced *by Φ_λ if Φ_Δ is both sound and complete.*

Example 5. Consider the formula $\Phi : \mathsf{cons}(x, y) = z$. The length constraint $|x| < |z| \wedge |y| < |z|$ is sound but it is not complete for Φ, as the integer assignment $\nu_\Delta : \{|x| = 3, |y| = 3, |z| = 4\}$ can not be realized. On the other hand, $|x| + |y| = |z| \wedge |x| > 5 \wedge |y| > 0$ is complete for Φ, but it is not sound because it does not satisfy the data assignment $\nu_\lambda : \{x = a, y = a, z = \mathsf{cons}(a, a)\}$. Finally, $|x| + |y| = |z| \wedge |x| > 0 \wedge |y| > 0$ is both sound and complete, and hence is the induced length constraint of Φ.

Main Theorem. *Let Φ be in the form $\Phi_\mathbb{Z} \wedge \Phi_\lambda$. Let Φ_Δ be the induced length constraint with respect to Φ_λ. Then Φ is satisfiable in $\mathfrak{B}$ if and only if $\Phi_\Delta \wedge \Phi_\mathbb{Z}$ is satisfiable in $\mathfrak{A}_\mathbb{Z}$ and Φ_λ is satisfiable in $\mathfrak{A}_\lambda$.*

Proof. ("$\Rightarrow$") Suppose that Φ is satisfiable and let ν be a satisfying assignment. ν divides into two disjoint parts: ν_λ for data variables and $\nu_\mathbb{Z}$ for integer variables. Φ_λ is obviously satisfiable under ν_λ. Let $\nu_\Delta = |\nu_\lambda|$. By soundness of Φ_Δ we know that ν_Δ satisfies Φ_Δ. Hence $\Phi_\Delta \wedge \Phi_\mathbb{Z}$ is satisfiable under the joint assignment of ν_Δ and $\nu_\mathbb{Z}$.
("$\Leftarrow$") Suppose that both $\Phi_\Delta \wedge \Phi_\mathbb{Z}$ and Φ_λ are satisfiable. Let ν be a satisfying assignment for $\Phi_\Delta \wedge \Phi_\mathbb{Z}$. ν divides into two disjoint parts: $\nu_\mathbb{Z}$ for pure integer

variables and ν_Δ for the generalized integer variables. By the completeness of Φ_Δ, ν_Δ can be realized by a satisfying assignment ν_λ such that $\nu_\Delta = |\nu_\lambda|$. Hence the joint assignment $\nu_\lambda \cup \nu_\mathbb{Z} \cup \nu_\Delta$ will satisfy Φ. $\square$

By the main theorem the decision problem for quantifier-free theories reduces to computing the induced length constraints in Presburger arithmetic. In the next two sections we show that this is possible for the two quantifier-free variants of our theory.

5 The Decision Procedure for $\mathsf{Th}^\forall(\mathfrak{B}^\omega)$

The first and easiest of the four variants considered is the quantifier-free combination with an infinite atom domain. In the structure $\mathfrak{B}^\omega$ the induced length constraints of a formula can be derived directly from the DAG for the formula. Before we present the algorithm we define the following predicates on terms in the DAG:

$$\begin{array}{ll}
\mathsf{Tree}(t) & : \exists x_1, \ldots, x_n \geq 0 \;\; (\; |t| = (\sum_{i=1}^{n}(d_i - 1)x_i) + 1) \\
\mathsf{Node}^\alpha(t, \bar{t}_\alpha) & : |t| = \sum_{i=1}^{\delta(\alpha)} |t_i| \\
\mathsf{Tree}^\alpha(t) & : \exists \bar{t}_\alpha \left(\mathsf{Node}^\alpha(t, \bar{t}_\alpha) \wedge \bigwedge_{i=1}^{\delta(\alpha)} \mathsf{Tree}(t_i) \right)
\end{array}$$

where $\bar{t}_\alpha$ stands for $t_1, \ldots, t_{\delta(\alpha)}$ and and $d_1, \ldots, d_n$ are the distinct arities of the constructors. The predicate $\mathsf{Tree}(t)$ is true iff $|t|$ is the length of a well-formed tree, since whenever a leaf expands one level with outgoing degree d, the length of the tree increases by $d - 1$. The second predicate forces the length of an α-typed node with known children to be the sum of the lengths of its children. The last predicate states the length constraint for an α-typed leaf. With these predicates the construction of the induced length constraint is given by the following algorithm.

Algorithm 3 (Construction of Φ_Δ in $\mathfrak{B}^\omega$). *Let Φ_λ be a (type-complete) data constraint, G_λ the DAG of Φ_λ and $R\Updownarrow$ the bidirectional closure obtained by Algorithm 1. Initially set $\Phi_\Delta = \emptyset$. For each term t add the following to Φ_Δ.*

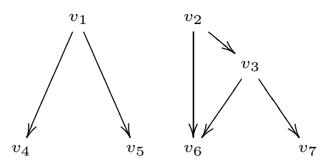
- $|t| = 1$, *if t is an atom;*
- $|t| = |s|$, *if $(t, s) \in R\Updownarrow$.*
- $\mathsf{Tree}(t)$ *if t is an untyped leaf vertex.*
- $\mathsf{Node}^\alpha(t, \bar{t}_\alpha)$ *if t is an α-typed vertex with children $\bar{t}_\alpha$.*
- $\mathsf{Tree}^\alpha(t)$ *if t is an α-typed leaf vertex.*

Proposition 2. *Φ_Δ obtained by Algorithm 3 is expressible in a quantifier-free Presburger formula linear in the size of Φ.*

Theorem 1. *Φ_Δ obtained by Algorithm 3 is the induced length constraint of Φ_λ.*

Algorithm 4 (Decision Procedure for $\mathsf{Th}^\forall(\mathfrak{B}^\omega)$). *Input: $\Phi_\lambda \wedge \Phi_\mathbb{Z}$.*

1. *Guess a type completion Φ'_λ of Φ_λ.*



$v_1 : x$ $\quad$ $v_5 : \mathsf{cdr}(x)$
$v_2 : \mathsf{cons}(\mathsf{car}(y), y)$ $\quad$ $v_6 : \mathsf{car}(y)$
$v_3 : y$ $\quad$ $v_7 : \mathsf{cdr}(y)$
$v_4 : \mathsf{car}(x)$

Fig. 2. The DAG of (2) under the assumption $\mathsf{Is}_{\mathsf{cons}}(y)$

2. *Call Algorithm 1 on Φ'_λ.*
   – *Return* FAIL *if Φ'_λ is unsatisfiable; continue otherwise.*
3. *Construct Φ_Δ from G'_λ using Algorithm 3.*
   – *Return* SUCCESS *if Φ_Δ is satisfiable and $\Phi_\mathbb{Z}$ is satisfiable.*
   – *Return* FAIL *otherwise.*

The correctness of the algorithm follows from Main Theorem and Theorem 1.

Example 6. Figure 2 shows the DAG of

$$x = \mathsf{cons}(\mathsf{car}(y), y) \wedge |\mathsf{cons}(\mathsf{car}(y), y)| < 2|\mathsf{car}(x)|, \tag{2}$$

assuming that y is not an atom. The computed $R\Updownarrow$ is $\{(v_1, v_2), (v_3, v_5), (v_4, v_6)\}$. By Algorithm 3 Φ_Δ includes the conjunction $|\mathsf{cons}(\mathsf{car}(y), y)| = |\mathsf{car}(y)| + |y| \wedge |\mathsf{car}(x)| = |\mathsf{car}(y)| \wedge |\mathsf{cdr}(x)| = |y|$ which implies $|\mathsf{cons}(\mathsf{car}(y), y)| \geq 2|\mathsf{car}(x)|$, contradicting $|\mathsf{cons}(\mathsf{car}(y), y)| < 2|\mathsf{car}(x)|$. If y is an atom, v_3, v_6, v_7 are merged. But still we have $|\mathsf{cons}(\mathsf{car}(y), y)| \geq 2|\mathsf{car}(x)|$. Therefore (2) is unsatisfiable.

6 The Decision Procedures for $\mathsf{Th}^\forall(\mathfrak{B}^{=k})$ and $\mathsf{Th}^\forall(\mathfrak{B}^{<\omega})$

Algorithm 3 cannot be used to construct the induced length constraints in structures with a finite number of atoms, $\mathfrak{B}^{=k}$, as illustrated by the following example.

Example 7. Consider $\mathfrak{B}^{=1}_{\mathsf{List}}$ (with atom a). The constraint

$$|x| = 3 \wedge \mathsf{Is}_A(y) \wedge x \neq \mathsf{cons}(\mathsf{cons}(y, y), y) \wedge x \neq \mathsf{cons}(y, \mathsf{cons}(y, y)) \tag{3}$$

is unsatisfiable while Φ_Δ obtained by Algorithm 3 is

$$|y| = 1 \wedge |\mathsf{cons}(y, y)| = 2 \wedge |\mathsf{cons}(\mathsf{cons}(y, y), y)| = 3 \wedge |\mathsf{cons}(y, \mathsf{cons}(y, y)| = 3$$

which is obviously satisfiable together with $|x| = 3$.

The reason is that if the atom domain is finite there are only finitely many terms of length n for any $n > 0$. If a term t is forced to be distinct from all of them, then t cannot have length n. Therefore Φ_Δ needs to include constraints that count the number of distinct terms of a certain length.

Definition 8 (Counting Constraint). *A* counting constraint *is a predicate* $\mathsf{CNT}^{\alpha}_{k,n}(x)$ *that is* true *if and only if there are at least* $n+1$ *different* α*-terms of length* x *in the language with exactly* $k > 0$ *distinct atoms.* $\mathsf{CNT}_{k,n}(x)$ *is similarly defined with* α*-terms replaced by* λ*-terms.*

The following two monotonicity properties are easily proven: for any $l \geq k > 0$ and $m \geq n > 0$, (i) $\mathsf{CNT}^{\alpha}_{k,n}(x) \rightarrow \mathsf{CNT}^{\alpha}_{l,n}(x)$ and (ii) $\mathsf{CNT}^{\alpha}_{k,m}(x) \rightarrow \mathsf{CNT}^{\alpha}_{k,n}(x)$.

Example 8. For $\mathfrak{B}^{=1}_{\mathsf{List}}$, $\mathsf{CNT}^{\mathsf{cons}}_{1,n}(x)$ is $x \geq m$ where m is the least number such that the m-th Catalan number $C_m = \frac{1}{m}\binom{2m-2}{m-1}$ is greater than n. This is not surprising as C_m gives the number of binary trees with m leaf vertices.

In general we have the following result.

Proposition 3. $\mathsf{CNT}^{\alpha}_{k,n}(x)$ *is expressible by a quantifier-free Presburger formula that can be computed in time* $O(n)$*.*

In order to construct counting constraints, we need equality information between terms.

Definition 9 (Equality Completion). *An* equality completion Φ'_{λ} *of* Φ_{λ} *is formula obtained from* Φ_{λ} *such that for any two terms* u *and* v *in* Φ_{λ} *either* $u = v$ *or* $u \neq v$*, and either* $|u| = |v|$ *or* $|u| \neq |v|$ *are in* Φ'_{λ}*.*

As before we present a nondeterministic algorithm; Φ is satisfiable if and only if at least one of the compatible (type and equality) completions of Φ is. By $\mathsf{NEQ}_n(x_0, x_1, \ldots, x_n)$ we shall mean that $x_0, \ldots, x_n$ have the same length but are pairwise distinct.

Algorithm 5 (Construction of Φ_{Δ} in $\mathfrak{B}^{=k}$).
Input: Φ_{λ} *(type and equality complete),* G_{λ} *and* $R\Updownarrow$*.*

1. *Call Algorithm 3 to obtain* Φ_{Δ}*.*
2. *Add* $\mathsf{CNT}^{\alpha}_{k,n}(|t|)$ *to* Φ_{Δ} *for each* t *occurring in* $\mathsf{NEQ}_n(t, t_1, \ldots, t_n)$*.*

Proposition 4. Φ_{Δ} *obtained by Algorithm 5 is expressible in a quantifier-free Presburger formula and the size of such a formula is linear in the size of* Φ*.*

Theorem 2. Φ_{Δ} *obtained by Algorithm 5 is the induced length constraint of* Φ_{λ}*.*

Algorithm 6 (Decision Procedure for $\mathsf{Th}^{\forall}(\mathfrak{B}^{=k})$). *Input :* $\Phi_{\lambda} \wedge \Phi_{\mathbb{Z}}$*.*

1. *Guess a type and equality completion* Φ'_{λ} *of* Φ_{λ}*.*
2. *Call Algorithm 1 on* Φ'_{λ}*.*
   - *Return* FAIL *if* Φ'_{λ} *is unsatisfiable; continue otherwise.*
3. *Construct* Φ_{Δ} *from* G'_{λ} *using Algorithm 5.*
   - *Return* SUCCESS *if* Φ_{Δ} *is satisfiable and* $\Phi_{\mathbb{Z}}$ *is satisfiable.*
   - *Return* FAIL *otherwise.*

The correctness of Algorithm 6 follows from the Main Theorem and Theorem 2. Notice that, when $\Phi_{\mathbb{Z}}$ is empty, the algorithm can be viewed as an extension of Oppen's original algorithm for structures with finite atom domain.

Example 9. Let us return to Example 7. Constraint (3) has exactly one compatible completion, namely $\mathsf{NEQ}_3(x, \mathsf{cons}(\mathsf{cons}(y, y), y), \mathsf{cons}(y, \mathsf{cons}(y, y)))$. This results in the counting constraint $|x| \geq 4$, contradicting $|x| = 3$.

Algorithm 4 is also a decision procedure for $\mathsf{Th}^{\forall}(\mathfrak{B}^{<\omega})$ according to the following theorem.

Theorem 3. $\mathsf{Th}^{\forall}(\mathfrak{B}^{<\omega}) = \mathsf{Th}^{\forall}(\mathfrak{B}^{\omega})$ *in the languages with no constants.*

7 The Decision Procedure for $\mathsf{Th}(\mathfrak{B}^{\omega})$

In this section and the next one we show the decidability of $\mathsf{Th}(\mathfrak{B}^{\omega})$, $\mathsf{Th}(\mathfrak{B}^{=k})$ and $\mathsf{Th}(\mathfrak{B}^{<\omega})$ by extending the quantifier elimination procedure for the theory of term algebras [13, 11, 8] to our combined theory. A structure is said to "admit quantifier elimination" if any formula can be equivalently (and effectively) transformed into a quantifier-free formula. We demonstrate a procedure by which a sentence of $\mathfrak{B}$ is reduced to a ground quantifier-free formula (i.e., with no variable occurrence) whose validity can be easily checked. Our elimination procedures induce decision procedures for quantifier-free theories as satisfiability of a quantifier-free formula is the same as validity of its existential closure. However, unfortunately, the complexity lower bound of the theory of term algebras is non-elementary [3, 7].

It is well-known that eliminating arbitrary quantifiers reduces to eliminating existential quantifiers from formulae in the form $\exists x(A_1(x) \wedge \ldots \wedge A_n(x))$, where $A_i(x)$ $(1 \leq i \leq n)$ are literals [8]. In the rest of the paper, we assume that all transformations are done on formulae of this form. We may also assume that $A_i's$ are not of the form $x = t$ as $\exists x(x = t \wedge \varphi(x, \bar{y}))$ simplifies to $\varphi(t, \bar{y})$, if x does not occur in t, to $\exists x \varphi(x, \bar{y})$ if $t \equiv x$, and to false by Axiom (A) if t properly contains x. As before, we present nondeterministic algorithms, but now a formula is valid if and only if it is true in every existential branch.

For our two-sorted language, we show how to eliminate quantifiers on integer variables as well as quantifiers on data variables. It is easy to see the soundness of transformations in the elimination procedures presented in this section and the next one. We leave the termination proofs to the extended version of this paper.

Eliminate Quantifiers on Integer Variables

We may assume formulae with quantifiers on integer variables are in the form

$$\exists x : \mathbb{Z}\ (\Phi_{\mathbb{Z}}(x, \bar{y}, \bar{z}) \wedge \Phi_{\lambda}(\bar{z})), \tag{4}$$

where $\bar{y}$ (resp. $\bar{z}$) denotes a sequence of integer variables (resp. data variables). Since $\Phi_{\lambda}(\bar{z})$ does not contain x, we can move it out of the scope of $\exists x$, and obtain

$$\exists x : \mathbb{Z}\ (\Phi_{\mathbb{Z}}(x, \bar{y}, \bar{z})) \wedge \Phi_{\lambda}(\bar{z}). \tag{5}$$

Notice that in $\Phi_{\mathbb{Z}}(x, \bar{y}, \bar{z})$, $\bar{z}$ only occurs inside generalized integer variables of the form $|t|$. Therefore $\exists x : \mathbb{Z}(\Phi_{\mathbb{Z}}(x, \bar{y}, \bar{z}))$ is essentially a Presburger formula and we can proceed to remove the quantifier using Cooper's method [4].

Eliminate Quantifiers on Data Variables

We may assume formulae with quantifiers on data variables are in the form

$$\exists x : \lambda \ (\Phi_{\mathbb{Z}}(x, \bar{y}, \bar{z}) \wedge \Phi_{\lambda}(x, \bar{z})), \tag{6}$$

where $\bar{y}$ (resp. $\bar{z}$) denotes a sequence of integer variables (resp. data variables). As before in $\Phi_{\mathbb{Z}}(\bar{y}, x, \bar{z})$, $x, \bar{z}$ only occur inside integer terms. We may assume that (6) does not contain constructors (see Section 3).

First we make sure that x does not appear properly inside any terms. Suppose otherwise that $s(x)$ occurs for some selector s. As in [8], by guessing that x is α-typed, (6) becomes

$$\exists x, x_1, \ldots, x_{\mathsf{ar}(\alpha)} : \lambda \ \Big[\mathsf{Is}_{\alpha}(x) \wedge \bigwedge_{1 \le i \le \mathsf{ar}(\alpha)} s_i^{\alpha}(x) = x_i \wedge \Phi_{\lambda}(x, \bar{z}) \wedge \Phi_{\mathbb{Z}}(x, \bar{y}, \bar{z})\Big]. \tag{7}$$

We simplify (7) as follows: replace $x \neq t$ by $\bigvee_{1 \le i \le \mathsf{ar}(\alpha)} s_i^{\alpha}(t) \neq x_i \vee \neg \mathsf{Is}_{\alpha}(t)$ (which will cause disjunctive splittings), replace $s_i^{\alpha}(x)$ by x_i, replace $s_j^{\beta}(x)$ (for $\alpha \not\equiv \beta$) by x and replace $|x|$ by $\Sigma_{i=1}^{\mathsf{ar}(\alpha)} |x_i|$. After the simplification no x occurs in Φ_{λ} and $\Phi_{\mathbb{Z}}$, so we can remove $\bigwedge_{1 \le i \le \mathsf{ar}(\alpha)} s_i^{\alpha}(x) = x_i$, $\mathsf{Is}_{\alpha}(x)$ and the quantifier $\exists x$. Repeat the process if all of $x_1 \ldots x_{\mathsf{ar}(\alpha)}$ occur inside selector functions.

Although the transformation from (6) to (7) introduces new quantified variables, those new variables appear in smaller terms (compared with x). Therefore the process will eventually terminate[2], producing formulae of the form

$$\exists x : \lambda \ \big(\bigwedge_{i<n} x \neq t_i \wedge \Phi_{\lambda}(\bar{z}) \wedge \Phi_{\mathbb{Z}}(x, \bar{y}, \bar{z}) \big), \tag{8}$$

where x does not appear in $\Phi_{\lambda}(\bar{z})$ and t_i $(i < n)$. (8) says that there exists an x which is not equal to any terms of $t_0, \ldots t_n$ and whose length is constrained by $\Phi_{\mathbb{Z}}(x, \bar{y}, \bar{z})$. But in $\mathfrak{B}^{\omega}$ for any $n > 0$ there are infinitely many terms of length n. It follows that $\bigwedge_{i<n} x \neq t_i$ can be ignored and thus (8) is equivalent to

$$\exists n : \mathbb{Z} \ \Big[\mathsf{Tree}(x)\big[|x|/n\big] \wedge \Phi_{\lambda}(\bar{z}) \wedge \Phi_{\mathbb{Z}}(x, \bar{y}, \bar{z})\big[|x|/n\big]\Big], \tag{9}$$

where $\varphi[|x|/n]$ stands for the formula obtained from φ by substituting all occurrences of $|x|$ for n. Now (9) can be handled by the elimination procedure for integer quantifiers. Note that if $\mathsf{Is}_{\alpha}(x)$ is in (8) we use $\mathsf{Tree}^{\alpha}(x)$ instead of $\mathsf{Tree}(x)$ in (9). Also, if we guessed that x is an atom, we would directly arrive at (8) by substituting x for $s(x)$, and then to (9) with n instantiated to 1.

8 The Decision Procedures for $\mathsf{Th}(\mathfrak{B}^{=k})$ and $\mathsf{Th}(\mathfrak{B}^{<\omega})$

In $\mathfrak{B}^{=k}$ the reduction from (8) to (9) is not sound for the same reason as in Section 6. However, the technique (of adding counting constraints) is still applicable here. We first introduce some new notations.

[2] To guarantee termination, we require the next round variable selection be restrained to $\{x_1 \ldots x_{\mathsf{ar}(\alpha)}\}$. In other words, the variable selection is done in depth-first manner.

Definition 10 (Partitioning Formula). *Let x be a data variable and S be the set of data terms $\{t_1, \ldots, t_n\}$. Let P be a partition of S and $Q \subseteq P$. By a partitioning formula, written* $\mathsf{PART}[x, S, P, Q]$*, we shall mean*

$$\mathsf{NE}_\lambda[x, S] \wedge \underbrace{\mathsf{NE}'_\lambda[S, P] \wedge \mathsf{EQ}_\lambda[S, P] \wedge \mathsf{NE}_\mathbb{Z}[x, S, Q] \wedge \mathsf{EQ}_\mathbb{Z}[x, S, Q]}_{\mathsf{PART}'[x,S,P,Q]}, \tag{10}$$

where

$$\mathsf{NE}_\lambda[x, S] \equiv \bigwedge_{t \in S} x \neq t, \qquad \mathsf{NE}_\mathbb{Z}[x, S, Q] \equiv \bigwedge_{t \in S_i \notin Q} |x| \neq |t|,$$

$$\mathsf{NE}'_\lambda[S, P] \equiv \bigwedge_{t \in S_i \in P, t' \in S_j \in P, S_i \neq S_j} t \neq t', \qquad \mathsf{EQ}_\mathbb{Z}[x, S, Q] \equiv \bigwedge_{t \in S_i \in Q} |x| = |t|,$$

$$\mathsf{EQ}_\lambda[S, P] \equiv \bigwedge_{t, t' \in S_i \in P} t = t'.$$

In fact $\mathsf{PART}[x, S, P, Q]$ is an equality completion of terms $\{x\} \cup S$.

Example 10. Let $S = \{t_1, t_2, t_3\}$, $P = \{\{t_1, t_2\}, \{t_3\}\}$ and $Q = \{\{t_1, t_2\}\}$. Then

$\mathsf{NE}_\lambda[x, S] : x \neq t_1 \wedge x \neq t_2 \wedge x \neq t_3$ $\qquad$ $\mathsf{NE}_\mathbb{Z}[x, S, Q] : |x| \neq |t_3|$

$\mathsf{NE}'_\lambda[S, P] : t_1 \neq t_3 \wedge t_2 \neq t_3$ $\qquad$ $\mathsf{EQ}_\mathbb{Z}[x, S, Q] : |x| = |t_1| \wedge |x| = |t_2|$

$\mathsf{EQ}_\lambda[S, P] : t_1 = t_2$

The elimination of quantifiers on integer variables is the same as before. We show how to eliminate quantifiers on data variables as follows. Starting from (8) we guess a partition P of $S = \{t_1, \ldots, t_n\}$ and a set $Q \subseteq P$, resulting in

$$\exists x : \lambda \left[\mathsf{NE}_\lambda[x, S] \wedge \mathsf{PART}'[x, S, P, Q] \wedge \Phi_\lambda(\bar{z}) \wedge \Phi_\mathbb{Z}(x, \bar{y}, \bar{z}) \right], \tag{11}$$

which says that there exists an x such that (i) x is not equal to any terms in S, and (ii) S is partitioned into $|P|$ equivalence classes among which there are $|Q|$ classes whose members have the same length as x. But this is exactly what the counting constraint $\mathsf{CNT}_{k,|Q|}(|x|)$ states, and so we can transform (11) to

$$\exists n : \mathbb{Z} \Big[\mathsf{Tree}(x)\big[|x|/n\big] \wedge \mathsf{CNT}_{k,|Q|}(n) \wedge \\ \mathsf{PART}'[x, S, P, Q]\big[|x|/n\big] \wedge \Phi_\lambda(\bar{z}) \wedge \Phi_\mathbb{Z}(|x|/n, \bar{y}, \bar{z}) \Big]. \tag{12}$$

Again, we are left with the known task of eliminating integer quantifiers. As before, in case of presence of $\mathsf{Is}_\alpha(x)$, we use $\mathsf{Tree}^\alpha(x)$ and $\mathsf{CNT}^\alpha_{k,|Q|}(n)$ instead of $\mathsf{Tree}(x)$ and $\mathsf{CNT}_{k,|Q|}(n)$, respectively, in (12). If we had guessed that x is an atom, we would have (12) with n instantiated to 1.

Example 11. Consider, in $\mathfrak{B}^{=1}_{\mathsf{List}}$ (with atom a), $\exists x : \lambda(x \neq \mathsf{cons}(a, a) \wedge |x| = 2)$. The only compatible completion is $\exists x : \lambda(\mathsf{Is}_{\mathsf{cons}}(x) \wedge |x| = |\mathsf{cons}(a, a)| \wedge x \neq \mathsf{cons}(a, a) \wedge |x| = 2)$. This produces the counting constraint $\mathsf{CNT}^{\mathsf{cons}}_{1,1}(|x|)$ which is $|x| > 2$, contradicting $|x| = 2$.

We can also derive a decision procedure for $\mathsf{Th}(\mathfrak{B}^{<\omega})$ using the above elimination procedure.

Theorem 4. $\mathsf{Th}(\mathfrak{B}^{<\omega})$ *(in the languages with no constants) is decidable.*

9 Complexity

In this section we briefly discuss complexity of the quantifier theories. Let n be the input size of Φ. First it is not hard to see that both $\mathsf{Th}^{\forall}(\mathfrak{B}^{\omega})$ and $\mathsf{Th}^{\forall}(\mathfrak{B}^{=k})$ are NP-hard as they are super theories of $\mathsf{Th}^{\forall}(\mathfrak{A}_{\lambda})$ and $\mathsf{Th}^{\forall}(\mathfrak{A}_{\mathbb{Z}})$, either of which is NP-complete. Second, Algorithm 3 computes Φ_{Δ} in $O(n)$ (see Proposition 2) and so does Algorithm 5 (see Proposition 4). Third, the size of any type and equality completion of Φ is bounded by $O(n^2)$ as there are at most n^2 pairs of terms. By the nondeterministic nature of our algorithms, we see that each branch of computation (in Algorithm 4 and Algorithm 6 respectively) is in P. Therefore both $\mathsf{Th}^{\forall}(\mathfrak{B}^{\omega})$ (or $\mathsf{Th}^{\forall}(\mathfrak{B}^{<\omega})$) and $\mathsf{Th}^{\forall}(\mathfrak{B}^{=k})$ are NP-complete.

10 Conclusion

We presented four classes of decision procedures for recursive data structures integrated with Presburger arithmetic. Our technique is based on the extraction of sound and complete integer constraints from data constraints. We believe that this technique may apply to arithmetic integration of various other theories.

We plan to extend our results in two directions. The first is to reason about the combination of recursive data structures with integers in richer languages such as the theory of recursive data structures with subterm relation $\preceq$ [20]. The second is to relax the restriction of unique construction of data objects to enable handling of structures in which a data object can be constructed in more than one way such as in the theory of queues [1, 19] and word concatenation [12].

Recently it came to our attention that the combination of Presburger arithmetic and term algebras has been used in [9, 10] to show that the quantifier-free theory of term algebras with Knuth-Bendix order is NP-complete. For quantifier-free theories in finite signatures, the decidability result is more general than ours, as the language also includes the (Knuth-Bendix) ordering predicate. But on the other hand, our quantifier elimination method presented in this paper can be readily extended to show the decidability of the first-order theory of Knuth-Bendix order [23]. We will compare the work with ours in the extended version of this paper.

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