Ordinal Arithmetics: An Introduction

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1 Preliminaries

Definition 1.1 (Partial Order). *A relation R is said to be a partial order on set S if it is reflexive, transitive and antisymmetric. That is*

$$\forall x (xRx).$$
$$\forall x \forall y \forall z (xRy \wedge yRz \implies xRz).$$
$$\forall x \forall y [(xRy) \wedge (yRx) \implies (x = y)].$$

Instead of xRy, we usually write $x \preceq y$.

Definition 1.2 (Linear Order). *A relation R is said to be a linear order if it is a partial order and complete. The last condition means that*

$$\forall x \forall y (xRy \vee yRx).$$

We say that x *strictly precedes* y (y *strictly succeeds* x) if

$$xRy \text{ and } x \neq y.$$

We also say that x is a *predecessor* of y (y is a *successor* of x). We write $x \prec_R y$ (or simply $x \prec y$ if R is clear from the context). We also write $y \succ_R x$ or $y \succ x$. If x has no predecessor (successor), we say that x is the *first* element (the *last* element) in A. If $x \in A$ and the set $\{y \mid x \prec y \text{ and } y \in A\}$ has a first element, then this element is called a *immediate successor* of x. The last element of $\{y \mid y \prec x \text{ and } y \in A\}$ (if one exists) is called a *immediate predecessor* of x. A proper subset of X of the set A is said to be an *initial segment* (a *final segment*) if $x \in X$ implies that every element preceding x belongs to X (every element after x belons to X).

Definition 1.3 (Well-Order). *We say a relation R well orders a set X if R linearly orders X and every non-empty subset of X contains a minimum element (with respect to the relation R.)*

Example 1.1.

(1) $\langle \mathbb{N}, \leq \rangle$ is a well-ordered set.

(2) $\langle \mathbb{Z}, \leq \rangle$ is a linearly ordered set, but not a well-ordered set.

(3) $\langle \mathbb{Z}, R \rangle$ is a well-ordered set where R is defined as.

$$R = \{\langle x, y \rangle \mid |x| < |y| \vee (|x| = |y| \wedge x \leq y)\}.$$

In this ordering numbers with smaller absolute value precedes numbers with larger absolute value. In case of tie, negative numbers precedes positive numbers.

Let $x \in A$. Define

$$O_R(x) = \{y \mid y \prec x\}$$

It is clearly that $O_R(x)$ is an initial segment. However, not every initial segment is of the form $O_R(x)$. If we require A be a well-ordered set, we have

Lemma 1.1. *Each initial segment X of a well-ordered set A is of the form $O(x)$ for some $x \in A$.*

Proof. Take the first element of the difference $A - X$, then $O(x) = X$. $\square$

Theorem 1.1 (Principle of Transfinite Induction). *If a set A is well-ordered, $B \subset A$ and if for every $x \in A$ the set B satisfies the condition*

$$(O(x) \subset B) \implies (x \in B)$$

then $B = A$.

Proof. Suppose that $A - B \neq \emptyset$. Let x be the first element in $A - B$. That means that if $y \prec x$ then $y \notin A - B$, that is, $y \in B$. Hence, $O(x) \subset B$. It follows from the assumption that $x \in B$, contradicting the hypothesis that $x \notin B$. $\square$

Definition 1.4 (Equivalence). *The set A is equivalent to the set B if there exists a* 1-1 *onto function $f : A \mapsto B$.*

Definition 1.5 (Isomorphism). *Two partially ordered sets $\langle A, \preceq_A \rangle$ and $\langle B, \preceq_B \rangle$ are said to be isomorphic if there exists a* 1-1 *onto function $f : A \mapsto B$ such that for any $x, y \in A$*

$$x \preceq_A y \iff f(x) \preceq_B f(y).$$

We say that f is an isomorphism of A and B, denoted by $A \simeq_f B$ (or simply $A \simeq B$).

Example 1.2.

(1) $J(x, y)$ (Enderton 3.3) establishes an equivalence of $\mathbb{N}^2$ *and* $\mathbb{N}$.

(2) $\langle \{-1\} \cup \mathbb{N}, \leq \rangle \simeq \langle \mathbb{N}, \leq \rangle$. *The isomorphism is established by the function*

$$f : \{-1\} \cup \mathbb{N} \mapsto \mathbb{N}, f(x) = x - 1.$$

Clearly isomorphic sets are equivalent. The converse holds for finite sets only.

Theorem 1.2. *Two finite linearly ordered equivalent sets are isomorphic.*

Proof Sketch. By induction on the size of the sets. $\square$

We list several properties of well-ordered sets which we will use to show ordinal properties in latter sections. We skip some proofs. See [1] Page 233-235 for details. Let $\langle A, \preceq_A \rangle$ be a linearly ordered set. A function f which establishes isomorphism of A and the set $f(A)$ contained in A is an *increasing function* if the following condition holds:

$$x \prec y \implies f(x) \prec f(y).$$

Theorem 1.3. *If a function f defined on a well-ordered set A is increasing, then for every x we have $x \preceq f(x)$.*

Proof. See [1], Page 230. $\square$

Theorem 1.4. *If the well-ordered sets A and B are isomorphic, then there exists only one function which establishes their isomorphism.*

Proof. See [1], Page 231. $\square$

Theorem 1.5. *No well-ordered set is isomorphic to any of its initial segments.*

Proof. Suppose that there exists a function f which establishes the isomorphism of A and $O(x)$ for some $x \in A$. Then f is increasing and $f(x) \in O(x)$, that is $f(x) \prec x$, which contradict Theorem 1.3. $\square$

Theorem 1.6. *No two distinct initial segments of a well-ordered set are isomorphic.*

Proof. By Theorem 1 and the observation that given any two distinct initial segments one is always an initial segment of the other. $\square$

Theorem 1.7. *Let A and B be two well-ordered sets. Then either*

(1) A and B are isomorphic, or

(2) A is isomorphic to an initial segment of B, or

(3) B is isomorphic to an initial segment of A.

Proof. This theorem is due to Cantor; see [1] Page 231. $\square$

Definition 1.6 (Ordinals). *Two isomorphic ordered systems are said to be of the same order type. By ordinal numbers (or ordinals) we mean the order types of well-ordered sets.*

Ordinals are defined as the order types of equivalence classes of well-ordered sets with respect to $\simeq$ relation. We denote by $\overline{A}$ the type of set A. From now on, we use *ordinals* and *order types* interchangablly.

Definition 1.7 (Ordinal Ordering). *An ordinal α is less than an ordinal β if any set of type α is isomorphic to an initial segment of a set of type β. We denote the relation by $\alpha \prec \beta$.*

We write $\alpha \preceq \beta$ if $\alpha \prec \beta$ or $\alpha = \beta$. We say ordinal α is the *immediate predecessor* of ordinal β (respetively, β is the *immediate successor* of α) if $\alpha \neq \beta$ and there is no ordinal γ such that $\alpha \prec \gamma \prec \beta$ or $\beta \prec \gamma \prec \alpha$. As any well-ordered sets with n elements are isomorphic, we can denote by $\mathbf{n}$ their order type.

Example 1.3.

(1) $\mathbf{0}$ is an ordinal that represents the order type of $\emptyset$.

(2) $\mathbf{n}$ is an ordinal that represents the order type of $\{0, 1, \dots, n-1\}$.

(3) ω is an ordinal that represents the order type of $\langle \mathbb{N}, \leq \rangle$.

There are several well-known results describing ordinal properties.

Theorem 1.8 (Transitivity). *For any ordinals α, β and γ if $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$.*

Proof Sketch. Let $\overline{A} = \alpha$, $\overline{B} = \beta$, $\overline{C} = \gamma$. The fact A is isomorphic to an initial segment of B and B is isomorphic to an initial segment of C implies that A is isomorphic to an initial segment of C. □

Theorem 1.9 (Antisymmetry). *For any ordinals α, β if $\alpha \preceq \beta$ and $\beta \preceq \alpha$, then $\alpha = \beta$. .*

Proof. If $\alpha \neq \beta$, then $\alpha \prec \beta$ and $\beta \prec \alpha$. By Theorem 1.8 $\alpha \prec \alpha$ which contradict Theorem 1. □

Theorem 1.10 (Trichotomy). *For any ordinals α and β one and only one of the formulas $\alpha \prec \beta$, $\alpha = \beta$, $\alpha \succ \beta$ holds.*

Proof. It follows directly from ordinal definition and Theorem 1.7. □

Obviously, $\alpha \preceq \alpha$. If we denote by $\mathcal{ORD}$ the class of all ordinals, $\langle \mathcal{ORD}, \preceq \rangle$ is linearly ordered (if we generalizes ordering relations to classes.)

Theorem 1.11. *If the well-ordered sets A and B are of type α and β and if the set A is isomorphic to a subset B' of B, then $\alpha \preceq \beta$.*

Proof. By Theorem 1.10, if this were not so, we would have $\beta \prec \alpha$, that is B is isomorphic to an initial segment of B'. However the existence of any increasing function which establishes the isomorphism contradicts Theorem 1.3. □

Theorem 1.12. *The set $W(\alpha)$ consisting of all ordinals less than α is well ordered by relation $\leq$. Moreover, the type of $W(\alpha)$ is α.*

Theorem 1.13. *Every set of ordinals is well ordered by the relation $\succeq$. In other words, in any non-empty set Z of ordinals there exists a smallest ordinals.*

Theorem 1.14. *For every set Z of ordinals there exists an ordinal greater than all ordinals belonging to Z.*

Corollary 1.1. *There exist no set of all ordinals.*

Corollary 1.2. *There exists a smallest ordinal not belonging to a given set Z.*

Theorem 1.15 (Cantor Normal Form). *If an ordinal $\alpha \succ 0$ then there exist a natural number n and sequences $\alpha_1 \ldots \alpha_n$ such that*

$$\alpha = \sum_{i=1}^{n} \omega^{\alpha_i} = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}.$$

where

$$\alpha_1 \succeq \ldots \succeq \alpha_n.$$

We can also write

$$\alpha = \sum_{i=1}^{n} \omega^{\alpha_i} \cdot a_i = \omega^{\alpha_1} \cdot a_1 + \ldots + \omega^{\alpha_n} \cdot a_n$$

where

$$\alpha_1 \succ \ldots \succ \alpha_n \text{ and } a_1, \ldots, a_n \prec \omega.$$

2 Set-theoretic interpretation of ordinal arithmetics

Now we define two operations on well-ordered sets.

Definition 2.1 (Sum of Well-ordered Sets). *A well-ordered set C is said to be the sum of two disjoint well-ordered sets $\langle A, \preceq_A\rangle$ and $\langle B, \preceq_B\rangle$ if*

(1) $C = A \cup B$, *and*

(2) $\preceq_C = \preceq_A \cup \preceq_B \cup \{\langle x, y\rangle \mid x \in A \text{ and } y \in B\}$

Basically C is $A \cup B$ ordered as follows: all elements of A precede all elements of B and the order in each of the sets A and B is preserved. The disjointness doesn't put significant restrictions as we can replace them by the sets $A \times \{1\}$ and $B \times \{2\}$ which are disjoint and isomorphic to A and B respectively. We denote by $A \oplus B$ the sum of A and B (which are implicitly assumed disjoint.) Similarly,

Definition 2.2 (Product of Well-ordered Sets). *A well-ordered set $\langle C, \preceq_C\rangle$ is said to be the product of two disjoint well-ordered sets $\langle A, \preceq_A\rangle$ and $\langle B, \preceq_B\rangle$ if*

(1) $C = A \times B$, *and*

(2) $\langle x_1, y_1\rangle \preceq_C \langle x_2, y_2\rangle$ *iff* $y_1 \preceq_B y_2$ *or* $y_1 = y_2$ *and* $x_1 \preceq_A x_2$

Basically C is $A \times B$ ordered antilexicographically. We denote by $A \otimes B$ the product of A and B. The following formulas hold for set sum and product operations:

Lemma 2.1.

$$(A \oplus B) \oplus C \simeq A \oplus (B \oplus C) \tag{2.1}$$

$$A \oplus \emptyset \simeq A \simeq \emptyset \oplus A \tag{2.2}$$

$$(A \otimes B) \otimes C \simeq A \otimes (B \otimes C) \tag{2.3}$$

$$A \otimes 1 \simeq A \simeq 1 \otimes A \tag{2.4}$$

$$A \otimes \emptyset \simeq \emptyset \simeq \emptyset \otimes A \tag{2.5}$$

$$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C) \tag{2.6}$$

Proof. We only prove Equation 2.6. Others are obvious. Note that

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

We only need to show that the induced well-ordering on $A \times (B \cup C)$ is exactly the induced well-ordering on $(A \times B) \cup (A \times C)$. Let

$$\langle D, \preceq_D \rangle = A \otimes (B \oplus C),$$

$$\langle E, \preceq_E \rangle = (A \otimes B) \oplus (A \otimes C).$$

We know that $D = E$. Let $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in D$ and $\langle x_1, y_1 \rangle \preceq_D \langle x_2, y_2 \rangle$. There are several cases:

(1) $y_1 \prec y_2$ and $y_1, y_2 \in B$ or C

(2) $y_1 \prec y_2$ and $y_1 \in B$ and $y_2 \in C$

(3) $y_1 = y_2$

It is easy to check that $\langle x_1, y_1 \rangle \preceq_E \langle x_2, y_2 \rangle$. Similar arguments apply in the other direction. □

Lemma 2.2. *Let $\langle A, \preceq_A \rangle$, $\langle B, \preceq_B \rangle$ $\langle C, \preceq_C \rangle$ and $\langle D, \preceq_D \rangle$ be well-ordered sets. If $A \simeq B$ and $C \simeq D$, then $A \oplus C \simeq B \oplus D$ and $A \otimes C \simeq B \otimes D$.*

Now we are able to define ordinal arithmetics formally:

Definition 2.3 (Ordinal Addition and Multiplication). *Let α and β be two ordinals and let A and B be two well-order sets such that $\overline{A} = \alpha$ and $\overline{B} = \beta$. The sum $\alpha + \beta$ is defined by*

$$\alpha + \beta = \overline{A \oplus B}$$

and the product $\alpha \cdot \beta$ is defined by

$$\alpha \cdot \beta = \overline{A \otimes B}$$

Note that the above definition is well-defined by Lemma 2.2. It follows directly from the definition that:

$$\begin{aligned} \overline{A \otimes B} &= \overline{A} \cdot \overline{B} \\ \overline{A \oplus B} &= \overline{A} + \overline{B} \end{aligned}$$

Lemma 2.3. *If $\alpha \succeq \beta$ then there exists exactly one ordinal γ such that $\alpha = \beta + \gamma$.*

Proof. Let $\overline{A} = \alpha$, let B be a initial segment of A of type β and let $\gamma = \overline{A - B}$. Clearly, $\alpha = \beta + \gamma$. The uniqueness follows from trichotomy and Lemma 3.1. □

Definition 2.4 (Ordinal Subtraction). *The difference of the ordinals α and β ($\alpha \succeq \beta$) is defined to be the unique ordinal γ such that $\alpha = \beta + \gamma$. The ordinal is denoted by $\alpha - \beta$.*

Theorem 2.1 (Ordinal Division). *If β is an ordinal and $\beta \succ 0$, then for each ordinal α there exist ordinals γ and ϱ such that*

$$\alpha = \beta \cdot \gamma + \varrho \text{ and } \varrho \prec \beta.$$

The ordinals γ and ϱ are uniquely determined are called quotient and reminder respectively.

Proof. See [1] Page 249. □

Definition 2.5 (Limit Ordinal). *An ordinal is said to be a limit ordinal if it has no immediate predecessor.*

Example 2.1.

(1) $\mathbf{0}$ *is a limit ordinal.*

(2) ω *is a limit ordinal.*

(3) $\mathbf{n}$ *are not limit ordinals whence* $n > 0$.

Definition 2.6 (Transfinite Sequence). *A transfinite sequence ($\alpha-$sequence) is a function ϕ whose domain is $W(\alpha)$ and whose reange is also a set of ordinals.*

If $\beta \prec \gamma \prec \alpha$ implies $\phi(\beta) \prec \phi(\gamma)$, then we say that the $\alpha-$sequence is *increasing*.

Definition 2.7 (Limit of Ordinal Sequence). *Given an $\alpha-$sequence ϕ, if α is a limit ordinal, there exist ordinals greater than all the ordinals $\phi(\beta)$ where $\beta \prec \alpha$. We call the smallest such ordinal the limit of the $\alpha-$sequence and denote it by $\lim_{\beta<\alpha} \phi(\beta)$.*

Example 2.2.

(1) $\lim_{n<\omega} n = \omega$

(2) $\lim_{n<\omega} 2^n = \omega$

(3) $\lim_{n<\omega} n^n = \omega$

Definition 2.8 (Exponentiation of Ordinals). *The operation of ordinal exponentiation is defined by (transfinite) induction as follows:*

- $\gamma^0 = 1$
- $\gamma^{\xi+1} = \gamma^\xi \cdot \gamma$
- $\gamma^\lambda = \lim_{\xi<\lambda} \gamma^\xi$ *where λ is a limit ordinal.*

We say that γ^α is the *power* of γ, γ is the *base* and α the *exponent*.

3 Arithmetic Rules

The following arithmetic rules are derived from the set-theoretic interpretation of ordinals. However, here we state them as definitions.

Definition 3.1 (Comparison Rules). *If $\alpha = \sum_{i=1}^{n} \omega^{\alpha_i}$ and $\beta = \sum_{i=1}^{m} \omega^{\beta_i}$ are two ordinals, then $\alpha \succ \beta$ iff for some $k \le n$, $\alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}$ and either $\alpha_k \succ \beta_k$ or $m = k - 1 < n$.*

Example 3.1.

(1) $\omega^{\omega^{10}+\omega^{10}+\omega^{10}} \prec \omega^{\omega^{\omega}}$

(2) $\omega^{\omega^5} + \omega^{\omega^4} + \omega^{\omega^3} \prec \omega^{\omega^6}$

(3) $\omega^{100} \prec \omega^{100} + 1$

Definition 3.2 (Addition Rules).

(1) $\alpha + \mathbf{0} \stackrel{\text{def}}{=} \mathbf{0} + \alpha \stackrel{\text{def}}{=} \alpha$

(2) $(\omega^{\alpha_1} + \ldots + \omega^{\alpha_k} + \omega^{\alpha_{k+1}} + \ldots + \omega^{\alpha_n}) + (\omega^{\beta_1} + \ldots + \omega^{\beta_m})$
$\stackrel{\text{def}}{=} (\omega^{\alpha_1} + \ldots + \omega^{\alpha_k} + \omega^{\beta_1} + \ldots + \omega^{\beta_m})$
where k is the maximal number such that $k \leq n$ and $\alpha_k \succeq \beta_1$.

Example 3.2.

(1) $(\omega^5 + \omega^4 + \omega^2 + \omega^2 + \omega + 5) + (\omega^3 + \omega^2) = \omega^5 + \omega^4 + \omega^3 + \omega^2$

(2) $(\omega^5 + \omega^4 + \omega^2 + \omega^2 + \omega + 5) + (\omega^2 + \omega^2) = \omega^5 + \omega^4 + \omega^2 + \omega^2 + \omega^2 + \omega^2$

(3) $(\omega^2 + \omega^2) + (\omega^5 + \omega^4 + \omega^2 + \omega^2 + \omega + 5) = \omega^5 + \omega^4 + \omega^2 + \omega^2 + \omega + 5$

Definition 3.3 (Multiplication Rules).

(1) $\alpha \cdot \mathbf{0} \stackrel{\text{def}}{=} \mathbf{0} \cdot \alpha \stackrel{\text{def}}{=} \mathbf{0}$.

(2) $\alpha \cdot \omega^x = \omega^{\alpha_1 + x}$ *where $x \succeq 1$ and α is in canonical form $\sum_{i=1}^{n} \omega^{\alpha_i}$.*

(3) $\alpha \cdot n \stackrel{\text{def}}{=} \underbrace{\alpha + \ldots + \alpha}_{n}$.

(4) $\alpha \cdot (\beta + \gamma) \stackrel{\text{def}}{=} \alpha \cdot \beta + \alpha \cdot \gamma$.

We prove the last rule as follows:

Proof. Let $\alpha = \overline{A},\ \beta = \overline{B},\ \gamma = \overline{C},\ B \cap C = \emptyset$. Then we have

$$\begin{array}{rcl}
\alpha \cdot (\beta + \gamma) & = & \overline{A} \cdot (\overline{B} + \overline{C}) \\
& = & \overline{A} \cdot \overline{B \oplus C} \\
& = & \overline{A \otimes (B \oplus C)} \\
& = & \overline{(A \otimes B) \oplus (A \otimes C)} \\
& = & \overline{A \otimes B} + \overline{A \otimes C} \\
& = & \overline{A} \cdot \overline{B} + \overline{A} \cdot \overline{C} \\
& = & \alpha \cdot \beta + \alpha \cdot \gamma
\end{array}$$

□

Example 3.3.

(1)

$$\begin{array}{rl}
& (\omega^2 + \omega + 1) \cdot (\omega^3 + \omega) \\
= & (\omega^2 + \omega + 1) \cdot \omega^3 + (\omega^2 + \omega + 1) \cdot \omega^1 \\
= & \omega^5 + \omega^3
\end{array}$$

(2)

$$
\begin{aligned}
& (\omega^{\omega+1}+\omega^{\omega}+1)\cdot(\omega^{\omega+1}+\omega^{\omega}+\omega) \\
= \; & (\omega^{\omega+1}+\omega^{\omega}+1)\cdot\omega^{\omega+1}+(\omega^{\omega+1}+\omega^{\omega}+1)\cdot\omega^{\omega} \\
& +(\omega^{\omega+1}+\omega^{\omega}+1)\cdot\omega \\
= \; & \omega^{(\omega+1)+(\omega+1)}+\omega^{(\omega+1)+(\omega)}+\omega^{(\omega+1)+1} \\
= \; & \omega^{\omega+\omega+1}+\omega^{\omega+\omega}+\omega^{\omega+2} \\
= \; & \omega^{\omega\cdot 2+1}+\omega^{\omega\cdot 2}+\omega^{\omega+2}
\end{aligned}
$$

Definition 3.4 (Exponentiation Rules).

(1) $\alpha^{\mathbf{0}} \stackrel{\text{def}}{=} \mathbf{1} \stackrel{\text{def}}{=} \omega^{\mathbf{0}}$

(2) $\alpha^{\mathbf{1}} \stackrel{\text{def}}{=} \alpha$

(3) $\mathbf{0}^{\alpha} \stackrel{\text{def}}{=} \mathbf{0}$ *for* $\alpha \neq \mathbf{0}$

(4) $\alpha^{\beta} \stackrel{\text{def}}{=} \omega^{\alpha_1\cdot\beta}$ *where* β *is a limit ordinal,* $\alpha \succeq \omega$ *and* α *is in canonical form* $\sum_{i=1}^{n}\omega^{\alpha_i}$.

(5) $\alpha^{\beta+\gamma} \stackrel{\text{def}}{=} \alpha^{\beta}\cdot\alpha^{\gamma}$

(6) $n^{\omega\cdot x} \stackrel{\text{def}}{=} \omega^{x}$

Example 3.4.

(1) $2^{\omega} = 2^{\omega\cdot 1} = \omega^{1} = \omega$

(2) $2^{\omega^2} = 2^{\omega\cdot\omega} = \omega^{\omega}$

(3) $2^{\omega^{\omega}} = 2^{\omega^{1+\omega}} = 2^{\omega^{1}\cdot\omega^{\omega}} = 2^{\omega\cdot\omega^{\omega}} = \omega^{\omega^{\omega}}$

(4) $(\omega+1)^{\omega} = (\omega^{1}+1)^{\omega} = \omega^{1\cdot\omega} = \omega^{\omega}$

(5) $(\omega^{\omega})^{\omega} = \omega^{\omega\cdot\omega} = \omega^{\omega^2}$

(6) $(\omega+1)^{n} = (\omega+1)\cdot(\omega+1)^{n-1} = (\omega^2+\omega+1)\cdot(\omega+1)^{n-2} = \ldots = \omega^{n}+\omega^{n-1}+\ldots+\omega+1$

Now we prove some laws concerning the properties of limits. First we summarize some monotonic laws of ordinal arithmetics. Proofs can be founded at [1] Page 247-250.

Lemma 3.1 (Monotonic Laws of Addition).

(1) $(\alpha \prec \beta) \implies (\gamma+\alpha \prec \gamma+\beta)$.

(2) $(0 \prec \beta) \implies (\gamma \prec \gamma+\beta)$.

(3) $(\alpha \preceq \beta) \implies (\alpha+\gamma \preceq \beta+\gamma)$.

(4) $\gamma \prec \beta+\gamma$.

Lemma 3.2 (Monotonic Laws of Subtraction).

(1) $\alpha = \beta+(\alpha-\beta)$ *if* $\alpha \succ \beta$.

(2) $(\alpha + \beta) - \alpha = \beta$.

(3) $(\alpha \prec \beta) \implies (\alpha - \gamma \prec \beta - \gamma)$.

(4) $(\gamma \prec \beta) \implies (\alpha - \beta \preceq \alpha - \gamma)$.

Lemma 3.3 (Monotonic Laws of Multiplication).

(1) $(\mathbf{0} \prec \alpha \prec \beta) \implies (\gamma \cdot \alpha \prec \gamma \cdot \beta)$.

(2) $(\alpha \preceq \beta) \implies (\alpha \cdot \gamma \preceq \beta \cdot \gamma)$.

(3) $(\alpha + \beta) \cdot \gamma \preceq \alpha \cdot \gamma + \beta \cdot \gamma$.

Lemma 3.4 (Monotonic Laws of Exponetiation).

(1) $(\mathbf{0} \prec \alpha \prec \beta) \implies \gamma^\alpha \prec \gamma^\beta$ *if* $\gamma \succ 1$.

Proof Sketch. It follows directly from definition of exponentiation by transfinite induction. □

Now we prove the properties of limit operation.

Theorem 3.1 (Continuity of Addition). *Assume that λ is a limit ordinal and the ϕ is an increasing λ-sequence. Then we have*

$$\lim_{\xi \prec \lambda} (\alpha + \phi(\xi)) = \alpha + \lim_{\xi \prec \lambda} \phi(\xi).$$

Proof. Note that $\alpha + \phi(\xi)$ is an increasing λ-sequence by Lemma 3.1. Thus the lefthand side is well-defined. Let $\beta = \lim_{\xi \prec \lambda} \phi(\xi)$. If $\xi \prec \lambda$, then $\phi(\xi) \prec \beta$ and therefore $\alpha + \phi(\xi) \prec \alpha + \beta$ by Lemma 3.1 again. Let $\zeta \prec \alpha + \beta$; we need to show that there exists $\xi \prec \lambda$ such that $\zeta \prec \alpha + \phi(\xi)$. If $\zeta \prec \alpha$, then $\zeta \prec \alpha + \phi(0)$. On the other hand, if $\zeta \succeq \alpha$, then $\zeta = \alpha + (\zeta - \alpha)$ and $\zeta - \alpha \prec (\alpha + \beta) - \alpha = \beta$ by Lemma 3.2. It follows that for some $\xi \prec \lambda$ we have $\zeta - \alpha \prec \phi(\xi)$ (since β is the limit of $\phi(\xi)$ where $\xi \prec \lambda$), thus $\zeta \prec \alpha + \phi(\xi)$ by Lemma 3.1 and 3.2. Hence the ordinal $\alpha + \beta$ is the smallest ordinal greater than all ordinals $\alpha + \phi(\xi)$ for $\xi \prec \lambda$. □

Theorem 3.2 (Continuity of Muliplication). *Assume that λ is a limit ordinal and the ϕ is an increasing λ-sequence. We have*

$$\lim_{\xi \prec \lambda} (\alpha \cdot \phi(\xi)) = \alpha \cdot \lim_{\xi \prec \lambda} \phi(\xi).$$

Proof. We assume $\alpha \neq 0$. (Case of $\alpha = 0$ is trivial.) Let $\beta = \lim_{\xi \prec \lambda} \phi(\xi)$. For $\xi \prec \lambda$ we have $\phi(\xi) \prec \beta$, thus $\alpha \cdot \phi(\xi) \prec \alpha \cdot \beta$. Let $\zeta \prec \alpha \cdot \beta$. By Thereom 2.1 there exist ordinals γ and ϱ such that $\zeta = \alpha \cdot \gamma + \varrho \prec \alpha \cdot \beta$ and $\varrho \prec \alpha$. By Lemma 3.3, we must have $\gamma \prec \beta$, which implies that for some $\xi \prec \lambda$ we have $\gamma \prec \phi(\xi)$ as β is the limit ordinal. Hence,

$$\zeta \preceq \alpha \cdot \phi(\xi) + \varrho \preceq \alpha \cdot \phi(\xi) + \alpha = \alpha \cdot (\phi(\xi) + 1) \preceq \alpha \cdot \phi(\xi + 1),$$

because ϕ is increasing. Since λ is a limit ordinal, we have $\xi + 1 \prec \lambda$ and the formula $\alpha \cdot \phi(\xi + 1)$ shows that $\alpha \cdot \beta$ is the smallest ordinal greater than all ordinals of the form $\alpha \cdot \phi(\eta)$ for $\eta \prec \lambda$. □

The following lemma follows directly from the definition of limit.

Lemma 3.5.

$$\lim_{\gamma \prec \lambda} \phi(\gamma) \succ \phi(\gamma) \textit{ for } \gamma \prec \lambda$$

Lemma 3.6.

$$\lim_{\gamma \prec \lambda} \phi(\gamma) \preceq \mu \textit{ iff } \phi(\gamma) \prec \mu \textit{ for all } \gamma \prec \lambda$$

Theorem 3.3 (Transitivity of Cofinality). *If ϕ and ψ are two increasing transfinite sequences, λ is a limit ordinal and $\xi = \lim_{\gamma \prec \lambda} \psi(\gamma)$, then*

$$\lim_{\delta \prec \xi} \phi(\delta) = \lim_{\gamma \prec \lambda} \phi(\psi(\gamma)).$$

Proof. If $\gamma \prec \lambda$ then by Lemma 3.5 $\phi(\gamma) \prec \xi$ and again by Lemma 3.5 $\phi(\psi(\gamma)) \prec \lim_{\delta \prec \xi} \phi(\delta)$. By Lemma 3.6 we have

$$\lim_{\gamma \prec \lambda} \phi(\psi(\gamma)) \preceq \lim_{\delta \prec \xi} \phi(\delta).$$

If $\delta \prec \xi$ then we have $\psi(\gamma) \succeq \delta$ for some ordinal $\gamma \prec \lambda$ by Lemma 3.6. Since the sequence ϕ is increasing, $\phi(\psi(\gamma)) \succeq \phi(\delta)$. By Lemma 3.6. Since the sequence ϕ is increasing, $\phi(\psi(\gamma)) \succeq \phi(\delta)$. Applying Lemma 3.5,

$$\lim_{\gamma \prec \lambda} \phi(\psi(\gamma)) \succ \phi(\psi(\gamma)) \succeq \phi(\delta).$$

Applying Lemma 3.6, we obtain

$$\lim_{\gamma \prec \lambda} \phi(\psi(\gamma)) \succeq \lim_{\delta \prec \xi} \phi(\delta).$$

This concludes our proof. $\square$

Theorem 3.4 (Continuity of Exponentiation). *Assume that λ is a limit ordinal and the ϕ is an increasing λ-sequence. We have*

$$\lim_{\xi \prec \lambda} (\alpha^{\phi(\xi)}) = \alpha^{\lim_{\xi \prec \lambda} \phi(\xi)}.$$

Proof. Let

$$\beta = \lim_{\xi \prec \lambda} \phi(\xi)$$

Define

$$\psi(\delta) = \alpha^{\delta}$$

By Lemma 3.4, $\phi(\delta)$ is increasing. Applying Theorem 3.3, we have

$$\begin{aligned} \lim_{\xi \prec \lambda} (\alpha^{\phi(\xi)}) &= \lim_{\delta \prec \beta} \alpha^{\delta} \\ &= \alpha^{\beta} \\ &= \alpha^{\lim_{\xi \prec \lambda} \phi(\xi)} \end{aligned}$$

$\square$

References

[1] K. Kuratowski and A. Mostowski, North-Holland **Set Theory**, 1968