Model Theory 290A: Automorphisms

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Abstract

This term paper is a summary of topological properties of automorphisms and model-theoretic consequences on countable structures. It is based on Section 4.1 of [1].

1 Introduction

In this section we recap some concepts in general topology and permutation groups. Most of materials can be found in [2].

Definition 1.1 (Topological Space)**.** *Given a set U, a **topology** on U is a collection $\mathcal{U}$ of subsets of U, called the open sets of U, such that $\emptyset \in \mathcal{U}$ and $U \in \mathcal{U}$ and such that $\mathcal{U}$ is closed under finite intersections and arbitrary (i.e., not necessarily finite or countable) unions. A set U with a topology $\mathcal{U}$ defined upon it is called a **topological space**, denoted by $(U,\mathcal{U})$.*

We say a set $X \subseteq U$ is **closed** if $U \setminus X$ is open. A set $X \subseteq U$ is said to be **dense** if any closed superset of X is U itself. A **neighborhood** of a point x is an open set containing x.

Lemma 1.1. *Let $(U,\mathcal{U})$ be a topological space. A subset X of U is open iff it contains a neighborhood of each of its elements.*

Proof. $(\Rightarrow)$ X itself is open and it is a neighborhood of any point in it.
$(\Leftarrow)$ Let Y be a union of neighborhoods of all points of X which are contained in X. Clearly, Y is open and $X \subseteq Y$. The assumption implies $Y \subseteq X$. Hence $X = Y$ and X is open. □

Lemma 1.2. *Let $(U,\mathcal{U})$ be a topological space. A subset X of U is closed iff for any $u \in U$, if all neighborhood of u intersects X, then $u \in X$.*

Proof. X is closed iff $U \setminus X$ is open, (by Lemma 1.1) iff for every $u \in U \setminus X$, there is a neighborhood of u which is contained in $U \setminus X$ iff for any $u \in U$, if all neighborhood of u intersects X, then $u \in X$. □

Definition 1.2 (Continuous Function)**.** *Let $f : (U,\mathcal{U}) \mapsto (V,\mathcal{V})$, f is said continuous if Z is an open set of $(V,\mathcal{V})$, then $f^{-1}[Y]$ is an open set of $(U,\mathcal{U})$.*

Lemma 1.3. *Let $f : (U,\mathcal{U}) \mapsto (V,\mathcal{V})$, f is continuous iff for each $x \in U$ and each neighborhood Y of $f(x)$, there exists a neighborhood X of x such that $f[X] \subseteq Y$.*

Proof. (⇐) Let $x \in U$ and Y be a neighborhood of $f(x)$. $f^{-1}[Y]$ is a neighborhood of x and $f[f^{-1}[Y]] = Y$.
(⇒) Let Z be an open set of $(V, \mathcal{V})$, x be any point of $f^{-1}[Z]$. Since Z is open, Z is a neighborhood of $f(x)$. By the assumption let X be the neighborhood of x such that $f[X] \subseteq Z$. Then we have $X \subseteq f^{-1}[Z]$. By Lemma 1.1 $f^{-1}[Z]$ is open. □

We write $H \preccurlyeq G$ to mean H is a subgroup of G. Let $H \preccurlyeq G$. The set gH is called the **(left) coset** of H in G. (Similarly, Hg is called the **(right) coset** of H in G.) It is well-known that the collection of cosets of F form a partition of G. The index of H in G is the number of distinct cosets of H, written $(G : H)$.

Lemma 1.4. *If $F \preccurlyeq H \preccurlyeq G$, then H is the union of cosets of F in G.*

Proof. Let gF be any coset of F in G. If $g \in H$, then $gF \subseteq H$. If $g \in G \setminus H$, then $gF \subseteq G \setminus H$. □

Lemma 1.5. *If $H \preccurlyeq G$, then $gH = kH$ iff $g^{-1}kH = H$ iff $g^{-1}k \in H$.*

Proof. Immediate. □

Definition 1.3 (Topological Group)**.** *A group G is called a topological group if G has a topology $\mathcal{G}$ and group multiplication and inversion are continuous functions of $(G, \mathcal{G})$.*

2 Topological properties of automorphisms

For any set Ω, the group of all permutations of Ω forms the **symmetric group** on Ω, written in $Sym(\Omega)$. For any structure A, all automorphism forms a group (written $Aut(A)$) which is a subgroup of $Sym(dom(A))$. Let $G \preccurlyeq Sym(\Omega)$. The **pointwise stabilizer** of a set $X \subseteq \Omega$ in G (written $G_{(X)}$) is the set $\{g \in G : g(a) = a \text{ for all } a \in X\}$.

An **orbit** of an element $a \in \Omega$ under G is the set $\{g(a) : g \in G\}$. We say G is **transitive** on Ω if the orbit of every element is Ω itself. A structure is **transitive** if $Aut(A)$ is transitive on $dom(A)$. A structure is called **rigid** if $\mathbf{1_A}$ is the only automorphism.

A subset S of $Sym(\Omega)$ is called **basic open** if there exists tuples $\bar{a}$ and $\bar{b}$ such that $S = \{g \in G : g(\bar{a}) = \bar{b}\}$. We write $S(\bar{a}, \bar{b})$ for such S. An **open set** of $Sym(\Omega)$ is a union of basic open sets. Also we say a set is **(basic) open** in $Aut(A)$ if it is an intersection of an (basic) open set in $Sym(\Omega)$ and $Aut(A)$.

The following lemma summarizes some topological properties of $Aut(A)$, and in particular $Sym(\Omega)$ if A is simply a set.

Lemma 2.1. *Write G for $Aut(A)$.*

(1) The open sets defined above form a topology of G. Under this topology, G is a topological group.

(2) A subgroup H of G is open iff $G_{(\bar{a})} \subseteq H$ for some tuple $\bar{a} \in dom(A)$.

(3) A subset H of G is closed iff H satisfies the following condition: if for any $g \in G$ and every tuple $\bar{a} \in dom(A)$ there is $h \in H$ such that $g\bar{a} = h\bar{a}$, then $g \in H$.

(4) A sugroup H of G is dense iff H and G and H has the same orbits on $(domA)^n$ for each positive n.

Proof. (1) Let two open sets

$$A = \bigcup_{i\in I}\{A_i\} = \bigcup_{i\in I}\{S(\bar{a}_i, \bar{b}_i)\}$$

$$B = \bigcup_{j\in J}\{B_j\} = \bigcup_{j\in J}\{S(\bar{a}_j, \bar{b}_j)\}$$

where $S(\bar{a}_k, \bar{b}_k)$ is a basic open set in G. It follows that

$$A \cap B = \bigcup_{i\in I, j\in J} S(\bar{a}_i, \bar{b}_i) \cap S(\bar{a}_j, \bar{b}_j)$$

Observe that

$$S(\bar{a}_i, \bar{b}_i) \cap S(\bar{a}_j, \bar{b}_j) = S(\bar{a}_i\bar{a}_j, \bar{b}_i\bar{b}_j)$$

So $A \cap B$ is open. Other conditions are satisfied immediately by definition.

If $g^{-1} \in S(\bar{a}, \bar{b})$, then $g \in S(\bar{b}, \bar{a})$ and $S^{-1}(\bar{a}, \bar{b}) = S(\bar{b}, \bar{a})$. Also if $gh \in S(\bar{a}, \bar{b})$, then $h \in S(\bar{a}, \bar{c})$ and $g \in S(\bar{c}, \bar{b})$ where $\bar{c} = h\bar{a}$. Moreover, $S(\bar{a}, \bar{c})S(\bar{c}, \bar{b}) \subseteq S(\bar{a}, \bar{b})$. By Lemma 1.3 multiplication and inversion are continuous, and hence G is a topological group.

(2) Let H be a subgroup of G containing $G_{(\bar{a})}$ for some $\bar{a} \in dom(A)$. Since $G_{(\bar{a})}$ is a subgroup of H, by Lemma 1.4, H is a union of cosets of $G_{(\bar{a})}$. Each cosets $gG_{(\bar{a})}$ is $S(\bar{a}, g\bar{a})$ which is basic open. It follows that H is open. Conversely, suppse that H is open. It contains $S(\bar{a}, \bar{b})$ for some $\bar{a}, \bar{b} \in dom(A)$. Then it contains $S(\bar{b}, \bar{a})$ and hence $S(\bar{a}, \bar{b})S(\bar{b}, \bar{a})$. It follows that $G_{(\bar{a})} = S(\bar{a}, \bar{a}) \subseteq S(\bar{a}, \bar{b})S(\bar{b}, \bar{a}) \subseteq H$.

(3) Note that any neighborhood of g contains a basic open set $S(\bar{a}, g\bar{a})$ for some $\bar{a} \in dom(A)$ and any basic open set $S(\bar{a}, g\bar{a})$ for any $\bar{a} \in dom(A)$ is a neighborhood of g. So the assumption can be translated to "for any $g \in G$, if every neighborhood of g intersects H, then $g \in H$", which by Lemma 1.2 is equivalent to "H is closed in G."

(4) If H and G has the same orbits, then for any $g \in G$ and every tuple $\bar{a} \in dom(A)$ there exists $h \in H$ such that $g(\bar{a}) = h(\bar{a})$. By (3), G is the only closed super set of H, that is, H is dense. Conversely, assuming that there exists $g \in G$ and $\bar{a} \in dom(A)$ such that there is no $h \in H$ such that $g(\bar{a}) = h(\bar{a})$, we have $S(\bar{a}, g(\bar{a})) \neq \emptyset$ and $S(\bar{a}, g(\bar{a})) \cap H = \emptyset$. Then H is contained in $G \setminus S(\bar{a}, g(\bar{a}))$ which is closed. So H is not dense.

□

Theorem 2.1. *Let Ω be a set, $H \preccurlyeq G \preccurlyeq Sym(\Omega)$. The following are equivalent:*

(1) H is closed in G.

(2) There is a structure A with $dom(A) = \Omega$ such that $H = G \cap Aut(A)$.

In particular, for any structure A with $dom(A) = \Omega$, $Aut(A)$ is a closed subgroup of $Sym(\Omega)$.

Proof. (1 ⇒ 2) For each orbit Δ of H on Ω^n with $0 < n < \omega$, define relation symbol R_Δ such that $R_\Delta^A = \Delta$. Obviously, $A \models R_\Delta(\bar{a}) \Leftrightarrow A \models R_\Delta(h\bar{a})$ for all $h \in H$, and hence $H \subseteq G \cap Aut(A)$. On the other hand, let $g \in G \cap Aut(A)$. Let $\bar{a}$ be any tuple of of Ω. There exists some orbit Δ of H such that $\bar{a} \in \Delta$. As g is an automorphism, $g\bar{a} \in \Delta$. By definition of Δ, there exists $h \in H$ such that $g\bar{a} = h\bar{a}$. It follows that $g \in H$ for H is closed, and hence $G \cap Aut(A) \subseteq H$.
(2 ⇒ 1) Let any $g \in G$ and $\bar{a} \in dom(A)$. Assume that there is $h \in H$ such that $g\bar{a} = h\bar{a}$. For any formula $\phi(\bar{x})$,

$$A \models \phi(\bar{a}) \Leftrightarrow A \models \phi(h\bar{a}) \Leftrightarrow A \models \phi(g\bar{a})$$

Hence $g \in Aut(A) \subseteq H$. It follows that H is closed. □

3 Model-theoretic consequences in countable structures

Theorem 3.1. *If G is a closed group of $Sym(\omega)$ and H is a closed subgroup of G, then the following are equivalent.*

(1) H *is open in* G.

(2) $(G : H) \leq \omega$.

(3) $(G : H) < 2^\omega$.

Proof. (1 ⇒ 2) By Lemma 2.1 since H is open in G, there exists $\bar{a} \in \omega$ such that $G_{(\bar{a})} \subseteq H$. For any two cosets gH and jH of H, $gH = jH$ iff $gj^{-1} \in H$. Thus if $g\bar{a} = j\bar{a}$, then $gj^{-1}\bar{a} = \bar{a}$, and hence $gj^{-1} \in G_{(\bar{a})} \subseteq H$. So $gH \neq jH$ iff $g\bar{a} \neq j\bar{a}$. As there are at most ω distinct outcomes of $g\bar{a}$, $(G : H) \leq \omega$.
(2 ⇒ 3) Immediate.
(3 ⇒ 1) We define sequences $(\bar{a}_i : i < \omega)$, $(\bar{b}_i : i < \omega)$ of tuples of ω and a sequence $(\bar{g}_i : i < \omega)$ of in G such that the following condition are satisfied.

(1) $\bar{b}_0 = \langle\rangle$; $\bar{b}_{i+1}$ is a concatenation of all sequences of the form

$$(k_i \cdots k_0)(\bar{a}_0{}^\wedge \cdots {}^\wedge \bar{a}_i)$$

where either $k_j = g_j$ or $k_j = \mathbf{1}$ for $0 \leq j \leq i$.

(2) $g_i\bar{b}_i = \bar{b}_i$.

(3) $h\bar{a}_i \neq g_i\bar{a}_i$ for all $h \in H$.

(4) $i \in \bar{a}_i$.

(5) $g_i \notin H$ and $g_i \in G$.

Condition (2) and (5) are satisfiable as H is not open, that is, for any tuple $\bar{a} \in \omega$, there is $g \notin H$ but $g \in G_{(\bar{a})}$. Since H is closed, by Condition (5), there exists $\bar{a}_i \in \omega$ such that $h\bar{a}_i \neq g_i\bar{a}_i$ for all $h \in H$. Then Condition (3) is satisfiable too. We can have Condition (4) by adding i into $\bar{a}_i$. Now we show that each subset of $\{g_i : i < \omega\}$ corresponds to a distinct (right) coset of H in G. Put $S \subseteq \{g_i : i < \omega\}$. For each $i < \omega$ define

$$g_i^S = \begin{cases} g_i & \text{if } i \in S \\ \mathbf{1} & \text{if } i \notin S \end{cases}$$

$$f_i^S = g_i^S g_{i-1}^S \cdots g_0^S$$

For all for $j > i$, because $g_i^S g_{i-1}^S \cdots g_0^S(\bar{a}_i) \subseteq \bar{b}_j$, we have $f_j^S(\bar{a}_i) = f_i^S(\bar{a}_i)$. So for each $S \subseteq \omega$, we can define

$$g^S(i) = f_i^S(i)$$

Obviously g^S is injective. For any $j < \omega$, put $i = (f_j^S)^{-1}(j)$. If $i \leq j$, then

$$g^S(i) = f_j^S(i) = f_j^S((f_j^S)^{-1}(j)) = j$$

Otherwise $i > j$, and

$$g^S(i) = f_i^S(i) = (g_i^S \cdots g_{j+1}^S f_j^S)((f_j^S)^{-1}(j)) = (g_i^S \cdots g_{j+1}^S)(j) = j$$

So g^S is subjective, and hence g^S is a permutation. For any $\bar{a} \in \omega$, let $j = max(\bar{a})$, then $g^S(\bar{a}) = f_j^S(\bar{a})$. Clearly, each $f_j^S \in G$, and hence $g^S \in G$ for G is closed. There are 2^ω subsets of ω. It suffices to show that if $S, T \subseteq \omega$, $S \neq T$, then $Hg^S \neq Hg^T$. Let i be the least element which differentiates S from T. Without loss, assume that $i \in S$, but $i \notin T$. Then $f_i^T = f_{i-1}^S$. Put $\bar{c} = (f_{i-1}^S)^{-1}(\bar{a}_i)$ and $j = max(\bar{c})$. We have

$$g^S(\bar{c}) = g^S(\bar{c}) = f_j^S(\bar{c}) = g_j^S \cdots g_{i+1}^S g_i f_{i-1}^S((f_{i-1}^S)^{-1}(\bar{a}_i)) = g_j^S \cdots g_{i+1}^S g_i(\bar{a}_i) = g_i(\bar{a}_i)$$
$$g^T(\bar{c}) = g^T(\bar{c}) = f_j^T(\bar{c}) = g_j^T \cdots g_{i+1}^T f_i^T((f_i^T)^{-1}(\bar{a}_i)) = g_j^T \cdots g_{i+1}^T(\bar{a}_i) = \bar{a}_i$$

Since for all $h \in H$, $g_i(\bar{a}_i) \neq h\bar{a}_i$, we have that for all $h \in H$, $g^S(\bar{c}) \neq hg^T(\bar{c})$. That is, $g^S \neq hg^T$ for all $h \in H$. So $Hg^S \neq Hg^T$ for $S \neq T$. The proof is finished. □

We can translate Theorem 3.1 to a model-theoretic version.

Theorem 3.2 (Kueker-Reyes Theorem). *Let L^- and L^+ be signatures with $L^- \subseteq L^+$. Let A be a countable structure of L^+ and B be the reduct $A|L^-$. Let $G = Aut(B)$. Then the following are equivalent.*

(1) There exists $\bar{a} \in dom(A)$ such that $G_{(\bar{a})} \subseteq Aut(A)$.

(2) There are at most countably infinite expansion of B which is isomorphic to A.

(3) The number of distinct expansions of B which are isomorphic to A is less than 2^ω.

Proof. Put $H = Aut(B)$. Let A' be an expansion of B with symbols in $L^+ \backslash L^-$ interpreted such that A is isomorphic to A' under mapping $f : A \mapsto A'$. It follows that $fS^B = fS^{A|L^-} = S^{A'|L^-} = S^B$. That is, f is an automorphism of B, i.e., $f \in G$. So all expansions of B which are isomorphic to A are in form gA with $g \in G$. Let gA, kA be two such expansions, we have

$$\begin{aligned} gA = kA &\iff gS^A = kS^A \text{ for every symbol in } L^+ \\ &\iff g^{-1}kS^A = S^A \text{ for every symbol in } L^+ \\ &\iff g^{-1}k \in H \\ &\iff gH = kH \end{aligned}$$

So the number of distinct expansions of B which are isomorphic to A is the same as the number of distinct coset of H in G, i.e., the index of H in G. Moreover A is countable structure, we can identify it with ω. Obviously, $H \preccurlyeq G \preccurlyeq Sym(\omega)$ with G closed in $Sym(\omega)$ and H closed in G by Lemma 2.1. Thus the theorem is a special case of Theorem 3.1. □

Corollary 3.1. *Let A be a countable structure. The following are equivalent.*

(1) There is a tuple $\bar{a} \in dom(A)$ such that $(A, \bar{a})$ is rigid.

(2) $|Aut(A)| \leq \omega$.

(3) $|Aut(A)| \leq 2^{\omega}$.

Proof. Put $G = Aut(A)$ and $H = \{\mathbf{1}\}$. (1) implies $G_{(\bar{a})} = \{\mathbf{1}\} = H$, and hence H is open subgroup of G. The corollary follows from Theorem 3.2 with observation that $|Aut(A)| = (G : H)$. □

References

[1] Wilfred Hodges, **A Shorter Model Theory**, 1997, Cambridge University Press.

[2] J. L. Kelley, **General Topology**, 1975, Springer-Verlag New York.