The Decidability of the First-order Theory of Knuth-Bendix Order

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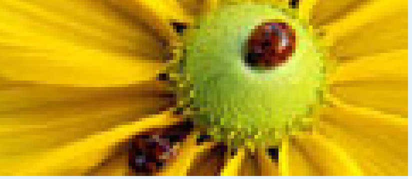

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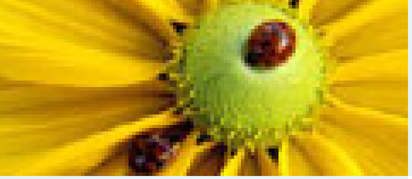

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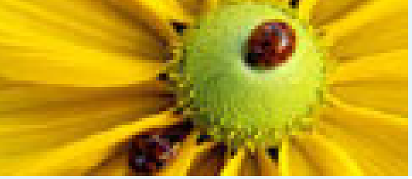
- Termination Proofs.



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- Termination Proofs.
- Ordered Resolution.

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- Termination Proofs.
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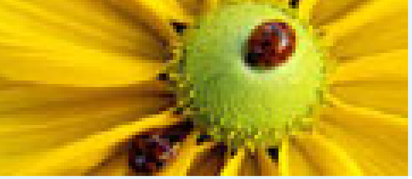
- Ordered Rewriting.



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- Termination Proofs.
- Ordered Resolution.
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☞ How to decide satisfi ability of order constraints?

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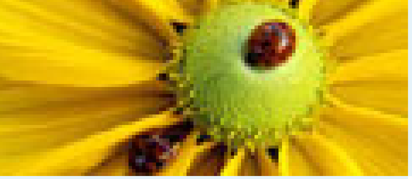

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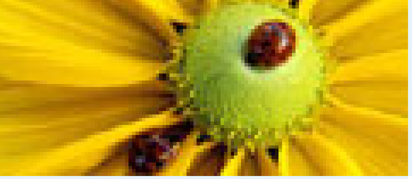

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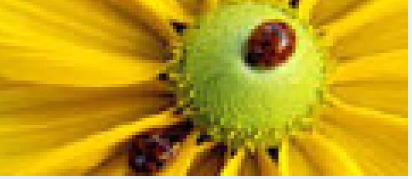

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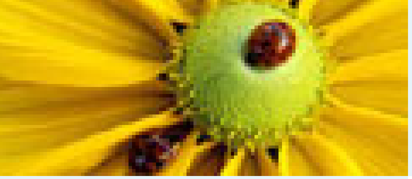

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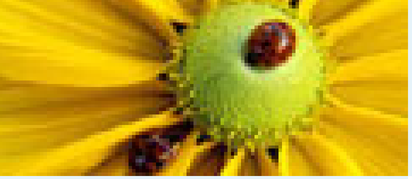

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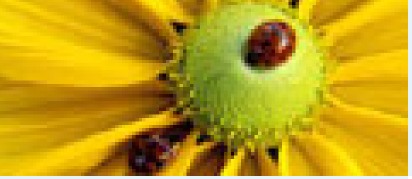
Two types of widely used orderings:

	Syntatic Nature		Hybrid Natur
	RPO		KBO
	MPO	LPO	
syntactic precedence			
multiset ordering			
lexicographical ordering			
numerical ordering			



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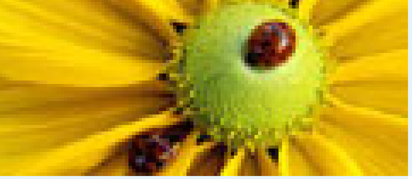
Two types of widely used orderings:

	Syntatic Nature		Hybrid Natur
	RPO		KBO
	MPO	LPO	
syntactic precedence	√		
multiset ordering			
lexicographical ordering			
numerical ordering			



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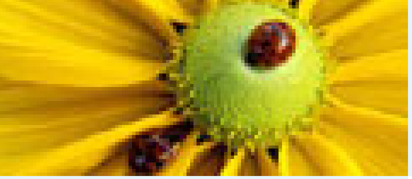
	RPO		KBO
	MPO	LPO	
syntactic precedence	√		
multiset ordering	√		
lexicographical ordering			
numerical ordering			



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Two types of widely used orderings:

	Syntatic Nature		Hybrid Natur
	RPO		KBO
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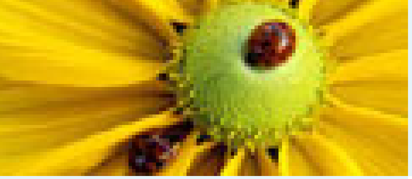
syntactic precedence	√	√	
multiset ordering	√		
lexicographical ordering			
numerical ordering			



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Two types of widely used orderings:

	Syntatic Nature		Hybrid Natur
	RPO		KBO
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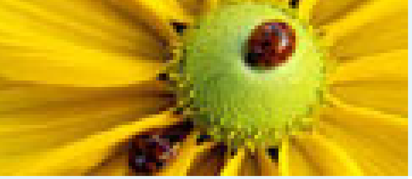
syntactic precedence	√	√	
multiset ordering	√		
lexicographical ordering		√	
numerical ordering			



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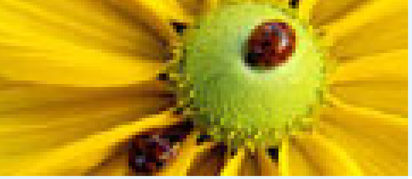
Two types of widely used orderings:

	Syntatic Nature		Hybrid Natur
	RPO		KBO
	MPO	LPO	
syntactic precedence	√	√	√
multiset ordering	√		
lexicographical ordering		√	
numerical ordering			



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	RPO		KBO
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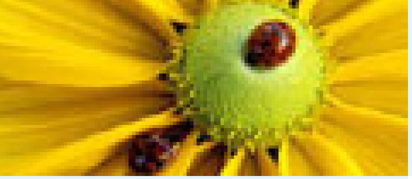
syntactic precedence	√	√	√
multiset ordering	√		
lexicographical ordering		√	√
numerical ordering			



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Two types of widely used orderings:

	Syntatic Nature		Hybrid Natur
	RPO		KBO
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syntactic precedence	√	√	√
multiset ordering	√		
lexicographical ordering		√	√
numerical ordering			√

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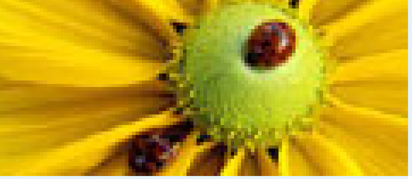
Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT			
UQT			
GQT			

QFT: Quantifier-free Theory.

UQT: Unary Quantified Theory.

GQT: General Quantified Theory.



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Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT	$\surd$ [JO91] [NRV99]		
UQT			
GQT			

QFT: Quantifier-free Theory.

UQT: Unary Quantified Theory.

GQT: General Quantified Theory.

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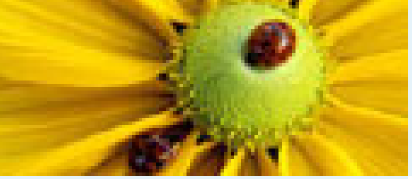
Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT	√ [JO91] [NRV99]		
UQT	√ [NR00]		
GQT			

QFT: Quantifier-free Theory.

UQT: Unary Quantified Theory.

GQT: General Quantified Theory.



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Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT	√ [JO91] [NRV99]		
UQT	√ [NR00]		
GQT	?		

QFT: Quantifi er-free Theory.

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Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT	$\surd$ [JO91] [NRV99]	$\surd$ [Com90] [Nie93]	
UQT	$\surd$ [NR00]		
GQT	?		

QFT: Quantifier-free Theory.

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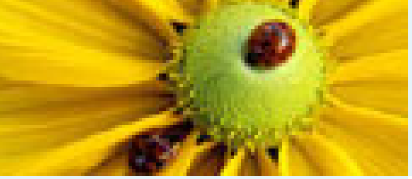

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Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT	√ [JO91] [NRV99]	√ [Com90] [Nie93]	
UQT	√ [NR00]	√ [NR00]	
GQT	?		

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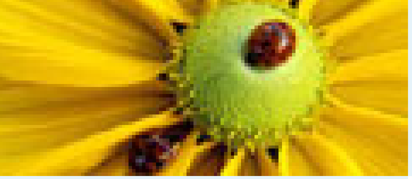

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Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT	√ [JO91] [NRV99]	√ [Com90] [Nie93]	
UQT	√ [NR00]	√ [NR00]	
GQT	?	× [Tre92, CT97]	

QFT: Quantifi er-free Theory.

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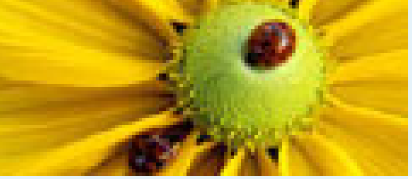

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Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT	√ [JO91] [NRV99]	√ [Com90] [Nie93]	√ [KV00] [KV01]
UQT	√ [NR00]	√ [NR00]	
GQT	?	× [Tre92, CT97]	

QFT: Quantifier-free Theory.

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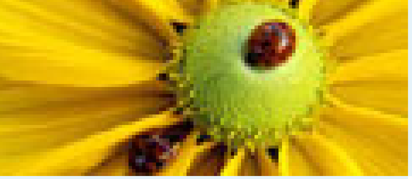
Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT	√ [JO91] [NRV99]	√ [Com90] [Nie93]	√ [KV00] [KV01]
UQT	√ [NR00]	√ [NR00]	√ [KV02]
GQT	?	× [Tre92, CT97]	

QFT: Quantifier-free Theory.

UQT: Unary Quantified Theory.

GQT: General Quantified Theory.



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Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT	√ [JO91] [NRV99]	√ [Com90] [Nie93]	√ [KV00] [KV01]
UQT	√ [NR00]	√ [NR00]	√ [KV02]
GQT	?	× [Tre92, CT97]	?

QFT: Quantifier-free Theory.

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Decidability Status (w.r.t. linear precedence):

	MPO	LPO	KBO
QFT	√ [JO91] [NRV99]	√ [Com90] [Nie93]	√ [KV00] [KV01]
UQT	√ [NR00]	√ [NR00]	√ [KV02]
GQT	?	× [Tre92, CT97]	√

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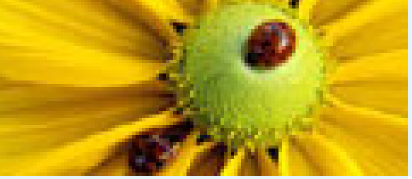
Our approach: $\mathrm{Th}(KBO) \;\rightarrow\; \mathrm{Th}(PA)$



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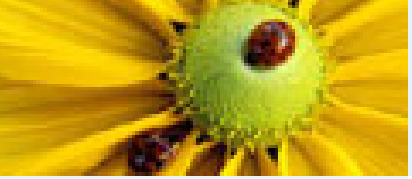
Our approach: $\mathrm{Th}(KBO) \rightarrow \mathrm{Th}(PA)$

- Reduce term constraints to integer constraints. [ZSM04a]



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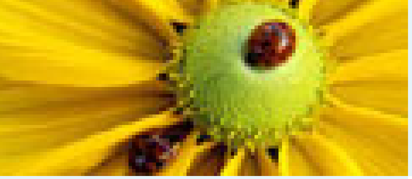
Our approach: $\mathrm{Th}(KBO) \rightarrow \mathrm{Th}(PA)$

- Reduce term constraints to integer constraints. [ZSM04a]
- Reduce term quantifiers to integer quantifiers. [ZSM04b]



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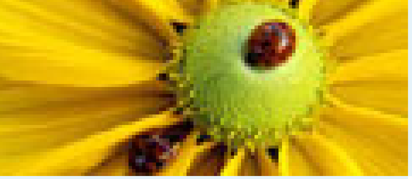
Our approach: $\mathrm{Th}(KBO) \;\rightarrow\; \mathrm{Th}(PA)$

- Reduce term constraints to integer constraints. [ZSM04a]
- Reduce term quantifiers to integer quantifiers. [ZSM04b]

☞ **Integers rule!**



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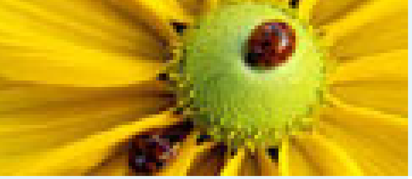

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A term algebra $\mathfrak{A}_{\mathsf{TA}} : \langle \mathsf{TA}; \mathcal{C}, \mathcal{A}, \mathcal{S}, \mathcal{T} \rangle$ consists of

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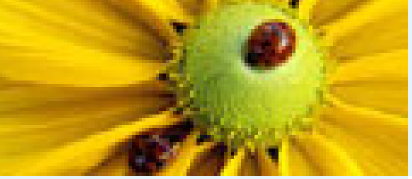
A term algebra $\mathfrak{A}_{\mathrm{TA}}: \langle \mathrm{TA}; \mathcal{C}, \mathcal{A}, \mathcal{S}, \mathcal{T} \rangle$ consists of

1. TA: The term domain.



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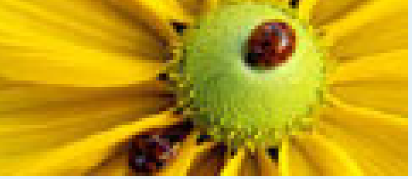

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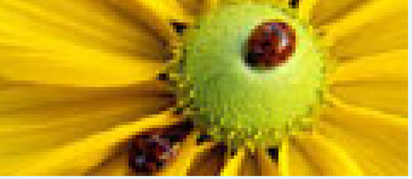

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5. $\mathcal{T}$: A finite set of testers. Is_α for $\alpha \in \mathcal{C}$.

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☞ TA is generated exclusively using $\mathcal{C}$.

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5. $\mathcal{T}$: A finite set of testers. Is_α for $\alpha \in \mathcal{C}$.

☞ TA is generated **exclusively** using $\mathcal{C}$.

☞ Each element of TA is **uniquely** generated.

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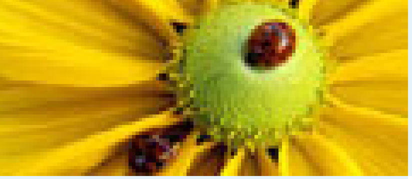
- Signature:

$$\langle \mathsf{list}; \{\mathsf{cons}, \mathsf{nil}\}; \{\mathsf{nil}\}; \{\mathsf{car}, \mathsf{cdr}\}; \{\mathsf{Is}_{\mathsf{nil}}, \mathsf{Is}_{\mathsf{cons}}\}\rangle$$



Example: LISP lists

- Signature:

$$\langle \mathsf{list}; \{\mathsf{cons}, \mathsf{nil}\}; \{\mathsf{nil}\}; \{\mathsf{car}, \mathsf{cdr}\}; \{\mathsf{Is}_{\mathsf{nil}}, \mathsf{Is}_{\mathsf{cons}}\} \rangle$$

- Axioms:

$$\begin{aligned}
\mathsf{Is}_{\mathsf{nil}}(x) &\leftrightarrow \neg \mathsf{Is}_{\mathsf{cons}}(x), \\
x &= \mathsf{car}(\mathsf{cons}(x, y)), \\
y &= \mathsf{cdr}(\mathsf{cons}(x, y)), \\
\mathsf{Is}_{\mathsf{nil}}(x) &\leftrightarrow \{\mathsf{car}, \mathsf{cdr}\}^+(x) = x, \\
\mathsf{Is}_{\mathsf{cons}}(x) &\leftrightarrow \mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x)) = x.
\end{aligned}$$

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We study KBO using selector language.

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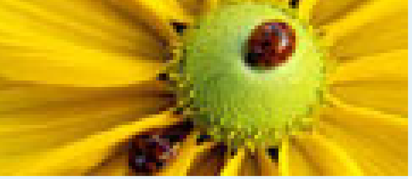
We study KBO using selector language.

- For $L = \mathsf{s}_1, \ldots, \mathsf{s}_n$, Lx stands for

$$\mathsf{s}_1(\ldots(\mathsf{s}_n(x)\ldots)).$$

  $|L|$ is called the depth of x in Lx.



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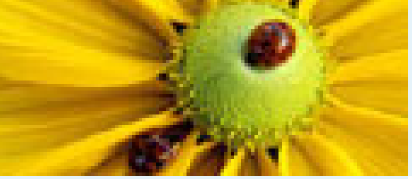
  $$\mathsf{s}_1(\ldots(\mathsf{s}_n(x)\ldots)).$$

  $|L|$ is called the **depth** of x in Lx.

- depth$\varphi(x)$: the maximum depth of x in φ.



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We study KBO using selector language.

- For $L = \mathsf{s}_1, \ldots, \mathsf{s}_n$, Lx stands for

$$\mathsf{s}_1(\ldots(\mathsf{s}_n(x)\ldots)).$$

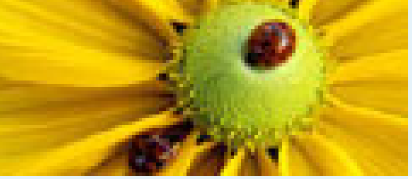
  $|L|$ is called the depth of x in Lx.

- depth$\varphi(x)$: the maximum depth of x in φ.
- Formulas are type-complete and selector terms are proper. For example,

$$\mathsf{car}(x) \neq \mathsf{cdr}(x)$$

  should be understood as

$$\mathsf{car}(x) \neq \mathsf{cdr}(x) \ \wedge \ \mathsf{Is}_{\mathsf{cons}}(x).$$



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A Knuth-Bendix order (KBO) $\prec^{\mathrm{kb}}$ is parametrically defi ned with

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A Knuth-Bendix order (KBO) $\prec^{\mathrm{kb}}$ is parametrically defined with

- $\mathsf{W} : \mathsf{TA} \to \mathbb{N}$: a weight function satisfying

$$\mathsf{W}(\alpha(t_1, \ldots, t_k)) = \mathsf{W}(\alpha) + \sum_{i=1}^{k} \mathsf{W}(t_i).$$

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A Knuth-Bendix order (KBO) $\prec^{\mathrm{kb}}$ is parametrically defined with

- $\mathsf{W} : \mathsf{TA} \to \mathbb{N}$: a weight function satisfying

$$\mathsf{W}(\alpha(t_1, \ldots, t_k)) = \mathsf{W}(\alpha) + \sum_{i=1}^{k} \mathsf{W}(t_i).$$

- $\prec^{\Sigma}$: a linear (precedence) order on C such that

$$\alpha_1 \succ^{\Sigma} \alpha_2 \succ^{\Sigma} \ldots \succ^{\Sigma} \alpha_{|C|}.$$

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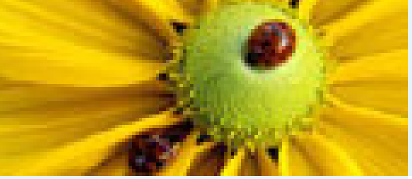
For $u, v \in$ TA, $u \prec^{\text{kb}} v$ if one of the following holds:



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For $u, v \in$ TA, $u \prec^{\text{kb}} v$ if one of the following holds:

- $\mathrm{W}(u) < \mathrm{W}(v)$.

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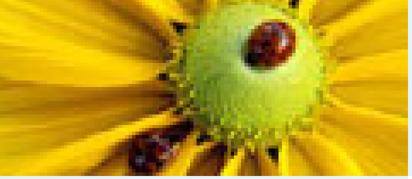
For $u, v \in \mathsf{TA}$, $u \prec^{\mathsf{kb}} v$ if one of the following holds:

- $\mathsf{W}(u) < \mathsf{W}(v)$.
- $\mathsf{W}(u) = \mathsf{W}(v)$ and $\mathsf{type}(u) \prec^{\Sigma} \mathsf{type}(v)$.



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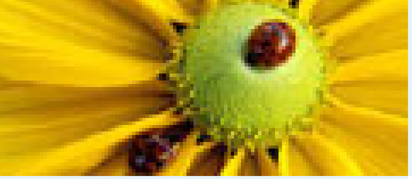
For $u, v \in$ TA, $u \prec^{\text{kb}} v$ if one of the following holds:

- $\text{W}(u) < \text{W}(v)$.
- $\text{W}(u) = \text{W}(v)$ and $\text{type}(u) \prec^{\Sigma} \text{type}(v)$.
- $\text{W}(u) = \text{W}(v)$, $u \equiv \alpha(u_1, \ldots, u_k)$, $v \equiv \alpha(v_1, \ldots, v_k)$, and

$$\exists i \Big[1 \leq i \leq k \wedge u_i \prec^{\text{kb}} v_i \wedge \forall j (1 \leq j < i \rightarrow u_j = v_j) \Big].$$



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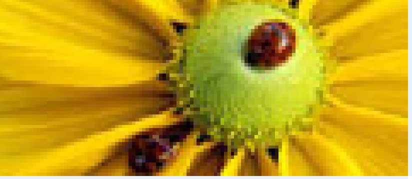

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- Suffi ces to eliminate ∃-quantifi ers from primitive formulas

$$\exists \bar{x}(A_1(\bar{x}) \wedge \ldots \wedge A_n(\bar{x})),$$

where $A_i(\bar{x})$ $(1 \leq i \leq n)$ are literals.

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- Suffices to eliminate ∃-quantifiers from primitive formulas

$$\exists \bar{x}(A_1(\bar{x}) \wedge \ldots \wedge A_n(\bar{x})),$$

  where $A_i(\bar{x})$ $(1 \leq i \leq n)$ are literals.

- Suffices to assume $A_i \not\equiv x = t$ if $x \notin t$, because

$$\exists x(x = t \wedge \varphi(x, \bar{y})) \leftrightarrow \varphi(t, \bar{y}).$$

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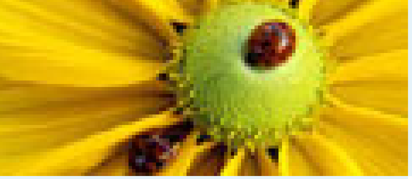

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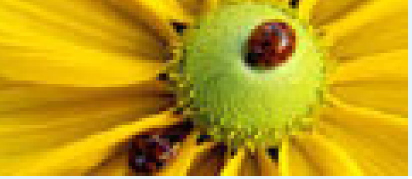
- Solved Form.

  Eliminating $\exists x$ from $(\exists x)\varphi(x, \bar{y})$ is straightforward once

  $\varphi(x, \bar{y})$ is solved in x.



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- Solved Form.

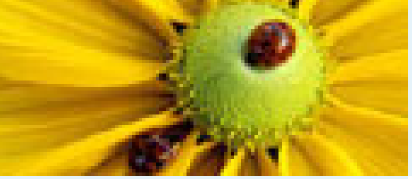
  Eliminating $\exists x$ from $(\exists x)\varphi(x, \bar{y})$ is straightforward once

  $\varphi(x, \bar{y})$ is solved in x.

- Depth Reduction.

  Depth reduction is to obtain solve forms as

  $\varphi(x, \bar{y})$ is solved in x iff $depth_\varphi(x) = 0$.



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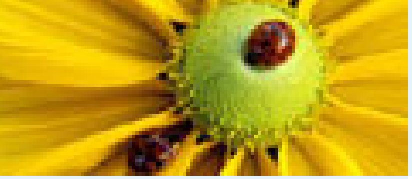
- $\varphi(x, \bar{y})$ is solved in x if it is in the form

$$\bigwedge_{i \leq m} u_i \prec^{\mathsf{kb}} x \;\wedge\; \bigwedge_{j \leq n} x \prec^{\mathsf{kb}} v_j \;\wedge\; \varphi'(\bar{y}),$$

where x does not appear in u_i, v_i and φ'.



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  where x does not appear in u_i, v_i and φ'.

- If $\varphi(x, \bar{y})$ is solved in x, then $(\exists x)\, \varphi(x, \bar{y})$ simplifi es to

$$\bigwedge_{i \le m, j \le n} u_i \prec_2^{\mathrm{kb}} v_j \;\wedge\; \varphi'(\bar{y})$$

  where $x \prec_n^{\mathrm{kb}} y$, called gap order, states there is an increasing chain from x to y of length at least n.

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Case 1: All occurrences of x have depth greater than 0.

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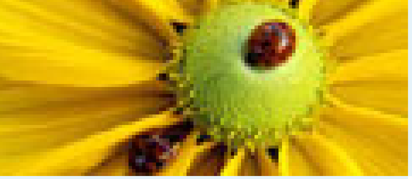
Case 1: All occurrences of x have depth greater than 0.

In this case, $\exists x \varphi(x, \bar{y})$ goes to

$$\exists x_1, \ldots, \exists x_k \varphi'(x_1, \ldots, x_k, \bar{y}),$$

where

$$\varphi'(x_1, \ldots, x_k, \bar{y}) \equiv \varphi(x, \bar{y})[x_i \leftarrow \mathsf{s}_i^{\alpha}(x)].$$



Depth Reduction (2)

Case 2: Some x have depth 0 and some do not.

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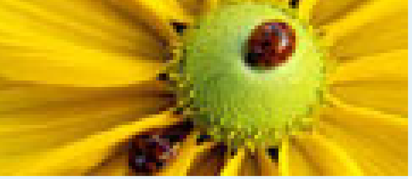

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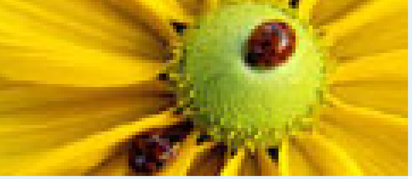
Case 2: Some x have depth 0 and some do not.

- Decompose 0-depth occurrences of x in terms of

$$\mathsf{s}_1^{\alpha}(x), \ldots, \mathsf{s}_k^{\alpha}(x).$$



Depth Reduction (2)

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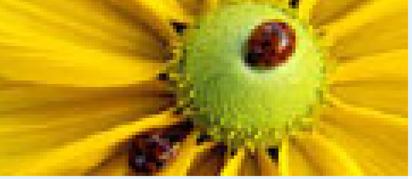
Case 2: Some x have depth 0 and some do not.

- Decompose 0-depth occurrences of x in terms of

$$\mathsf{s}_1^{\alpha}(x), \ldots, \mathsf{s}_k^{\alpha}(x).$$

- This amounts to expressing $x \prec_n^{\mathsf{kb}} t$ and $t \prec_n^{\mathsf{kb}} x$ using

$$\mathsf{s}_1^{\alpha}(x), \ldots, \mathsf{s}_k^{\alpha}(x).$$



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Case 2: Some x have depth 0 and some do not.

- Decompose 0-depth occurrences of x in terms of

$$\mathsf{s}_1^\alpha(x), \ldots, \mathsf{s}_k^\alpha(x).$$

- This amounts to expressing $x <_n^{\mathsf{kb}} t$ and $t <_n^{\mathsf{kb}} x$ using

$$\mathsf{s}_1^\alpha(x), \ldots, \mathsf{s}_k^\alpha(x).$$

- Then apply the reduction as in Case 1!

Depth Reduction (2)

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Case 2: Some x have depth 0 and some do not.

- Decompose 0-depth occurrences of x in terms of

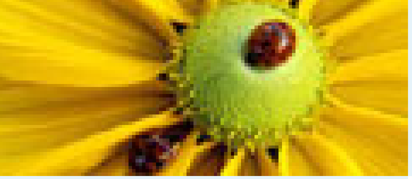
$$\mathsf{s}_1^{\alpha}(x), \ldots, \mathsf{s}_k^{\alpha}(x).$$

- This amounts to expressing $x \prec_n^{\mathsf{kb}} t$ and $t \prec_n^{\mathsf{kb}} x$ using

$$\mathsf{s}_1^{\alpha}(x), \ldots, \mathsf{s}_k^{\alpha}(x).$$

- Then apply the reduction as in Case 1!

☞ In order to do that, we need to extend the language.



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1. Decompose $\prec^{kb}$ into three disjoint suborders $\prec^{w}$, $\prec^{p}$ and $\prec^{l}$.

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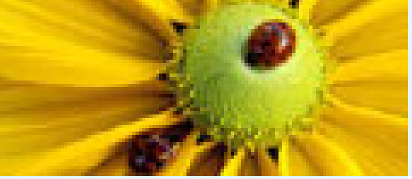
1. Decompose $\prec^{\mathsf{kb}}$ into three disjoint suborders $\prec^{\mathsf{w}}$, $\prec^{\mathsf{p}}$ and $\prec^{\mathsf{l}}$.

2. Extend $\prec^{\mathsf{w}}$, $\prec^{\mathsf{p}}$ and $\prec^{\mathsf{l}}$ to $\prec^{\mathsf{w}}_n$, $\prec^{\mathsf{p}}_n$ and $\prec^{\mathsf{l}}_n$, respectively.



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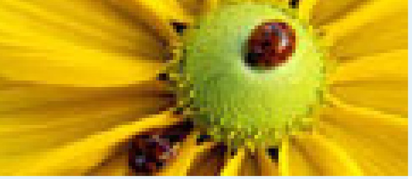
1. Decompose $\prec^{\mathsf{kb}}$ into three disjoint suborders $\prec^{\mathsf{w}}$, $\prec^{\mathsf{p}}$ and $\prec^{\mathsf{l}}$.
2. Extend $\prec^{\mathsf{w}}$, $\prec^{\mathsf{p}}$ and $\prec^{\mathsf{l}}$ to $\prec^{\mathsf{w}}_n$, $\prec^{\mathsf{p}}_n$ and $\prec^{\mathsf{l}}_n$, respectively.
3. Add Presburger arithmetic explicitly to represent weight.



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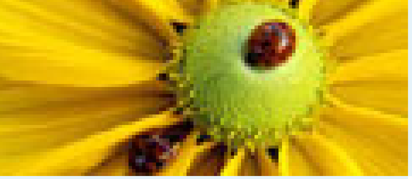
2. Extend $\prec^{\mathrm{w}}$, $\prec^{\mathrm{p}}$ and $\prec^{\mathrm{l}}$ to $\prec^{\mathrm{w}}_n$, $\prec^{\mathrm{p}}_n$ and $\prec^{\mathrm{l}}_n$, respectively.

3. Add Presburger arithmetic explicitly to represent weight.

4. Define counting constraints to count terms of certain weight.



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1. Decompose $\prec^{\mathsf{kb}}$ into three disjoint suborders $\prec^{\mathsf{w}}$, $\prec^{\mathsf{p}}$ and $\prec^{\mathsf{l}}$.
2. Extend $\prec^{\mathsf{w}}$, $\prec^{\mathsf{p}}$ and $\prec^{\mathsf{l}}$ to $\prec^{\mathsf{w}}_n$, $\prec^{\mathsf{p}}_n$ and $\prec^{\mathsf{l}}_n$, respectively.
3. Add Presburger arithmetic explicitly to represent weight.
4. Define **counting constraints** to count terms of certain weight.
5. Define **boundary functions** to **delineate** gap orders.

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1. Decompose $\prec^{\mathsf{kb}}$ into three disjoint suborders $\prec^{\mathsf{w}}$, $\prec^{\mathsf{p}}$ and $\prec^{\mathsf{l}}$.
2. Extend $\prec^{\mathsf{w}}$, $\prec^{\mathsf{p}}$ and $\prec^{\mathsf{l}}$ to $\prec^{\mathsf{w}}_n$, $\prec^{\mathsf{p}}_n$ and $\prec^{\mathsf{l}}_n$, respectively.
3. Add Presburger arithmetic explicitly to represent weight.
4. Define counting constraints to count terms of certain weight.
5. Define boundary functions to delineate gap orders.
6. Extend all aforementioned notions to tuples of terms.

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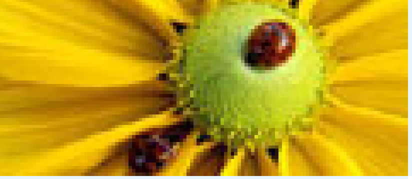
1. Weight Order $\prec^{\mathsf{w}}$:

$$u \prec^{\mathsf{w}} v \Leftrightarrow \mathsf{W}(u) < \mathsf{W}(v).$$



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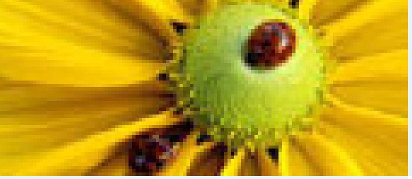
1. Weight Order $\prec^{\mathsf{w}}$:

$$u \prec^{\mathsf{w}} v \Leftrightarrow \mathsf{W}(u) < \mathsf{W}(v).$$

2. Precedence Order $\prec^{\mathsf{p}}$:

$$u \prec^{\mathsf{p}} v \Leftrightarrow \mathsf{W}(u) = \mathsf{W}(s) \ \ \& \ \ \mathsf{type}(u) \prec^{\Sigma} \mathsf{type}(v).$$



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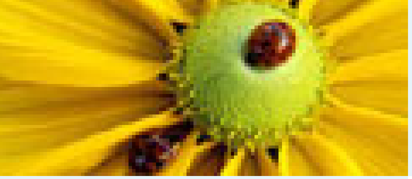
2. Precedence Order $\prec^{\mathsf{p}}$:

$$u \prec^{\mathsf{p}} v \Leftrightarrow \mathsf{W}(u) = \mathsf{W}(s) \;\;\&\;\; \mathsf{type}(u) \prec^{\Sigma} \mathsf{type}(v).$$

3. Lexicographical Order $\prec^{\mathsf{l}}$:

$$u \prec^{\mathsf{l}} v \Leftrightarrow \mathsf{W}(u) = \mathsf{W}(v) \;\;\&\;\; \mathsf{type}(u) = \mathsf{type}(v) \;\;\&\;\; u \prec^{\mathsf{kb}} v.$$



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$$u \prec^{\mathsf{p}} v \Leftrightarrow \mathsf{W}(u) = \mathsf{W}(s) \ \ \& \ \ \mathsf{type}(u) \prec^{\Sigma} \mathsf{type}(v).$$

3. Lexicographical Order $\prec^{\mathsf{l}}$:

$$u \prec^{\mathsf{l}} v \Leftrightarrow \mathsf{W}(u) = \mathsf{W}(v) \ \ \& \ \ \mathsf{type}(u) = \mathsf{type}(v) \ \ \& \ \ u \prec^{\mathsf{kb}} v.$$

☞ Abbreviations:

$$u \prec^{\mathsf{pl}} v \Leftrightarrow u \prec^{\mathsf{p}} v \ \vee \ u \prec^{\mathsf{l}} v,$$

$$u \prec^{\mathsf{kb}} v \Leftrightarrow u \prec^{\mathsf{w}} v \ \vee \ u \prec^{\mathsf{p}} v \ \vee \ u \prec^{\mathsf{l}} v.$$

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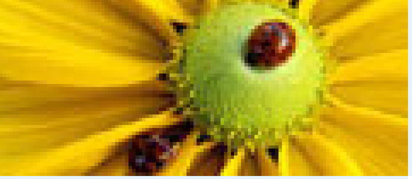
- Gap Order $\prec_n^{\mathsf{kb}}$:

$$u \prec_n^{\mathsf{kb}} v \leftrightarrow (\exists u_1, \ldots, \exists u_n)\Big[u \prec^{\mathsf{kb}} u_1 \prec^{\mathsf{kb}} \ldots \prec^{\mathsf{kb}} u_n \preceq^{\mathsf{kb}} v\Big].$$



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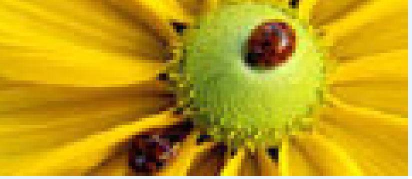
- **Gap Order** $\prec_n^{\mathsf{kb}}$:

$$u \prec_n^{\mathsf{kb}} v \leftrightarrow (\exists u_1, \ldots, \exists u_n)\left[u \prec^{\mathsf{kb}} u_1 \prec^{\mathsf{kb}} \ldots \prec^{\mathsf{kb}} u_n \preceq^{\mathsf{kb}} v\right].$$

- **Weight Gap Order** $\prec_n^{\mathsf{w}}$:

$$u \prec_n^{\mathsf{w}} v \leftrightarrow u \prec_n^{\mathsf{kb}} v \ \wedge \ u \prec^{\mathsf{w}} v.$$



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- **Gap Order** $\prec_n^{\mathsf{kb}}$:

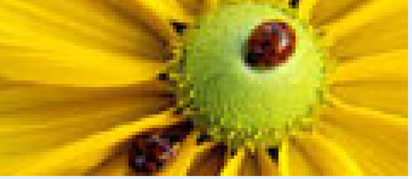
$$u \prec_n^{\mathsf{kb}} v \leftrightarrow (\exists u_1, \ldots, \exists u_n)\big[u \prec^{\mathsf{kb}} u_1 \prec^{\mathsf{kb}} \ldots \prec^{\mathsf{kb}} u_n \preceq^{\mathsf{kb}} v\big].$$

- **Weight Gap Order** $\prec_n^{\mathsf{w}}$:

$$u \prec_n^{\mathsf{w}} v \leftrightarrow u \prec_n^{\mathsf{kb}} v \ \wedge\ u \prec^{\mathsf{w}} v.$$

- **Precedence Gap Order** $\prec_n^{\mathsf{p}}$:

$$u \prec_n^{\mathsf{p}} v \leftrightarrow u \prec_n^{\mathsf{kb}} v \ \wedge\ u \prec^{\mathsf{p}} v.$$



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- Gap Order $\prec_n^{\mathsf{kb}}$:

$$u \prec_n^{\mathsf{kb}} v \leftrightarrow (\exists u_1, \ldots, \exists u_n)\big[u \prec^{\mathsf{kb}} u_1 \prec^{\mathsf{kb}} \ldots \prec^{\mathsf{kb}} u_n \preceq^{\mathsf{kb}} v\big].$$

- Weight Gap Order $\prec_n^{\mathsf{w}}$:

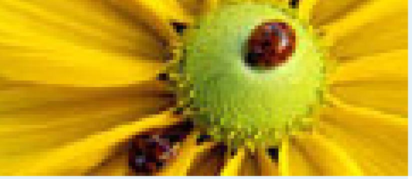
$$u \prec_n^{\mathsf{w}} v \leftrightarrow u \prec_n^{\mathsf{kb}} v \ \wedge \ u \prec^{\mathsf{w}} v.$$

- Precedence Gap Order $\prec_n^{\mathsf{p}}$:

$$u \prec_n^{\mathsf{p}} v \leftrightarrow u \prec_n^{\mathsf{kb}} v \ \wedge \ u \prec^{\mathsf{p}} v.$$

- Lexicographical Gap Order $\prec_n^{\mathsf{l}}$:

$$u \prec_n^{\mathsf{l}} v \leftrightarrow u \prec_n^{\mathsf{kb}} v \ \wedge \ u \prec^{\mathsf{l}} v.$$



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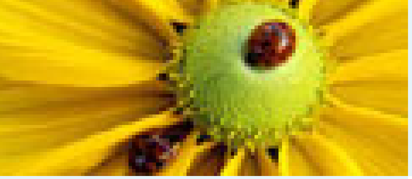
$0^{\mathsf{w}}, 1^{\mathsf{w}} : \mathbb{N} \rightarrow \mathsf{TA}; 0^{\mathsf{p}}, 1^{\mathsf{p}} : \mathbb{N}^2 \rightarrow \mathsf{TA}$ such that

- $0^{\mathsf{w}}(n)$: the smallest term of weight n.



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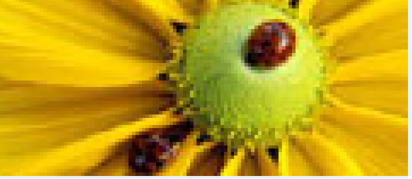
$0^{\mathsf{w}}, 1^{\mathsf{w}} : \mathbb{N} \to \mathsf{TA}; 0^{\mathsf{p}}, 1^{\mathsf{p}} : \mathbb{N}^2 \to \mathsf{TA}$ such that

- $0^{\mathsf{w}}(n)$: the smallest term of weight n.
- $0^{\mathsf{p}}(n, p)$: the smallest term of weight n and type α_p.



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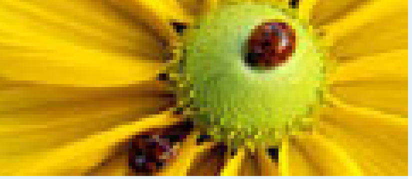
$0^{\mathsf{w}}, 1^{\mathsf{w}} : \mathbb{N} \rightarrow \mathsf{TA}; 0^{\mathsf{p}}, 1^{\mathsf{p}} : \mathbb{N}^2 \rightarrow \mathsf{TA}$ such that

- $0^{\mathsf{w}}(n)$: the smallest term of weight n.
- $0^{\mathsf{p}}(n, p)$: the smallest term of weight n and type α_p.
- $1^{\mathsf{w}}(n)$: the largest term of weight n.



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$0^{\mathsf{w}}, 1^{\mathsf{w}} : \mathbb{N} \rightarrow \mathsf{TA}; 0^{\mathsf{p}}, 1^{\mathsf{p}} : \mathbb{N}^2 \rightarrow \mathsf{TA}$ such that

- $0^{\mathsf{w}}(n)$: the smallest term of weight n.
- $0^{\mathsf{p}}(n, p)$: the smallest term of weight n and type α_p.
- $1^{\mathsf{w}}(n)$: the largest term of weight n.
- $1^{\mathsf{p}}(n, p)$: the largest term of weight n and type α_p.

Boundary Functions

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$0^{\mathsf{w}}, 1^{\mathsf{w}} : \mathbb{N} \to \mathsf{TA}; 0^{\mathsf{p}}, 1^{\mathsf{p}} : \mathbb{N}^2 \to \mathsf{TA}$ such that

- $0^{\mathsf{w}}(n)$: the **smallest** term of weight n.
- $0^{\mathsf{p}}(n, p)$: the **smallest** term of weight n and type α_p.
- $1^{\mathsf{w}}(n)$: the **largest** term of weight n.
- $1^{\mathsf{p}}(n, p)$: the **largest** term of weight n and type α_p.

☞ Example of Use:

$$u \prec_5^{\mathsf{w}} v \leftrightarrow \bigvee_{n_1+n_2+n_3=5} u \prec_{n_1}^{\mathsf{pl}} 1^{\mathsf{w}}_{(u^{\mathsf{w}})} \prec_{n_2}^{\mathsf{w}} 0^{\mathsf{w}}_{(v^{\mathsf{w}})} \prec_{n_3}^{\mathsf{pl}} v.$$

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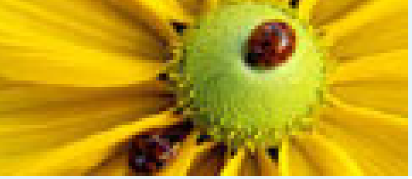
- $\text{CNT}_n(x)$ states that

  "*there are at least* $n+1$ *distinct* TA-*terms of weight* x."



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- $\mathrm{CNT}_n(x)$ states that

  "there are at least $n+1$ distinct TA*-terms of weight x."*

- $\mathrm{CNT}_0(x)$ (or $\mathrm{Tree}(x)$) states that

  x is a legitimate weight of a term.

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- $\text{CNT}_n(x)$ is expressible in Presburger arithmetic.

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  x *is a legitimate weight of a term.*

- $\text{CNT}_n(x)$ is expressible in Presburger arithmetic.

☞ Example of Use:

$$0^{\mathsf{w}}_{(x)} \prec^{\mathsf{pl}}_{n} 1^{\mathsf{w}}_{(x)} \;\leftrightarrow\; \text{CNT}_n(x).$$

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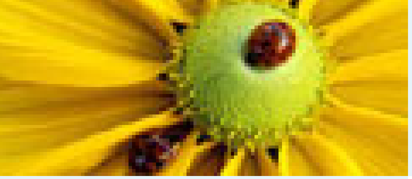
Extended structure:

$$\begin{aligned}\mathfrak{A}^{\mathbb{Z}}_{\mathsf{kb}^+} = \langle \quad & \mathfrak{A}_{\mathsf{TA}}; \mathfrak{A}_{\mathbb{Z}}; (.)^{\mathsf{w}}; \\ & \prec^{\sharp}_{n}, \sharp \in \{\mathsf{kb}, \mathsf{w}, \mathsf{p}, \mathsf{l}, \mathsf{pl}\}, \\ & 0^{*}(\ldots), 1^{*}(\ldots), * \in \{\mathsf{w}, \mathsf{p}\} \quad \rangle.\end{aligned}$$



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- $\mathfrak{A}_{\mathsf{TA}}$: Term algebras.

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- $\mathfrak{A}_{\mathsf{TA}}$: Term algebras.
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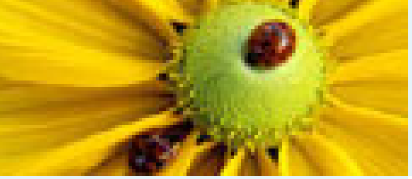

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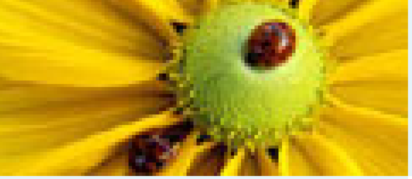
- $\mathfrak{A}_{\mathsf{TA}}$: Term algebras.
- $\mathfrak{A}_{\mathbb{Z}}$: Presburger arithmetic.
- $(.)^{\mathsf{w}}$: Weight function.



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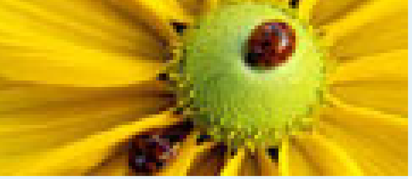
- $\mathfrak{A}_{\mathsf{TA}}$: Term algebras.
- $\mathfrak{A}_{\mathbb{Z}}$: Presburger arithmetic.
- $(.)^{\mathsf{w}}$: Weight function.
- $\prec^{\sharp}_{n}$: Gap orders.



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$$\prec^{\sharp}_{n}, \sharp \in \{\mathsf{kb}, \mathsf{w}, \mathsf{p}, \mathsf{l}, \mathsf{pl}\},$$
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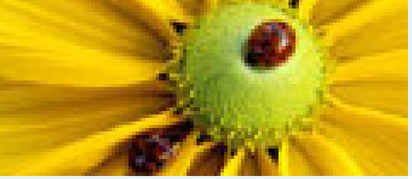
- $(.)^{\mathsf{w}}$: Weight function.
- $\prec^{\sharp}_{n}$: Gap orders.
- $0^{*}(\ldots), 1^{*}(\ldots)$: Boundary terms.



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Extended structure:

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- $\mathfrak{A}_{\mathsf{TA}}$: Term algebras.
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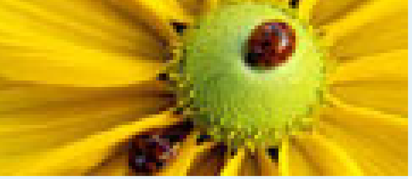
- $(.)^{\mathsf{w}}$: Weight function.
- $\prec^{\sharp}_{n}$: Gap orders.
- $0^{*}(...), 1^{*}(...)$: Boundary terms.

☞ Example of Use:

$$(\exists x : \mathsf{TA})\Big[\, 0^{\mathsf{w}}_{(x^{\mathsf{w}})} \prec^{\mathsf{l}}_{2} x \prec^{\mathsf{l}}_{3} 1^{\mathsf{w}}_{(x^{\mathsf{w}})} \,\Big]$$



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Input: $(\exists \bar{x})\ \varphi(\bar{x}, \bar{y})$.

While $\bar{x} \neq \emptyset$.

- While $(\forall x \in \bar{x})\ depth_{\varphi}(x) > 0$.

  Depth Reduction.

  - VARIABLE SELECTION.
  - DECOMPOSITION.
  - SIMPLIFICATION.

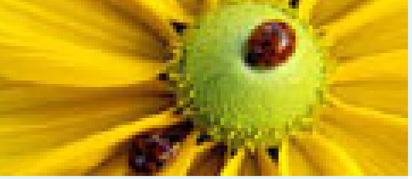
  Done.

- While $(\exists x \in \bar{x})\ depth_{\varphi}(x) = 0$.

  Elimination.

  Done.

Done.



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- Select a variable $x \in \bar{x}$ such that $\mathsf{s}_i^{\alpha}(x)$ appears in $\varphi(\bar{x}, \bar{y})$.

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- Select a variable $x \in \bar{x}$ such that $s_i^{\alpha}(x)$ appears in $\varphi(\bar{x}, \bar{y})$.

☞ The variable selection is done in depth-fi rstmanner.

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- Select a variable $x \in \bar{x}$ such that $s_i^\alpha(x)$ appears in $\varphi(\bar{x}, \bar{y})$.

☞ The variable selection is done in depth-fi rstmanner.

☞ I.e., choose variables generated in the previous round.

Decomposition

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Rewrite $(\exists \bar{x})\ \varphi(\bar{x}, \bar{y})$ to

$$\exists x_1 \ldots \exists x_k \exists \bar{x} \left[\mathsf{Is}_\alpha(x) \wedge \bigwedge_{1 \le i \le k} \mathsf{s}_i^\alpha(x) = x_i \wedge \varphi(\bar{x}, \bar{y}) \right].$$

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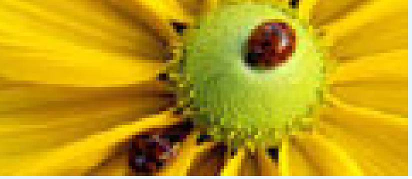
- Apply the following rules to each occurrence of x.



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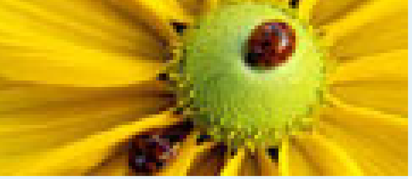
- Apply the following rules to each occurrence of x.

1. Replace $x \prec_n^\sharp t$ (or $t \prec_n^\sharp x$) by a quantifier-free formula

$$\varphi'(\mathsf{s}_1^\alpha(x), \ldots, \mathsf{s}_k^\alpha(x), \mathsf{s}_1^\alpha(t), \ldots, \mathsf{s}_k^\alpha(t)).$$



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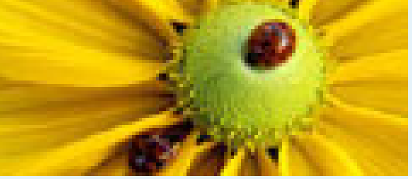
- Apply the following rules to each occurrence of x.

1. Replace $x <_n^\sharp t$ (or $t <_n^\sharp x$) by a quantifi er-free formula

$$\varphi'(\mathsf{s}_1^\alpha(x), \ldots, \mathsf{s}_k^\alpha(x), \mathsf{s}_1^\alpha(t), \ldots, \mathsf{s}_k^\alpha(t)).$$

2. Replace $\mathsf{s}_i^\alpha(x)$ in $\varphi(\bar{x}, \bar{y})$ by x_i $(1 \le i \le k)$.



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1. Replace $x <_n^\sharp t$ (or $t <_n^\sharp x$) by a quantifier-free formula

$$\varphi'(\mathsf{s}_1^\alpha(x), \ldots, \mathsf{s}_k^\alpha(x), \mathsf{s}_1^\alpha(t), \ldots, \mathsf{s}_k^\alpha(t)).$$

2. Replace $\mathsf{s}_i^\alpha(x)$ in $\varphi(\bar{x}, \bar{y})$ by x_i $(1 \leq i \leq k)$.

- Denote the result of this simplification as

$$\exists x_1 \ldots \exists x_k \exists(\bar{x} \setminus x) \left[\varphi'(\bar{x} \setminus x, x_1, \ldots, x_k, \bar{y}) \right].$$

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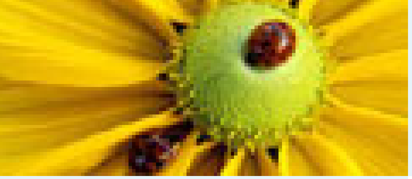
- Now we must have

$$\exists x \Big[\bigwedge_{i \le m} u_i \prec^{\mathsf{kb}} x \;\wedge\; \bigwedge_{j \le n} x \prec^{\mathsf{kb}} v_j \;\wedge\; \varphi'(\bar{y}) \Big],$$

where x appears none of u_i, v_j and φ'.



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- Now we must have

$$\exists x\Big[\bigwedge_{i\le m} u_i \prec^{\mathrm{kb}} x \;\wedge\; \bigwedge_{j\le n} x \prec^{\mathrm{kb}} v_j \;\wedge\; \varphi'(\bar{y})\Big],$$

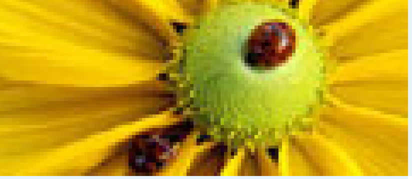
  where x appears none of u_i, v_j and φ'.

- Guess a gap order completion, we rewrite it to

$$u_{i'} \prec_2^{\mathrm{kb}} v_{j'} \;\wedge\; \varphi'(\bar{y})$$

$$\wedge \text{ “}u_{i'} \textit{ is the greatest of } \{u_i \mid i \le m\}\text{”}$$

$$\wedge \text{ “}v_{j'} \textit{ is the smallest of } \{v_j \mid j \le n\}\text{”}.$$



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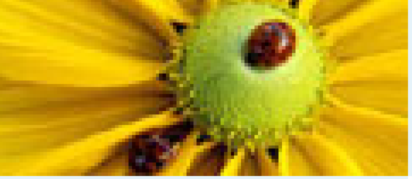
Termination is subtle as some complexity measures increase.



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Termination is subtle as some complexity measures increase.

- Depth reduction increases the depth of other variables.

  For example, $x \neq t$ becomes

$$\bigvee_{1 \leq i \leq k} \mathsf{s}_i^{\alpha}(t) \neq x_i \vee \neg \mathsf{Is}_{\alpha}(t).$$

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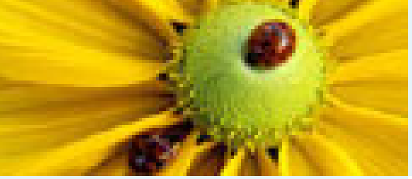
  For example, $x \neq t$ becomes

$$\bigvee_{1 \leq i \leq k} \mathsf{s}_i^{\alpha}(t) \neq x_i \vee \neg \mathsf{Is}_{\alpha}(t).$$

- Depth reduction introduces more existential quantifi ers.

  For example, $(\exists \bar{x})\ \varphi(\bar{x}, \bar{y})$ becomes

$$\exists x_1 \ldots \exists x_k \exists \bar{x} \left[\mathsf{Is}_{\alpha}(x) \wedge \bigwedge_{1 \leq i \leq k} \mathsf{s}_i^{\alpha}(x) = x_i \wedge \varphi(\bar{x}, \bar{y}) \right].$$



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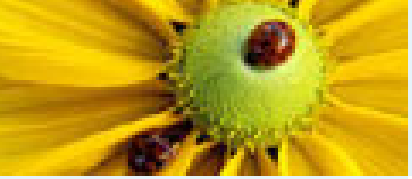
- Depth reduction introduces more order literals.

  For example, $u \prec_5^{\mathsf{w}} v$ becomes

$$\bigvee_{n_1+n_2+n_3=5} u \prec_{n_1}^{\mathsf{pl}} 1_{(u^{\mathsf{w}})}^{\mathsf{w}} \prec_{n_2}^{\mathsf{w}} 0_{(v^{\mathsf{w}})}^{\mathsf{w}} \prec_{n_3}^{\mathsf{pl}} v.$$



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- Depth reduction introduces more order literals.

  For example, $u \prec_5^{\mathsf{w}} v$ becomes

$$\bigvee_{n_1+n_2+n_3=5} u \prec_{n_1}^{\mathsf{pl}} 1^{\mathsf{w}}_{(u^{\mathsf{w}})} \prec_{n_2}^{\mathsf{w}} 0^{\mathsf{w}}_{(v^{\mathsf{w}})} \prec_{n_3}^{\mathsf{pl}} v.$$

- Depth reduction introduces more equalities.

  For example, $x \prec^{\mathsf{l}} t$ could produce

$$\mathsf{car}(x) = \mathsf{car}(t) \;\wedge\; \mathsf{cdr}(x) \prec^{\mathsf{kb}} \mathsf{cdr}(t).$$

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- Depth reduction introduces more order literals.

  For example, $u \prec^{\mathsf{w}}_{5} v$ becomes

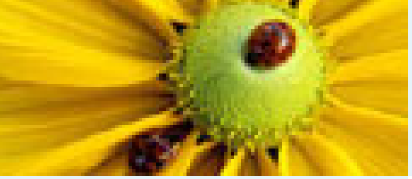
$$\bigvee_{n_1+n_2+n_3=5} u \prec^{\mathsf{pl}}_{n_1} 1^{\mathsf{w}}_{(u^{\mathsf{w}})} \prec^{\mathsf{w}}_{n_2} 0^{\mathsf{w}}_{(v^{\mathsf{w}})} \prec^{\mathsf{pl}}_{n_3} v.$$

- Depth reduction introduces more equalities.

  For example, $x \prec^{\mathsf{l}} t$ could produce

$$\mathsf{car}(x) = \mathsf{car}(t) \ \wedge\ \mathsf{cdr}(x) \prec^{\mathsf{kb}} \mathsf{cdr}(t).$$

☞ Does it terminate???



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☞ Real measure: $\sharp$ **of open gap order literals (OGOL).**

OGOL: a gap order relation between ordinary terms.

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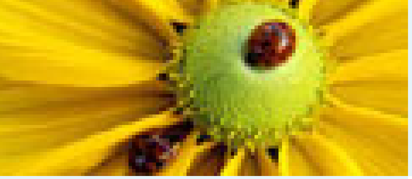
- Real measure: $\sharp$ **of open gap order literals (OGOL).**

  OGOL: a gap order relation between ordinary terms.

- No transformation generates new OGOLs.



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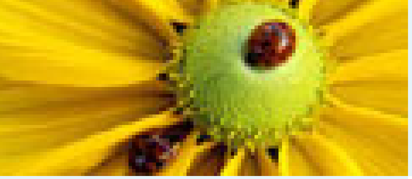
- Real measure: $\sharp$ **of open gap order literals (OGOL).**

  OGOL: a gap order relation between ordinary terms.

- No transformation generates new OGOLs.
- The fi nal elimination step removes at least one OGOL.



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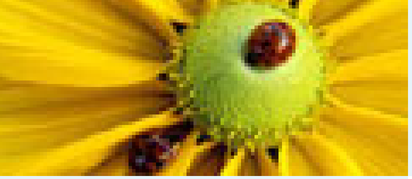
- Real measure: $\sharp$ **of open gap order literals (OGOL).**

  OGOL: a gap order relation between ordinary terms.

- No transformation generates new OGOLs.

- The final elimination step removes at least one OGOL.

- Without OGOLs, the depths of terms strictly decrease!



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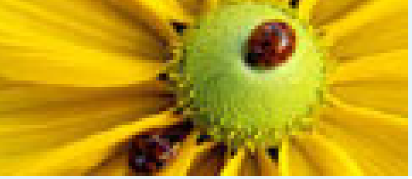
- Consider the KBO on LISP list structure parameterized with

$$\mathrm{W}(\mathrm{cons}) = \mathrm{W}(\mathrm{nil}) = 1 \ \&\ \mathrm{nil} \prec^{\Sigma} \mathrm{cons}.$$



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- Consider the KBO on LISP list structure parameterized with

$$\mathrm{W(cons)} = \mathrm{W(nil)} = 1 \ \&\ \mathrm{nil} \prec^{\Sigma} \mathrm{cons}.$$

- Consider the formula

$$(\exists x)\left[\,\mathrm{car}(x) \prec^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x)) \ \wedge\ \mathrm{cdr}(\mathrm{cdr}(\mathrm{car}(x))) \prec^{!}_{3} y\,\right],$$

where $depth(x) = 3$.

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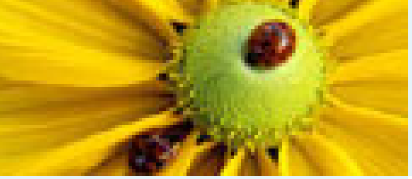
$$(\exists x)\left[\ \mathsf{car}(x) \prec^{!}_{2} \mathsf{cdr}(\mathsf{cdr}(x)) \ \wedge\ \mathsf{cdr}(\mathsf{cdr}(\mathsf{car}(x))) \prec^{!}_{3} y\ \right],$$



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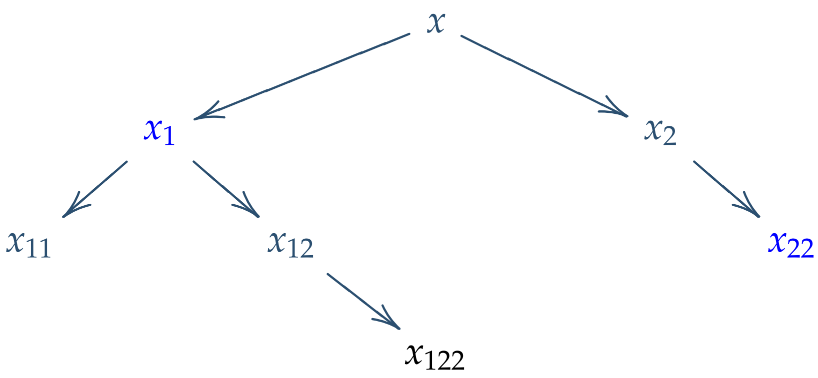
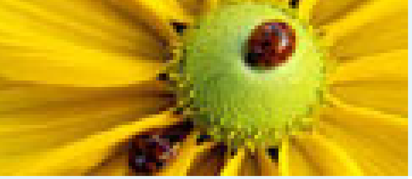
$$(\exists x)\left[\, \mathrm{car}(x) \prec^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x)) \;\wedge\; \mathrm{cdr}(\mathrm{cdr}(\mathrm{car}(x))) \prec^{!}_{3} y \,\right],$$





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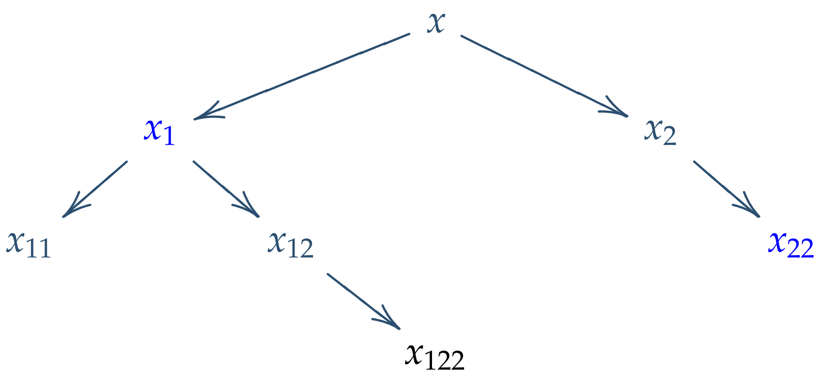
$$(\exists x)\left[\ \mathrm{car}(x) \prec^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x)) \ \wedge\ \mathrm{cdr}(\mathrm{cdr}(\mathrm{car}(x))) \prec^{!}_{3} y\ \right],$$



x_1 : car(x),

x_2 : cdr(x),

x_{11} : car(car(x)),

x_{12} : cdr(car(x)),

x_{22} : cdr(cdr(x)),

x_{122} : cdr(cdr(car(x)))

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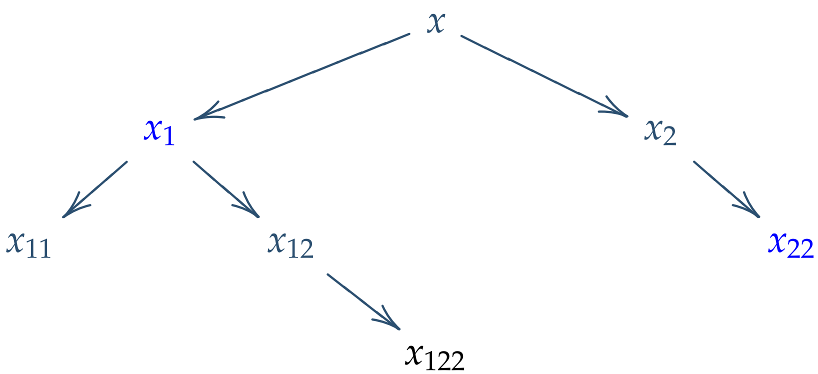
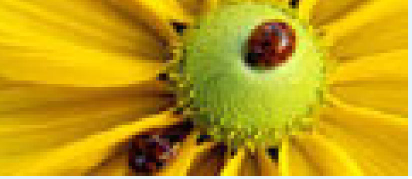
$$(\exists x)\left[\ \mathrm{car}(x) \prec^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x)) \ \land \ \mathrm{cdr}(\mathrm{cdr}(\mathrm{car}(x))) \prec^{!}_{3} y\ \right],$$



x_1 : $\mathrm{car}(x)$,
x_2 : $\mathrm{cdr}(x)$,
x_{11} : $\mathrm{car}(\mathrm{car}(x))$,
x_{12} : $\mathrm{cdr}(\mathrm{car}(x))$,
x_{22} : $\mathrm{cdr}(\mathrm{cdr}(x))$,
x_{122} : $\mathrm{cdr}(\mathrm{cdr}(\mathrm{car}(x)))$

Solution: $x = ?$



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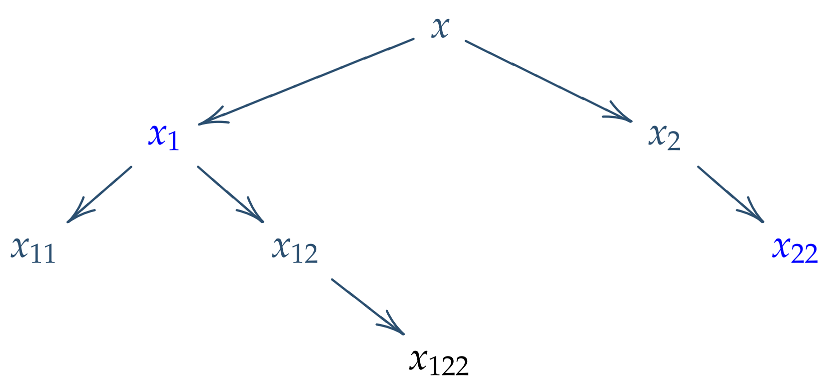
$$(\exists x)\left[\ \mathsf{car}(x) \prec^{!}_{2} \mathsf{cdr}(\mathsf{cdr}(x)) \ \wedge\ \mathsf{cdr}(\mathsf{cdr}(\mathsf{car}(x))) \prec^{!}_{3} y\ \right],$$



x_1 : car(x),

x_2 : cdr(x),

x_{11} : car(car(x)),

x_{12} : cdr(car(x)),

x_{22} : cdr(cdr(x)),

x_{122} : cdr(cdr(car(x)))

Solution: $x = ?$

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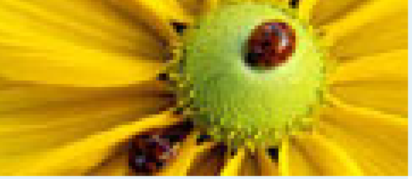
- Select variable x.



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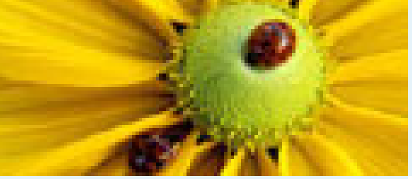
- Select variable x.
- Decompose x in terms of car(x) and cdr(x). We have

$$(\exists x \exists x_1 \exists x_2) \Big[\text{car}(x) = x_1 \ \wedge \ \text{cdr}(x) = x_2 \\ \wedge \ \text{car}(x) <^{!}_{2} \text{cdr}(\text{cdr}(x)) \ \wedge \ \text{cdr}(\text{cdr}(\text{car}(x))) <^{!}_{3} y \Big].$$



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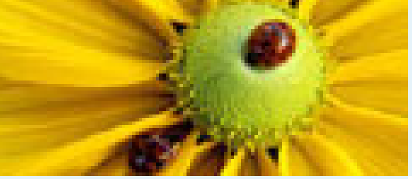
- Select variable x.
- Decompose x in terms of car(x) and cdr(x). We have

$$(\exists x \exists x_1 \exists x_2) \Big[\text{car}(x) = x_1 \ \wedge \ \text{cdr}(x) = x_2 \ \wedge \ \text{car}(x) <^!_2 \text{cdr}(\text{cdr}(x)) \ \wedge \ \text{cdr}(\text{cdr}(\text{car}(x))) <^!_3 y \Big].$$

- Simplification.

$$(\exists x_1 \exists x_2) \Big[x_1 <^!_2 \text{cdr}(x_2) \ \wedge \ \text{cdr}(\text{cdr}(x_1)) <^!_3 y \Big],$$

where $depth(x_1) = 2$ and $depth(x_2) = 1$,



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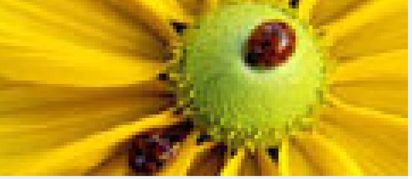
Continue with

$$(\exists x_1 \exists x_2)\Big[\; x_1 \prec^{!}_{2} \mathrm{cdr}(x_2) \;\wedge\; \mathrm{cdr}(\mathrm{cdr}(x_1)) \prec^{!}_{3} y \;\Big].$$



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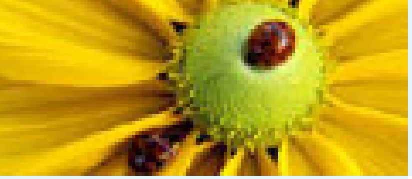
Continue with

$$(\exists x_1 \exists x_2)\Big[\, x_1 \prec^{!}_{2} \mathrm{cdr}(x_2) \ \wedge\ \mathrm{cdr}(\mathrm{cdr}(x_1)) \prec^{!}_{3} y \,\Big].$$

- Select variable x_1.



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Continue with

$$(\exists x_1 \exists x_2)\Big[\; x_1 \prec^{|}_{2} \mathrm{cdr}(x_2) \;\wedge\; \mathrm{cdr}(\mathrm{cdr}(x_1)) \prec^{|}_{3} y \;\Big].$$

- Select variable x_1.
- Decompose x_1. Replace $x_1 \prec^{|}_{2} \mathrm{cdr}(x_2)$ by

$$\mathrm{car}(x_1) = \mathrm{car}(\mathrm{cdr}(x_2)) \;\wedge\; \mathrm{cdr}(x_1) \prec^{|}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)).$$

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Continue with

$$(\exists x_1 \exists x_2)\Big[\, x_1 \prec^!_2 \mathrm{cdr}(x_2) \;\wedge\; \mathrm{cdr}(\mathrm{cdr}(x_1)) \prec^!_3 y \,\Big].$$

- Select variable x_1.
- Decompose x_1. Replace $x_1 \prec^!_2 \mathrm{cdr}(x_2)$ by

$$\mathrm{car}(x_1) = \mathrm{car}(\mathrm{cdr}(x_2)) \;\wedge\; \mathrm{cdr}(x_1) \prec^!_2 \mathrm{cdr}(\mathrm{cdr}(x_2)).$$

- Simplifi cation.

$$(\exists x_2 \exists x_{11} \exists x_{12})\Big[\, x_{11} = \mathrm{car}(\mathrm{cdr}(x_2)) \;\wedge\; x_{12} \prec^!_2 \mathrm{cdr}(\mathrm{cdr}(x_2)) \;\wedge\; \mathrm{cdr}(x_{12}) \prec^!_3 y \,\Big],$$

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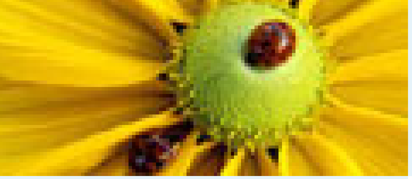
Continue with

$$(\exists x_2 \exists x_{11} \exists x_{12}) \Big[\, x_{11} = \mathrm{car}(\mathrm{cdr}(x_2)) \;\wedge\; x_{12} <^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \;\wedge\; \mathrm{cdr}(x_{12}) <^{!}_{3} y \,\Big],$$



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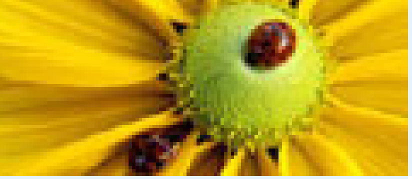
Continue with

$$(\exists x_2 \exists x_{11} \exists x_{12}) \left[x_{11} = \text{car}(\text{cdr}(x_2)) \;\land\; x_{12} <^{!}_{2} \text{cdr}(\text{cdr}(x_2)) \right.$$
$$\left. \land\; \text{cdr}(x_{12}) <^{!}_{3} y \right],$$

- Elimination. Since $\text{depth}(x_{11}) = 0$, we have

$$(\exists x_2 \exists x_{12}) \left[x_{12} <^{!}_{2} \text{cdr}(\text{cdr}(x_2)) \;\land\; \text{cdr}(x_{12}) <^{!}_{3} y \right],$$



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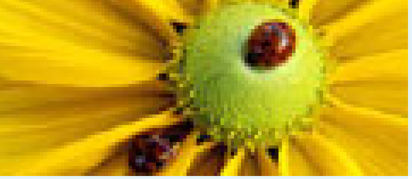
Continue with

$$(\exists x_2 \exists x_{12}) \left[\, x_{12} <^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \;\wedge\; \mathrm{cdr}(x_{12}) <^{!}_{3} y \,\right].$$



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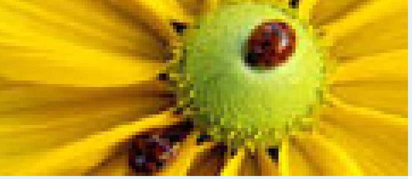
Continue with

$$(\exists x_2 \exists x_{12}) \left[\, x_{12} <^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \;\wedge\; \mathrm{cdr}(x_{12}) <^{!}_{3} y \,\right].$$

- Select variable x_{12}.



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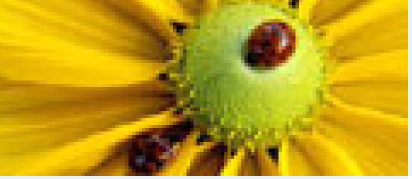
Continue with

$$(\exists x_2 \exists x_{12}) \left[\; x_{12} \prec^{|}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \;\wedge\; \mathrm{cdr}(x_{12}) \prec^{|}_{3} y \; \right].$$

- Select variable x_{12}.
- Decompose x_{12}. Replace $x_{12} \prec^{|}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2))$ by

$$x_{121} = \mathrm{car}(\mathrm{cdr}(\mathrm{cdr}(x_2))) \;\wedge\; x_{122} \prec^{|}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)).$$



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Continue with

$$(\exists x_2 \exists x_{12}) \left[\, x_{12} <^!_2 \mathrm{cdr}(\mathrm{cdr}(x_2)) \;\wedge\; \mathrm{cdr}(x_{12}) <^!_3 y \, \right].$$

- Select variable x_{12}.
- Decompose x_{12}. Replace $x_{12} <^!_2 \mathrm{cdr}(\mathrm{cdr}(x_2))$ by

$$x_{121} = \mathrm{car}(\mathrm{cdr}(\mathrm{cdr}(x_2))) \;\wedge\; x_{122} <^!_2 \mathrm{cdr}(\mathrm{cdr}(x_2)).$$

- Simplifi cation.

$$(\exists x_2 \exists x_{121} \exists x_{122}) \left[\, x_{121} = \mathrm{car}(\mathrm{cdr}(\mathrm{cdr}(x_2))) \;\wedge\; x_{122} <^!_2 \mathrm{cdr}(\mathrm{cdr}(x_2)) \;\wedge\; x_{122} <^!_3 y \, \right],$$

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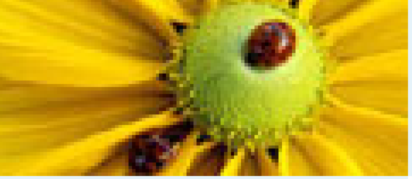
Continue with

$$(\exists x_2 \exists x_{121} \exists x_{122}) \Big[\ x_{121} = \mathrm{car}(\mathrm{cdr}(\mathrm{cdr}(x_2))) \ \wedge\ x_{122} \prec^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \ \wedge\ x_{122} \prec^{!}_{3} y\ \Big],$$



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Continue with

$$(\exists x_2 \exists x_{121} \exists x_{122}) \Big[\, x_{121} = \mathrm{car}(\mathrm{cdr}(\mathrm{cdr}(x_2))) \ \wedge \ x_{122} <^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \\ \wedge \ x_{122} <^{!}_{3} y \, \Big],$$

- Elimination. Since $\mathrm{depth}(x_{121}) = 0$, we have

$$(\exists x_2 \exists x_{122}) \Big[\, x_{122} <^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \ \wedge \ x_{122} <^{!}_{3} y \, \Big].$$

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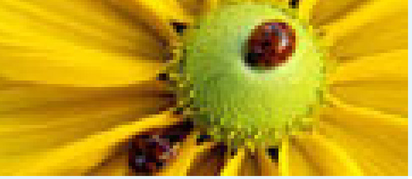
Continue with

$$(\exists x_2 \exists x_{122}) \left[\, x_{122} \prec^{|}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \;\wedge\; x_{122} \prec^{|}_{3} y \,\right].$$



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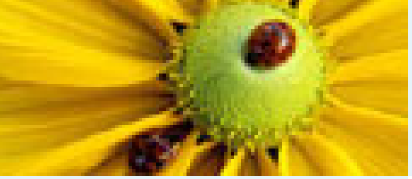
Continue with

$$(\exists x_2 \exists x_{122})\left[\; x_{122} \prec^{|}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \;\wedge\; x_{122} \prec^{|}_{3} y \;\right].$$

- Elimination.



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Continue with

$$(\exists x_2 \exists x_{122}) \left[\; x_{122} \prec_2^{\mid} \mathrm{cdr}(\mathrm{cdr}(x_2)) \;\wedge\; x_{122} \prec_3^{\mid} y \;\right].$$

- Elimination.
  - Take a gap order completion

$$x_{122} \prec_2^{\mid} \mathrm{cdr}(\mathrm{cdr}(x_2)) \prec_1^{\mid} y.$$

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Continue with

$$(\exists x_2 \exists x_{122}) \left[\; x_{122} \prec^{!}_{2} \mathsf{cdr}(\mathsf{cdr}(x_2)) \;\wedge\; x_{122} \prec^{!}_{3} y \;\right].$$

- Elimination.
  - Take a gap order completion

$$x_{122} \prec^{!}_{2} \mathsf{cdr}(\mathsf{cdr}(x_2)) \prec^{!}_{1} y.$$

  - We have

$$(\exists x_2 \exists x_{122}) \left[\; x_{122} \prec^{!}_{2} \mathsf{cdr}(\mathsf{cdr}(x_2)) \prec^{!}_{1} y \;\right],$$

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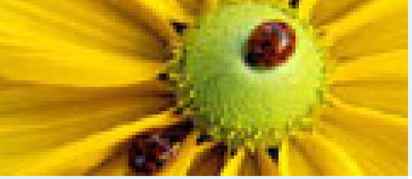

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Continue with

$$(\exists x_2 \exists x_{122})\left[\, x_{122} \prec^{|}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \ \wedge\ x_{122} \prec^{|}_{3} y \,\right].$$

- Elimination.
  - Take a gap order completion

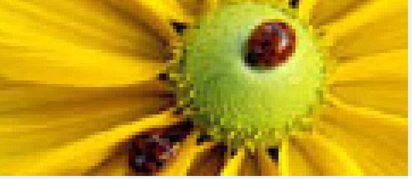
$$x_{122} \prec^{|}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \prec^{|}_{1} y.$$

  - We have

$$(\exists x_2 \exists x_{122})\left[\, x_{122} \prec^{|}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \prec^{|}_{1} y \,\right],$$

  - which simplifi es to

$$(\exists x_2)\left[\, 0^{\mathrm{w}}_{((\mathrm{cdr}(\mathrm{cdr}(x_2)))^{\mathrm{w}})} \prec^{|}_{2} \mathrm{cdr}(\mathrm{cdr}(x_2)) \prec^{|}_{1} y \,\right].$$



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Continue with

$$(\exists x_2 \exists x_{122}) \left[\; x_{122} <^{!}_{2} \mathsf{cdr}(\mathsf{cdr}(x_2)) \;\wedge\; x_{122} <^{!}_{3} y \; \right].$$

- Elimination.
  - Take a gap order completion

$$x_{122} <^{!}_{2} \mathsf{cdr}(\mathsf{cdr}(x_2)) <^{!}_{1} y.$$

  - We have

$$(\exists x_2 \exists x_{122}) \left[\; x_{122} <^{!}_{2} \mathsf{cdr}(\mathsf{cdr}(x_2)) <^{!}_{1} y \; \right],$$

  - which simplifi es to

$$(\exists x_2) \left[\; 0^{\mathsf{w}}_{((\mathsf{cdr}(\mathsf{cdr}(x_2)))^{\mathsf{w}})} <^{!}_{2} \mathsf{cdr}(\mathsf{cdr}(x_2)) <^{!}_{1} y \; \right].$$

☞ The number of OGOLs reduced to 1!

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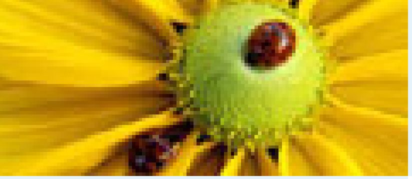
Continue with

$$(\exists x_2)\left[\, 0^{\mathsf{w}}_{((\mathsf{cdr}(\mathsf{cdr}(x_2)))^{\mathsf{w}})} \prec^{\mathsf{l}}_2 \mathsf{cdr}(\mathsf{cdr}(x_2)) \prec^{\mathsf{l}}_1 y \,\right].$$



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Continue with

$$(\exists x_2)\left[\ 0^{\mathsf{w}}_{((\mathsf{cdr}(\mathsf{cdr}(x_2)))^{\mathsf{w}})} \prec^{\mathsf{l}}_2 \mathsf{cdr}(\mathsf{cdr}(x_2)) \prec^{\mathsf{l}}_1 y\ \right].$$

- Depth Reduction. Repeating twice the DEPTH-REDUCTION subprocedure, we have

$$(\exists x_{222})\left[\ 0^{\mathsf{w}}_{(x^{\mathsf{w}}_{222})} \prec^{\mathsf{l}}_2 x_{222} \prec^{\mathsf{l}}_1 y\ \right].$$

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Continue with

$$(\exists x_{222}) \left[\, 0^{\mathsf{w}}_{(x^{\mathsf{w}}_{222})} \prec^{\mathsf{l}}_{2} x_{222} \prec^{\mathsf{l}}_{1} y \, \right].$$

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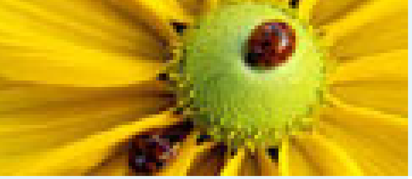
Continue with

$$(\exists x_{222}) \left[\, 0^{\mathsf{w}}_{(x^{\mathsf{w}}_{222})} \prec^{\mathsf{l}}_{2} x_{222} \prec^{\mathsf{l}}_{1} y \, \right].$$

- Reduce term quantifi ers to integer quantifi ers.

$$(\exists z) \left[\, 0^{\mathsf{w}}_{(z)} \prec^{\mathsf{l}}_{3} y \wedge \mathsf{Tree}^{\mathsf{cons}}(z) \, \right].$$



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Continue with

$$(\exists x_{222})\left[\, 0^{\mathsf{w}}_{(x^{\mathsf{w}}_{222})} \prec^{\mathsf{l}}_{2} x_{222} \prec^{\mathsf{l}}_{1} y \,\right].$$

- Reduce term quantifi ers to integer quantifi ers.

$$(\exists z)\left[\, 0^{\mathsf{w}}_{(z)} \prec^{\mathsf{l}}_{3} y \wedge \mathsf{Tree}^{\mathsf{cons}}(z) \,\right].$$

- Eliminate integer quantifi ers.

$$0^{\mathsf{w}}_{(y^{\mathsf{w}})} \prec^{\mathsf{l}}_{3} y \wedge \mathsf{Tree}^{\mathsf{cons}}(y^{\mathsf{w}}).$$

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Continue with

$$(\exists x_{222}) \left[\, 0^{\mathsf{w}}_{(x^{\mathsf{w}}_{222})} \prec^{\mathsf{l}}_{2} x_{222} \prec^{\mathsf{l}}_{1} y \,\right].$$

- Reduce term quantifi ers to integer quantifi ers.

$$(\exists z) \left[\, 0^{\mathsf{w}}_{(z)} \prec^{\mathsf{l}}_{3} y \wedge \mathrm{Tree}^{\mathsf{cons}}(z) \,\right].$$

- Eliminate integer quantifi ers.

$$0^{\mathsf{w}}_{(y^{\mathsf{w}})} \prec^{\mathsf{l}}_{3} y \wedge \mathrm{Tree}^{\mathsf{cons}}(y^{\mathsf{w}}).$$

- As $0^{\mathsf{w}}_{(y^{\mathsf{w}})} \prec^{\mathsf{l}}_{3} y$ implies $\mathrm{Tree}^{\mathsf{cons}}(y^{\mathsf{w}})$, we have

$$0^{\mathsf{w}}_{(y^{\mathsf{w}})} \prec^{\mathsf{l}}_{3} y.$$

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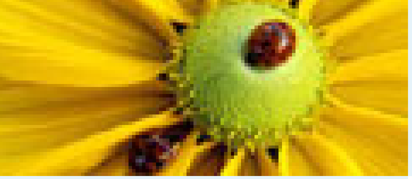
In summary,

$$0^{\mathsf{w}}_{(y^{\mathsf{w}})} \prec^{\mathsf{l}}_{3} y \implies$$

$$(\exists x) \left[\mathsf{car}(x) \prec^{\mathsf{l}}_{2} \mathsf{cdr}(\mathsf{cdr}(x)) \;\wedge\; \mathsf{cdr}(\mathsf{cdr}(\mathsf{car}(x))) \prec^{\mathsf{l}}_{3} y \right],$$



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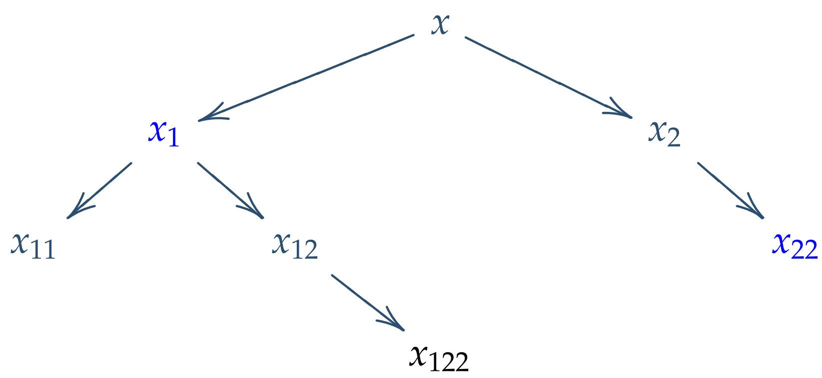
In summary,

$$0^{\mathsf{w}}_{(y^{\mathsf{w}})} \prec^{!}_{3} y \implies$$

$$(\exists x)\left[\, \mathrm{car}(x) \prec^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x)) \ \wedge\ \mathrm{cdr}(\mathrm{cdr}(\mathrm{car}(x))) \prec^{!}_{3} y \,\right],$$



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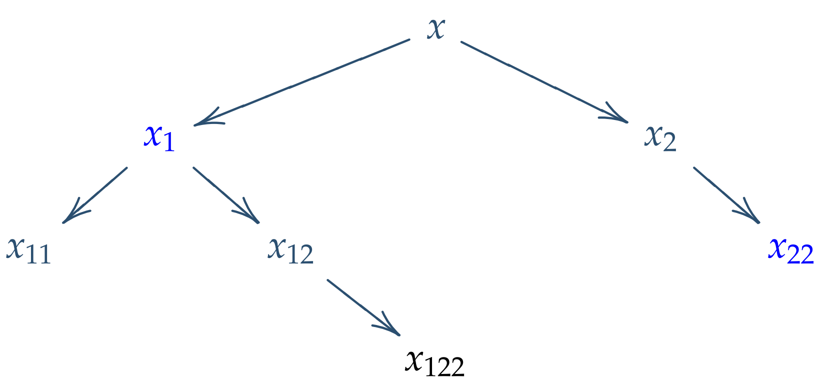
In summary,

$$0^{\mathsf{w}}_{(y^{\mathsf{w}})} \prec^{\mathsf{l}}_{3} y \implies$$

$$(\exists x)\left[\ \mathsf{car}(x) \prec^{\mathsf{l}}_{2} \mathsf{cdr}(\mathsf{cdr}(x)) \ \wedge\ \mathsf{cdr}(\mathsf{cdr}(\mathsf{car}(x))) \prec^{\mathsf{l}}_{3} y \ \right],$$



$x_1 : \mathsf{car}(x)$,

$x_2 : \mathsf{cdr}(x)$,

$x_{11} : \mathsf{car}(\mathsf{car}(x))$,

$x_{12} : \mathsf{cdr}(\mathsf{car}(x))$,

$x_{22} : \mathsf{cdr}(\mathsf{cdr}(x))$,

$x_{122} : \mathsf{cdr}(\mathsf{cdr}(\mathsf{car}(x)))$

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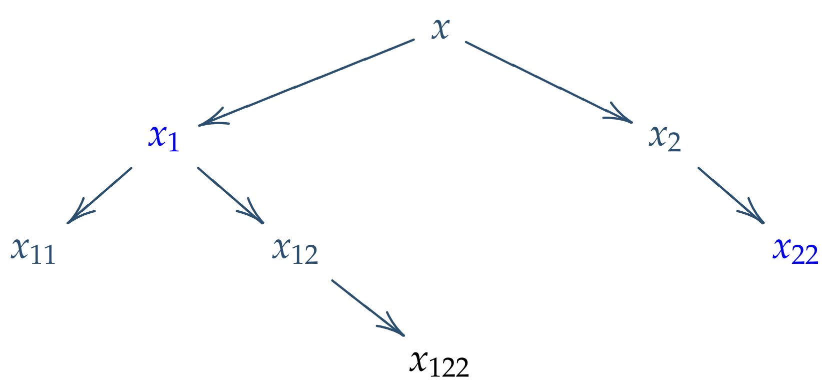
In summary,

$$0^{\mathsf{w}}_{(y^{\mathsf{w}})} \prec^{!}_{3} y \Longrightarrow$$

$$(\exists x)\left[\,\mathrm{car}(x) \prec^{!}_{2} \mathrm{cdr}(\mathrm{cdr}(x)) \;\wedge\; \mathrm{cdr}(\mathrm{cdr}(\mathrm{car}(x))) \prec^{!}_{3} y\,\right],$$



x_1 : car(x),

x_2 : cdr(x),

x_{11} : car(car(x)),

x_{12} : cdr(car(x)),

x_{22} : cdr(cdr(x)),

x_{122} : cdr(cdr(car(x)))

Solution: $x_{122} = 0^{\mathsf{w}}_{(y^{\mathsf{w}})}$!

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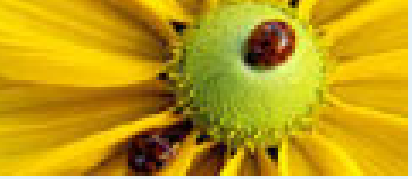

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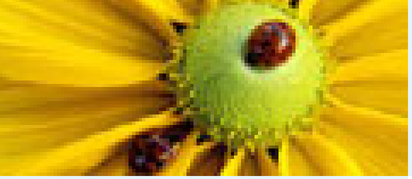
- Smallest extensions for quantifier elimination.

  More expressive power induces higher complexity.



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- Smallest extensions for quantifier elimination.

  More expressive power induces higher complexity.

- Block-wise quantifier elimination.

  Small quantifier alternations in real life.

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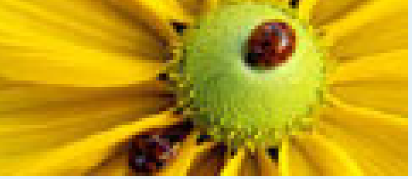
- Smallest extensions for quantifi er elimination.

  More expressive power induces higher complexity.

- Block-wise quantifi er elimination.

  Small quantifi er alternations in real life.

- Decidability of KBO on term domain with variables.



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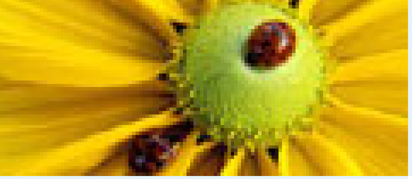

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- Elimination of Equalities.

$$\exists x\Big[\ x = 0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}\ \wedge\ \mathsf{car}(x) \prec^{\mathsf{p}}_{4} \mathsf{cdr}(x)\ \Big].$$

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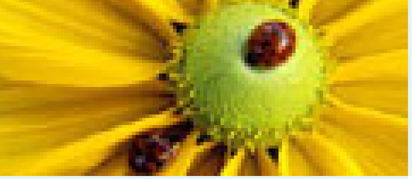

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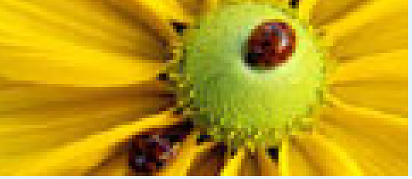
- Elimination of Equalities.

$$\exists x\left[\, x = 0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})} \;\wedge\; \mathsf{car}(x) \prec^{\mathsf{p}}_{4} \mathsf{cdr}(x)\,\right].$$

- Simplifi cation of Selector Terms.

$$\mathsf{car}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}).$$



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- Elimination of Equalities.

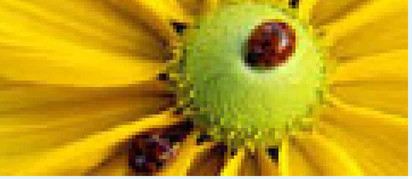
$$\exists x \Big[\; x = 0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})} \;\wedge\; \mathsf{car}(x) \prec^{\mathsf{p}}_{4} \mathsf{cdr}(x) \; \Big].$$

- Simplifi cation of Selector Terms.

$$\mathsf{car}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}).$$

- Elimination of Negations.

$$\neg\Big(\mathsf{car}(x) \prec^{\mathsf{w}}_{3} \mathsf{cdr}(x)\Big).$$



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Continue with

$$\exists x \Big[\; x = 0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})} \;\wedge\; \mathsf{car}(x) \prec^{\mathsf{p}}_{4} \mathsf{cdr}(x) \; \Big].$$

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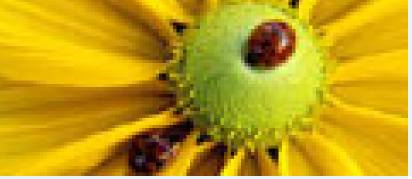
Continue with

$$\exists x\left[\ x = 0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})} \ \wedge\ \mathsf{car}(x) \prec^{\mathsf{p}}_{4} \mathsf{cdr}(x)\ \right].$$

- Substitution.

$$\exists x\left[\ x = 0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})} \ \wedge\ \mathsf{car}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}) \prec^{\mathsf{p}}_{4} \mathsf{cdr}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})})\ \right].$$



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Continue with

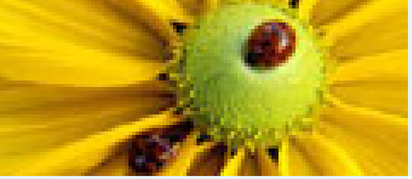
$$\exists x \big[\ x = 0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}\ \wedge\ \mathsf{car}(x) \prec^{\mathsf{p}}_4 \mathsf{cdr}(x)\ \big].$$

- Substitution.

$$\exists x \big[\ x = 0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}\ \wedge\ \mathsf{car}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}) \prec^{\mathsf{p}}_4 \mathsf{cdr}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})})\ \big].$$

- Reduction to Integer Quantifi ers.

$$\exists (\mathsf{car}(x))^{\mathsf{w}} \big[\ \mathsf{Tree}((\mathsf{car}(x))^{\mathsf{w}})\ \wedge\ \mathsf{car}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}) \prec^{\mathsf{p}}_4 \mathsf{cdr}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})})\ \big].$$



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Continue with

$$\exists x\Big[\ x = 0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}\ \wedge\ \mathsf{car}(x) \prec^{\mathsf{p}}_{4} \mathsf{cdr}(x)\ \Big].$$

- Substitution.

$$\exists x\Big[\ x = 0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}\ \wedge\ \mathsf{car}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}) \prec^{\mathsf{p}}_{4} \mathsf{cdr}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})})\ \Big].$$

- Reduction to Integer Quantifi ers.

$$\exists(\mathsf{car}(x))^{\mathsf{w}}\Big[\ \mathsf{Tree}((\mathsf{car}(x))^{\mathsf{w}})\ \wedge\ \mathsf{car}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})}) \prec^{\mathsf{p}}_{4} \mathsf{cdr}(0^{\mathsf{w}}_{((\mathsf{car}(x))^{\mathsf{w}})})\ \Big].$$

- Renaming.

$$\exists z\Big[\ \mathsf{Tree}(z)\ \wedge\ \mathsf{car}(0^{\mathsf{w}}_{(z)}) \prec^{\mathsf{p}}_{4} \mathsf{cdr}(0^{\mathsf{w}}_{(z)})\ \Big].$$

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$$\mathrm{car}(0^{\mathrm{w}}_{((\mathrm{car}(x))^{\mathrm{w}})})$$

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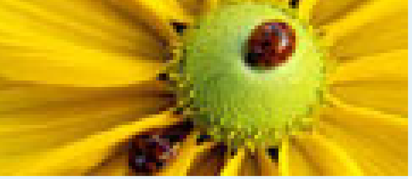
$$\mathrm{car}(0^{w}_{((\mathrm{car}(x))^{w})})$$

simplifi es to

$$0^{w}_{f_{\mathrm{car}}((\mathrm{car}(x))^{w})},$$



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$$\mathrm{car}(0^{\mathrm{w}}_{((\mathrm{car}(x))^{\mathrm{w}})})$$

simplifi es to

$$0^{\mathrm{w}}_{f_{\mathrm{car}}((\mathrm{car}(x))^{\mathrm{w}})},$$

where

$$f_{\mathrm{car}}(\cdot)$$

is an integer function from Presburger arithmetic.

Elimination of Negations

$$\neg\left(\mathsf{car}(x) \prec_3^{\mathsf{w}} \mathsf{cdr}(x)\right).$$

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$$\neg\Big(\mathsf{car}(x) \prec_3^{\mathsf{w}} \mathsf{cdr}(x)\Big).$$

simplifi es to

$$\mathsf{cdr}(x) \prec^{\mathsf{w}} \mathsf{car}(x) \;\vee\; (\mathsf{cdr}(x))^{\mathsf{w}} = (\mathsf{car}(x))^{\mathsf{w}} \;\vee$$
$$\mathsf{car}(x) \leqq_1^{\mathsf{w}} \mathsf{cdr}(x) \;\vee\; \mathsf{car}(x) \leqq_2^{\mathsf{w}} \mathsf{cdr}(x).$$

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