Model Theory 290B: Interpolation in Classic Predicate Logic and Quantified Modal Logic S5(S5B)

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Abstract

Interpolation theorem has an important role in classic predicate logic. Two prominent applications are Robinson's joint consistency theorem and Beth's definability theorem (though both were proved originally in different ways). It also have close relation to cut-elimination of sequent calculus. However, interpolation theorem fails in quantified modal logic $S5$ and related extensions. In this term paper we compare the positive and negative results. In Section 1, we prove interpolation theorem in classic logic and in Section 2 we show the failure of interpolation theorem in $S5$ and $S5B$ respectively. This work is based on [1, 2].

1 Interpolation Theorem in Classic Predicate Logic

Throughout this paper we treat equality as a logical constant.

Theorem 1.1 (Interpolation Theorem)**.** *Let ϕ_1, ϕ_2 be two first-order sentences. If $\phi_1 \models \phi_2$, then there exists a sentence θ such that $\phi_1 \models \theta$, $\theta \models \phi_2$ and θ only contains nonlogical parameters that appear both in ϕ_1 and ϕ_2. In this situation θ is called the interpolant of ϕ_1 and ϕ_2.*

Proof. We prove it by showing that if there is no interpolant for ϕ_1 and ϕ_2, then $\phi_1 \wedge \neg\phi_2$ is consistent. Let Σ_i denote signatures of ϕ_i $(i = 1, 2)$. Let $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\Sigma_0 = \Sigma_1 \cap \Sigma_2$. Let $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_0$ and $\mathscr{L}$ be languages in corresponding signatures. We also assume that each language is augmented with an set C of infinite new constant symbols.

Let φ_1, φ_2 be sentences in $\mathscr{L}_1$, $\mathscr{L}_2$ respectively. We say φ_1 and φ_2 are *inseparable* if there is no sentence φ in $\mathscr{L}_0$ such that $\varphi_1 \models \varphi$ and $\varphi_2 \models \neg\varphi$. If ϕ_1 and ϕ_2 don't have interpolant then ϕ_1 and $\neg\phi_2$ are inseparable. For suppose there is θ such that $\phi_1 \models \theta$ and $\neg\phi_2 \models \neg\theta$. Then $\phi_1 \models \theta$ and $\theta \models \phi_2$, i.e., θ is an interpolant.

Let $\delta_0, \delta_1, \ldots,$ $\psi_0, \psi_1, \ldots$ enumerate all sentences in $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. Construct increasing sequences of theories $T_0, T_1, \ldots$ and $U_0, U_1, \ldots$ as follows. Let $T_0 = \{\phi_1\}$ and $U_0 = \{\neg\phi_2\}$. At each step first set

$$X = \begin{cases} T_i \cup \{\delta_i\} & \text{if } T_i \cup \{\delta_i\} \text{ is inseparable from } U_i \\ T_i & \text{otherwise.} \end{cases}$$

$$Y = \begin{cases} U_i \cup \{\psi_i\} & \text{if } U_i \cup \{\psi_i\} \text{ is inseparable from } T_{i+1} \\ U_i & \text{otherwise.} \end{cases}$$

Then let

$$T_{i+1} = \begin{cases} X \cup \{\sigma(c)\} & \text{for some } c \in C, \text{ if } \delta_i = \exists x \sigma(x) \in X \\ X & \text{otherwise.} \end{cases}$$

$$U_{i+1} = \begin{cases} Y \cup \{\lambda(d)\} & \text{for some } d \in C, \text{ if } \psi_i = \exists x \lambda(x) \in Y \\ Y & \text{otherwise.} \end{cases}$$

Note that each time we use a fresh constant as the witness. Finally let $T_\omega = \bigcup_{i<\omega} T_i$ and $U_\omega = \bigcup_{i<\omega} U_i$. As $\phi_1 \in T_\omega$ and $\phi_2 \in U_\omega$ it suffices to show that $T_\omega \cup U_\omega$ is consistent. We proceed in the following steps.

(1) T_i and U_i are inseparable for $i < \omega$.

This is obvious from the construction.

(2) T_ω and U_ω are inseparable.

Suppose there exists θ such that $T_\omega \models \theta$ and $U_\omega \models \neg\theta$. By compactness there is $j < \omega$ such that $T_j \models \theta$ and $U_j \models \neg\theta$. Then T_j and U_j are separable.

(3) T_ω and U_ω are both consistent.

If T_ω is not consistent, then $T_\omega \models x \neq x$ while $U_\omega \models x = x$. Then T_ω and U_ω are separable. Similarly we have contradiction if U_ω is not consistent.

(4) T_ω and U_ω are both complete.

Suppose neither $T_\omega \models \theta$ nor $T_\omega \models \neg\theta$ for some $\theta \in \mathscr{L}_1$. Without loss assume θ is δ_i and $\neg\theta$ is δ_j. Certainly $\theta \notin T_{i+1}$ and $\neg\theta \notin T_{j+1}$. By the construction there exists θ_1 and θ_2 in $\mathscr{L}$ respectively such that

$$T_i \cup \{\delta_i\} \models \theta_1 \text{ and } U_i \models \neg\theta_1$$
$$T_j \cup \{\delta_j\} \models \theta_2 \text{ and } U_j \models \neg\theta_2$$

Then we have

$$T_\omega \models \theta_1 \vee \theta_2 \text{ and } U_\omega \models \neg\theta_1 \wedge \neg\theta_2$$

So T_ω and U_ω are separable, a contradiction. Similarly U_ω is complete.

(5) $T_\omega \cap U_\omega$ is complete.

Since both T_ω and U_ω are complete. Let θ be a sentence in $\mathscr{L}$. We have

$$\theta \in T_\omega \text{ or } \neg\theta \in T_\omega$$

and

$$\theta \in U_\omega \text{ or } \neg\theta \in U_\omega$$

Since T_ω and U_ω are inseparable, either $\theta \in T_\omega \cap U_\omega$ or $\neg\theta \in T_\omega \cap U_\omega$.

(6) $T_\omega \cup U_\omega$ is consistent.

Let $\mathfrak{A} = \langle A; c_0, c_1, \ldots \rangle$, $\mathfrak{B} = \langle B; c_0, c_1, \ldots \rangle$ be models of T_ω and U_ω respectively. Because we add one witness for each existential sentence we can assume that every element in domain A and B is named by a constant. Let $\mathfrak{A}'$, $\mathfrak{B}'$ be $\mathscr{L}$-reduct of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Since $T_\omega \cap U_\omega$ is complete, $diag(A) = diag(B)$. So $\mathfrak{A}'$ and $\mathfrak{B}'$ are isomorphic under mapping $f : c_i^{\mathfrak{A}'} \to c_i^{\mathfrak{B}'}$ for $i < \omega$. Hence we can "paste" $\mathfrak{A}'$ and $\mathfrak{B}'$ together to have a model $\mathfrak{C}$ of $T_\omega \cup U_\omega$ such that $\mathfrak{A}'$ and $\mathfrak{B}'$ are $\mathscr{L}_1$-reduct and $\mathscr{L}_2$-reduct of $\mathfrak{C}$ respectively.

□

2 Failure of Interpolation in Quantified $S5$ and $S5B$

2.1 Semantics of Quantified Modal Logics

We briefly review some relevant concepts of quantified modal logic. We follow Fine's notation in [2]. By $S5B$ we mean $S5$ with constant domain. A structure $\mathfrak{A} = \langle W, A, \bar{A}, R, \nu \rangle$ is defined as follows:

(1) W is a nonempty set of worlds.

(2) A is a nonempty domain.

(3) $\bar{A}$ assigns an individual domain $\bar{A}_w$ to each $w \in W$.

(4) R is an accessibility relation on $W \times W$.

(5) ν is an interpretation of nonlogical parameters on W and A. More precisely, $\nu(c) \in A$ for each constant c, $\nu(P) \subseteq W \times A^n$ for each n-ary predicate symbol P, $\nu(c) \in A$ and $\nu(F) : W \times A^n \to A$ for each n-ary function symbol F.

Semantics is defined as usual. For simplicity we assume that we have a set of constants each of which names an element in A.

(1) $(\mathfrak{A}, w) \models c = d$ iff $\nu(c) = \nu(d)$.

(2) $(\mathfrak{A}, w) \models P(c_0, \dots, c_{n-1})$ iff $\langle w, \nu(c_0), \dots, \nu(c_{n-1}) \rangle \in \nu(P)$.

(3) $(\mathfrak{A}, w) \models \forall x \phi(x)$ iff $(\mathfrak{A}, w) \models \phi(c)$ for all c such that $\nu(c) \in \bar{A}_w$.

(4) $(\mathfrak{A}, w) \models \Box\phi$ iff $(\mathfrak{A}, v) \models \phi$ for all v such that Rwv.

2.2 Isomorphisms Between $S5$ Structures

Let $\mathfrak{A} = \langle W, A, \bar{A}, R, \nu \rangle$ be a $S5$-structure and w a world of $\mathfrak{A}$. A *projection* $\mathfrak{A}_w$ of $\mathfrak{A}$ on w is the strcuture $\langle A, \bar{A}_w, \nu_w \rangle$, where

$$\nu_w(F) = \{\langle a_1, \dots, a_n \rangle \in A^n : \langle w, a_1, \dots, a_n \rangle \in \nu(F)\}$$

for every nonlogical parameter F. Similarly, a *inner projection* $\bar{\mathfrak{A}}_w$ of $\mathfrak{A}$ on w is the strcuture $\langle \bar{A}_w, \bar{\nu}_w \rangle$, where

$$\bar{\nu}_w(F) = \{\langle a_1, \dots, a_n \rangle \in \bar{A}_w^n : \langle w, a_1, \dots, a_n \rangle \in \nu(F)\}$$

for every nonlogical parameter F. Note that for the inner projection, interpretations of nonlogical symbols are completely confined to the corresponding individual domain. Since projections (of both types) are first-order structures, we have standard notion of isomorphism aomong them.

Let $\mathfrak{A} = \langle W, A, \bar{A}, R, \nu \rangle$ and $\mathfrak{A} = \langle V, B, \bar{B}, S, \mu \rangle$ be two $S5$-structures. We say a one-to-one function σ from A to B is an *isomorphism* between $\mathfrak{A}$ and $\mathfrak{B}$, written $\sigma : \mathfrak{A} \cong \mathfrak{B}$, if

(1) $\forall w \in W \exists v \in V(\sigma : \mathfrak{A}_w \cong \mathfrak{B}_v)$, and

(2) $\forall v \in V \exists w \in W(\sigma : \mathfrak{A}_w \cong \mathfrak{B}_v)$

2.3 Two Easy Lemmas

Lemma 2.1. *Suppose that: (i)* $\sigma : \mathfrak{A} \cong \mathfrak{B}$ *and (ii) for* $w \in W$, $v \in V$, $\sigma : \mathfrak{A}_{\mathfrak{w}} \cong \mathfrak{B}_{\mathfrak{v}}$. *Then for any formula* $\phi(x_1, \ldots, x_n)$ *with free variables* $x_1, \ldots, x_n$ *and for any tuple* $a_1, \ldots, a_n \in A$,

$$(\mathfrak{A}, w) \models \phi[a_1, \ldots, a_n] \textit{ iff } (\mathfrak{B}, v) \models \phi[\sigma(a_1), \ldots, \sigma(a_n)]$$

Proof. (1) $\phi(x_1, \ldots, x_n) = P(x_1, \ldots, x_n)$.

$$\begin{aligned} (\mathfrak{A}, w) \models P[a_1, \ldots, a_n] &\Leftrightarrow \langle w, a_1, \ldots, a_n \rangle \in P^{\mathfrak{A}} \\ &\Leftrightarrow \langle a_1, \ldots, a_n \rangle \in P^{\mathfrak{A}_{\mathfrak{w}}} \\ &\Leftrightarrow \langle \sigma(a_1), \ldots, \sigma(a_n) \rangle \in P^{\mathfrak{B}_{\mathfrak{v}}} \\ &\Leftrightarrow \langle v, \sigma(a_1), \ldots, \sigma(a_n) \rangle \in P^{\mathfrak{B}} \\ &\Leftrightarrow (\mathfrak{B}, v) \models P[a_1, \ldots, a_n] \end{aligned}$$

(2) $\phi(x_1, \ldots, x_n) = \Box\varphi(x_1, \ldots, x_n)$.

Assume that $(\mathfrak{A}, w) \models \Box\varphi[a_1, \ldots, a_n]$. Then for any $w' \in W$, $(\mathfrak{A}, w') \models \varphi[a_1, \ldots, a_n]$. Since $\sigma : \mathfrak{A} \cong \mathfrak{B}$, for any $v' \in V$ there is $w' \in W$ such that $\sigma : \mathfrak{A}_{\mathfrak{w}'} \cong \mathfrak{B}_{\mathfrak{v}'}$. By induction hypothesis $(\mathfrak{B}, v') \models \varphi[\sigma(a_1), \ldots, \sigma(a_n)]$ for any $v' \in V$. That is, $(\mathfrak{B}, v) \models \Box\varphi[\sigma(a_1), \ldots, \sigma(a_n)]$. By symmetry, $(\mathfrak{B}, v) \models \Box\varphi[\sigma(a_1), \ldots, \sigma(a_n)]$ implies $(\mathfrak{A}, w) \models \Box\varphi[a_1, \ldots, a_n]$.

(3) $\phi(x_1, \ldots, x_n) = \forall x_0 \varphi(x_0, x_1, \ldots, x_n)$.

$$\begin{aligned} (\mathfrak{A}, w) \models \forall x_0 \varphi[x_0, a_1, \ldots, a_n] &\Leftrightarrow (\mathfrak{A}, w) \models \varphi[a_0, a_1, \ldots, a_n] \text{ for all } a_0 \in \bar{A}_w \\ &\Leftrightarrow (\mathfrak{B}, v) \models \varphi[\sigma(a_0), \sigma(a_1), \ldots, \sigma(a_n)] \text{ for all } \sigma(a_0) \in \bar{B}_v \\ &\Leftrightarrow (\mathfrak{B}, v) \models \forall x_0 \varphi[x_0, \sigma(a_1), \ldots, \sigma(a_n)] \end{aligned}$$

□

Lemma 2.2. *Suppose that: (i)* $\rho : \bar{\mathfrak{A}}_w \cong \bar{\mathfrak{B}}_v$ *and (ii)* $(\forall$ *finite* $\rho' \subseteq \rho)(\exists\sigma \supseteq \rho')(\sigma : \mathfrak{A} \cong \mathfrak{B})$. *Then for any formula* $\phi(x_1, \ldots, x_n)$ *with free variables* $x_1, \ldots, x_n$ *and for any tuple* $a_1, \ldots, a_n \in \bar{A}_w$,

$$(\mathfrak{A}, w) \models \phi[a_1, \ldots, a_n] \textit{ iff } (\mathfrak{B}, v) \models \phi[\rho(a_1), \ldots, \rho(a_n)]$$

Proof. (1) $\phi(x_1, \ldots, x_n) = P(x_1, \ldots, x_n)$.

$$\begin{aligned} (\mathfrak{A}, w) \models P[a_1, \ldots, a_n] &\Leftrightarrow \langle w, a_1, \ldots, a_n \rangle \in P^{\mathfrak{A}} \\ &\Leftrightarrow \langle a_1, \ldots, a_n \rangle \in P^{\bar{\mathfrak{A}}_{\mathfrak{w}}} \\ &\Leftrightarrow \langle \rho(a_1), \ldots, \rho(a_n) \rangle \in P^{\bar{\mathfrak{B}}_{\mathfrak{v}}} \\ &\Leftrightarrow \langle v, \rho(a_1), \ldots, \rho(a_n) \rangle \in P^{\mathfrak{B}} \\ &\Leftrightarrow (\mathfrak{B}, v) \models P[a_1, \ldots, a_n] \end{aligned}$$

(2) $\phi(x_1, \ldots, x_n) = \Box\varphi(x_1, \ldots, x_n)$.

Assume that $(\mathfrak{A}, w) \models \Box\varphi[a_1, \ldots, a_n]$. Then for any $w' \in W$, $(\mathfrak{A}, w') \models \varphi[a_1, \ldots, a_n]$. Since $a_1, \ldots, a_n$ are finite, by the assumption there is σ such that $\sigma : \mathfrak{A} \cong \mathfrak{B}$ and $\sigma \upharpoonright \{a_1, \ldots, a_n\} = \rho'$. So for any $v' \in V$ there is $w' \in W$ such that $\sigma : \mathfrak{A}_{\mathfrak{w}'} \cong \mathfrak{B}_{\mathfrak{v}'}$ By Lemma 2.1 $(\mathfrak{B}, v') \models \varphi[\sigma(a_1), \ldots, \sigma(a_n)]$ for all $v' \in V$. Since σ and ρ agree on $a_1, \ldots, a_n$ we have $(\mathfrak{B}, v') \models \varphi[\rho(a_1), \ldots, \rho(a_n)]$. Hence $(\mathfrak{B}, v) \models \Box\varphi[\rho(a_1), \ldots, \rho(a_n)]$. As ρ is one-to-one, by symmetry, $(\mathfrak{B}, v) \models \Box\varphi[\rho(a_1), \ldots, \rho(a_n)]$ implies $(\mathfrak{A}, w) \models \Box\varphi[a_1, \ldots, a_n]$.

(3) $\phi(x_1, \dots, x_n) = \forall x_0 \varphi(x_0, x_1, \dots, x_n)$.

$$\begin{aligned}(\mathfrak{A}, w) \models \forall x_0 \varphi[x_0, a_1, \dots, a_n] \quad &\Leftrightarrow \quad (\mathfrak{A}, w) \models \varphi[a_0, a_1, \dots, a_n] \text{ for all } a_0 \in \bar{A}_w \\ &\Leftrightarrow \quad (\mathfrak{B}, v) \models \varphi[\rho(a_0), \rho(a_1), \dots, \rho(a_n)] \text{ for all } \rho(a_0) \in \bar{B}_v \\ &\Leftrightarrow \quad (\mathfrak{B}, v) \models \forall x_0 \varphi[x_0, \rho(a_1), \dots, \rho(a_n)]\end{aligned}$$

□

2.4 Failure of Interpolation Theorem in Quantified S5

To show the failure of interpolation theorem it suffices to construct a counterexample.

Lemma 2.3. *Let*

$$\varphi_1 = p \wedge \Box \forall x \Box (p \to \exists y (y = x)) \textit{ and } \varphi_2 = q \to \Box \forall x \Diamond (q \wedge \exists y (y = x))$$

Then $\varphi_1 \models_{S5} \varphi_2$.

Proof. Let $\mathfrak{A} = \langle W, A, \bar{A}, \nu \rangle$ be any $S5$-structure. Let $w \in W$. $(\mathfrak{A}, w) \models \varphi_1$ iff p is true in w and for any world $w' \in W$ if p is true in w', then $\bar{A}_{w'} = A$. Also $(\mathfrak{A}, w) \models \varphi_2$ iff either q is not true in w or there exists a world $w' \in W$ such that q is true in w' and $\bar{A}_{w'} = A$. Now suppose $(\mathfrak{A}, w) \models \varphi_1$. Then $\bar{A}_w = A$. If q is not true in w, then $(\mathfrak{A}, w) \models \varphi_2$. If q is true in w, since $\bar{A}_w = A$, $(\mathfrak{A}, w) \models \varphi_2$. Therefore $\varphi_1 \models_{S5} \varphi_2$. □

Theorem 2.1. *Let* φ_1, φ_2 *as above. There is no interpolant between* φ_1 *and* φ_2.

Proof. Let $\mathbb{N}$ denote the set of natural numbers. Define a $S5$-structure $\mathfrak{A} = \langle W, A, \bar{A}, p, q \rangle$ as follows:

(1) $A = \mathbb{N}$,

(2) $W = \{w_i : i < \omega\}$,

(3) $\bar{A}_{w_0} = \mathbb{N}$, $\bar{A}_{w_i} = \mathbb{N} - \{i\}$ for $0 < i < \omega$,

(4) $V(w_i, p) = 1$ iff $i = 0$ and $V(w_i, q) = 1$ iff $i = 1$.

Since p is only true at w_0 and $\bar{A}_{w_0} = A$,

$$(\mathfrak{A}, w_i) \models p \wedge \Box \forall x \Box (p \to \exists y (y = x)) \text{ iff } i = 0$$

Similarly since q is only true at w_1 yet $\bar{A}_{w_1} \neq A$,

$$(\mathfrak{A}, w_i) \models q \to \Box \forall x \Diamond (q \wedge \exists y (y = x)) \text{ iff } i \neq 1$$

Let $\mathfrak{A}'$ be the reduct of $\mathfrak{A}$ with no nonlogical parameters. Define $\rho : A \to A$ by $\rho(0) = 0$ and $\rho(i) = i+1$ for $i > 0$. Obviously $\rho : \bar{\mathfrak{A}}'_{w_0} \cong \bar{\mathfrak{A}}'_{w_1}$. Also since $\mathfrak{A}'$ is equational, ρ is an automorphism of $\mathfrak{A}'$. Let $\sigma = \rho$ and quote Lemma 2.2 we have

$$(\mathfrak{A}', w_0) \models \theta[i_1, \dots, i_n] \text{ iff } (\mathfrak{A}', w_1) \models \theta[\rho(i_1), \dots, \rho(i_n)]$$

where $\theta(x_1, \ldots, x_n)$ is a formula without nonlogical parameters. Now suppose there is a sentence θ such that $\varphi_1 \models \theta$ and $\theta \models \varphi_2$. Then

$$\begin{aligned}(\mathfrak{A}, w_0) \models \varphi_1 &\Rightarrow (\mathfrak{A}, w_0) \models \theta \Rightarrow (\mathfrak{A}', w_0) \models \theta \\ &\Rightarrow (\mathfrak{A}', w_1) \models \theta \Rightarrow (\mathfrak{A}, w_1) \models \theta \Rightarrow (\mathfrak{A}, w_1) \models \varphi_2\end{aligned}$$

A contradiction. Note that though we used propositional letters in the above proof, it doesn't put essential restriction on the applicability of this theorem. We can encode the properties of propositional letters using unary predicates. More precisely, let

$$\varphi_1 = \Box \forall u \forall v (P(u) \leftrightarrow P(v)) \land \forall z (P(z) \land \Box \forall x \Box (P(z) \to \exists y (y = x)))$$

$$\varphi_2 = \Box \forall u \forall v (Q(u) \leftrightarrow Q(v)) \to \forall z (Q(z) \to \Box \forall x \Diamond (Q(z) \land \exists y (y = x)))$$

The proof still go through. □

2.5 Failure of Interpolation Theorem in Quantified S5B

Surprisingly, interpolation theorem even fails in quantified $S5B$. As before, we circumvent the Beth's theorem and construct a counterexample directly using the same idea in [2]. This time we work with a language whose nonlogical parameters are proposition symbols p, q and a unary predicate symbol P. The motivation to introduce P is to fake variable "inner domains" in order to save the previous counterexample.

Lemma 2.4. *Let*

$$\varphi_1 = p \land \Box \forall x \Box (p \to P(x)) \text{ and } \varphi_2 = q \to \Box \forall x \Diamond (q \land P(x))$$

Then $\varphi_1 \models_{S5B} \varphi_2$.

Proof. Let $\mathfrak{A} = \langle W, A, \bar{A}, \nu \rangle$ be any $S5$-structure. Let $w \in W$. $(\mathfrak{A}, w) \models \varphi_1$ iff p is true in w and for any world $w' \in W$ if p is true in w', then $P^{\mathfrak{A}}_{w'} = A$. Also $(\mathfrak{A}, w) \models \varphi_2$ iff either q is not true in w or there exists a world $w' \in W$ such that q is true in w' and $P^{\mathfrak{A}}_{w'} = A$. Now suppose $(\mathfrak{A}, w) \models \varphi_1$. Then $P^{\mathfrak{A}}_{w} = A$. If q is not true in w, then clearly $(\mathfrak{A}, w) \models \varphi_2$. If q is true in w, since $P^{\mathfrak{A}}_{w} = A$, $(\mathfrak{A}, w) \models \varphi_2$. Therefore $\varphi_1 \models_{S5B} \varphi_2$. □

Theorem 2.2. *Let* φ_1, φ_2 *as above. There is no interpolant between* φ_1 *and* φ_2.

Proof. Let $\mathbb{N}$, $\mathbb{O}$ be the set of natural numbers and the set of odd natural numbers respectively. We say a permutation τ on $\mathbb{N}$ is *essentially finite* if the set $\{a : \tau(a) \neq a\}$ is finite. Let I denote the identity permutation. Define a $S5$-structure $\mathfrak{A} = \langle W, A, \bar{A}, p, q, P \rangle$ as follows.

(1) $A = \mathbb{N}$.

(2) $W = \{w_{\langle k, \tau \rangle} : k = 0, 1; \tau \text{ is a essentially finite permutation } \}$

(3) For any $w_{\langle k, \tau \rangle} \in W$, let

$$P^{\mathfrak{A}}_{w_{\langle k, \tau \rangle}} = \begin{cases} \tau[\mathbb{N}] & \text{if } k = 0 \\ \tau[\mathbb{O}] & \text{if } k = 1 \end{cases}$$

(4) $V(w, p) = 1$ iff $w = w_0$ and $V(w, q) = 1$ iff $w = w_1$, where w_0 and w_1 abbreviates $w_{\langle 0, I \rangle}$ and $w_{\langle 1, I \rangle}$ respectively.

Since p is only true at w_0 and $P^{\mathfrak{A}}_{w_1} = \mathbb{N} = A$,

$$(\mathfrak{A}, w_0) \models p \wedge \Box\forall x \Box(p \to P(x))$$

Similarly since q is only true at w_1 yet $P^{\mathfrak{A}}_{w_1} = \mathbb{O} \neq \mathbb{N} = A$,

$$(\mathfrak{A}, w_1) \not\models q \to \Box\forall x \Diamond(q \wedge P(x))$$

Let $\mathfrak{A}'$ be the reduct of $\mathfrak{A}$ with only the unary predicate P. Let $\rho : \mathbb{N} \to \mathbb{N}$ be a permutation such that $P[\mathbb{N}] = \mathbb{O}$. Then clearly $\rho : \mathfrak{A}'_{w_0} \cong \mathfrak{A}'_{w_1}$. Since any finite $\rho' \subseteq \rho$ is an essentially finite permutation, there always exists an essentially finite permutation σ such that $\rho' \subseteq \sigma$. It is easily to verify that

$$\sigma : \mathfrak{A}'_{\langle k,\tau\rangle} \cong \mathfrak{A}'_{\langle k,\sigma\circ\tau\rangle} \text{ and } \sigma : \mathfrak{A}'_{\langle k,\sigma^{-1}\circ\tau\rangle} \cong \mathfrak{A}'_{\langle k,\tau\rangle}$$

Hence two conditions in the premise of Lemma 2.2 are satisfied. It follows that

$$(\mathfrak{A}', w_0) \models \theta[i_1, \ldots, i_n] \text{ iff } (\mathfrak{A}', w_1) \models \theta[\rho(i_1), \ldots, \rho(i_n)]$$

where $\theta(x_1, \ldots, x_n)$ is a formula with P as the only nonlogical parameters. The rest of the proof remains unchanged. As before propositional letters can be replaced by predicate symbols. $\Box$

References

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