Ordinal Arithmetic: Addition, Multiplication, Exponentiation and Limit

Ting Zhang
Stanford
May 11, 2001

Outline

- Ordinal Classification
- Ordinal Addition
- Ordinal Multiplication
- Ordinal Exponentiation
- Ordinal Limit
- Continuity

Definition 0.1 (Limit Ordinal). *An ordinal is said to be a limit ordinal if it has no immediate predecessor.*

Example 0.1.

1. $\mathbf{0}$ *is a limit ordinal.*

2. ω *is a limit ordinal.*

3. $\mathbf{n}$ *are not limit ordinals whence* $n > 0$.

Remark 0.1.

$$\forall y(y \neq 0 \rightarrow \exists x(y = Sx))$$

is not deducible from A_E.

Theorem 0.1 (Cantor Normal Form). *If an ordinal $\alpha \succ 0$ then there exists a natural number n and sequences $\alpha_1 \ldots \alpha_n$ such that*

$$\alpha = \sum_{i=1}^{n} \omega^{\alpha_i} = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}.$$

where

$$\alpha_1 \succeq \ldots \succeq \alpha_n.$$

Notation 0.1.

1. $\mathbf{1} \stackrel{\text{def}}{=} \omega^0$
2. $\mathbf{n} \stackrel{\text{def}}{=} \underbrace{\omega^0 + \ldots + \omega^0}_{n}$
3. $\omega \stackrel{\text{def}}{=} \omega^1$

Definition 0.2. *If*

$$\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}.$$

and

$$\beta = \omega^{\beta_1} + \ldots + \omega^{\beta_m}.$$

are two ordinals, then $\alpha \succ \beta$ *iff for some* $k \leq n$*,*

$$\alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}$$

and either $\alpha_k \succ \beta_k$ *or* $m = k - 1 < n$*.*

Notation 0.2.

1. $\alpha \prec \beta \stackrel{\text{def}}{=} \beta \succ \alpha$
2. $\alpha \succeq \beta \stackrel{\text{def}}{=} \alpha \succ \beta$ *or* $\alpha = \beta$

Exercise 0.1. *Which of the following is the canonical representation form?*

1. $\omega + 1$
2. $1 + \omega$
3. $\omega^{\omega^2+\omega} + \omega^{\omega^2+5} + \omega + 1$

Exercise 0.2. *Which of the following relations is true?*

1. $\omega^{\omega^\omega} \succ \omega^{\omega^{10}+\omega^{10}+\omega^{10}}$
2. $\omega^{\omega^5} + \omega^{\omega^4} + \omega^{\omega^3} \succ \omega^{\omega^6}$
3. $\omega^{100} \succ \omega^{100} + 1$

Definition 0.3 (Ordinal Addition).

1. $\alpha + \mathbf{0} \stackrel{\text{def}}{=} \mathbf{0} + \alpha \stackrel{\text{def}}{=} \alpha$
2. $(\omega^{\alpha_1} + \ldots + \omega^{\alpha_k} + \omega^{\alpha_{k+1}} + \ldots + \omega^{\alpha_n}) + (\omega^{\beta_1} + \ldots + \omega^{\beta_m}) \stackrel{\text{def}}{=}$ $(\omega^{\alpha_1} + \ldots + \omega^{\alpha_k} + \omega^{\beta_1} + \ldots + \omega^{\beta_m})$
   where k is the maximal number such that $k \leq n$ and $\alpha_k \succeq \beta_1$.

Exercise 0.3.

1. $(\omega^5 + \omega^4 + \omega^2 + \omega^2 + \omega + 5) + (\omega^3 + \omega^2) = \omega^5 + \omega^4 + \omega^3 + \omega^2$
2. $(\omega^5 + \omega^4 + \omega^2 + \omega^2 + \omega + 5) + (\omega^2 + \omega^2) = \omega^5 + \omega^4 + \omega^2 + \omega^2 + \omega^2 + \omega^2$
3. $(\omega^2 + \omega^2) + (\omega^5 + \omega^4 + \omega^2 + \omega^2 + \omega + 5) = \omega^5 + \omega^4 + \omega^2 + \omega^2 + \omega + 5$

Definition 0.4 (Ordinal Multiplication)**.**

1. $\alpha \cdot \mathbf{0} \stackrel{\text{def}}{=} \mathbf{0} \cdot \alpha \stackrel{\text{def}}{=} \mathbf{0}$

2. $\alpha \cdot \omega^x = \omega^{\alpha_1 + x}$ *where* $x \succeq 1$ *and* α *is of canonical form* $\sum_{i=1}^{n} \omega^{\alpha_i}$.

3. $\alpha \cdot n \stackrel{\text{def}}{=} \underbrace{\alpha + \ldots + \alpha}_{n}$

4. $\alpha \cdot (\beta + \gamma) \stackrel{\text{def}}{=} \alpha \cdot \beta + \alpha \cdot \gamma$

Remark 0.2. *We can also write*

$$\alpha = \sum_{i=1}^{n} \omega^{\alpha_i} \cdot a_i = \omega^{\alpha_1} \cdot a_1 + \ldots + \omega^{\alpha_n} \cdot a_n$$

where

$$\alpha_1 \succ \ldots \succ \alpha_n \text{ and } a_1, \ldots, a_n \prec \omega$$

Exercise 0.4.

1.

$$
\begin{aligned}
& (\omega^2 + \omega + 1) \cdot (\omega^3 + \omega) \\
= \quad & (\omega^2 + \omega + 1) \cdot \omega^3 + (\omega^2 + \omega + 1) \cdot \omega^1 \\
= \quad & \omega^5 + \omega^3
\end{aligned}
$$

2.

$$
\begin{aligned}
& (\omega^{\omega+1} + \omega^\omega + 1) \cdot (\omega^{\omega+1} + \omega^\omega + \omega) \\
= \quad & (\omega^{\omega+1} + \omega^\omega + 1) \cdot \omega^{\omega+1} + (\omega^{\omega+1} + \omega^\omega + 1) \cdot \omega^\omega \\
& +(\omega^{\omega+1} + \omega^\omega + 1) \cdot \omega \\
= \quad & \omega^{(\omega+1)+(\omega+1)} + \omega^{(\omega+1)+\omega} + \omega^{(\omega+1)+1} \\
= \quad & \omega^{\omega+\omega+1} + \omega^{\omega+\omega} + \omega^{\omega+2} \\
= \quad & \omega^{\omega\cdot 2+1} + \omega^{\omega\cdot 2} + \omega^{\omega+2}
\end{aligned}
$$

Definition 0.5 (Ordinal Exponentiation)**.**

1. $\alpha^{\mathbf{0}} \stackrel{\text{def}}{=} \mathbf{1} \stackrel{\text{def}}{=} \omega^{\mathbf{0}}$
2. $\alpha^{\mathbf{1}} \stackrel{\text{def}}{=} \alpha$
3. $\mathbf{0}^{\alpha} \stackrel{\text{def}}{=} \mathbf{0}$ *for* $\alpha \neq \mathbf{0}$
4. $\alpha^{\beta} \stackrel{\text{def}}{=} \omega^{\alpha_1 \cdot \beta}$ *where* β *is a limit ordinal and* α *is of canonical form* $\sum_{i=1}^{n} \omega^{\alpha_i}$ *and* $\alpha \succeq \omega$*.*
5. $\alpha^{\beta+\gamma} \stackrel{\text{def}}{=} \alpha^{\beta} \cdot \alpha^{\gamma}$
6. $n^{\omega \cdot x} \stackrel{\text{def}}{=} \omega^{x}$

Exercise 0.5. *Prove that* $(\omega^3 + \omega)^5 = (\omega^5 + \omega^3)^3$.

Proof.

$$\begin{aligned}
& (\omega^5 + \omega^3)^3 \\
= \;& (\omega^5 + \omega^3) \cdot (\omega^5 + \omega^3) \cdot (\omega^5 + \omega^3) \\
= \;& ((\omega^5 + \omega^3) \cdot \omega^5 + (\omega^5 + \omega^3) \cdot \omega^3) \cdot (\omega^5 + \omega^3) \\
= \;& (\omega^{10} + \omega^8) \cdot (\omega^5 + \omega^3) \\
= \;& (\omega^{10} + \omega^8) \cdot \omega^5 + (\omega^{10} + \omega^8) \cdot \omega^3 \\
= \;& \omega^{15} + \omega^{13}
\end{aligned}$$

$$
\begin{aligned}
& (\omega^3+\omega)^5 \\
= \quad & (\omega^3+\omega)\cdot(\omega^3+\omega)\cdot(\omega^3+\omega)^3 \\
= \quad & ((\omega^3+\omega)\cdot\omega^3+(\omega^3+\omega)\cdot\omega)\cdot(\omega^3+\omega)^3 \\
= \quad & (\omega^6+\omega^4)\cdot(\omega^3+\omega)^3 \\
= \quad & (\omega^6+\omega^4)\cdot(\omega^3+\omega)\cdot(\omega^3+\omega)^2 \\
= \quad & ((\omega^6+\omega^4)\cdot\omega^3+(\omega^6+\omega^4)\cdot\omega)\cdot(\omega^3+\omega)^2 \\
= \quad & (\omega^9+\omega^7)\cdot(\omega^3+\omega)^2 \\
= \quad & (\omega^9+\omega^7)\cdot(\omega^3+\omega)\cdot(\omega^3+\omega) \\
= \quad & ((\omega^9+\omega^7)\cdot\omega^3+(\omega^9+\omega^7)\cdot\omega)\cdot(\omega^3+\omega) \\
= \quad & (\omega^{12}+\omega^{10})\cdot(\omega^3+\omega) \\
= \quad & (\omega^{12}+\omega^{10})\cdot\omega^3+(\omega^{12}+\omega^{10})\cdot\omega \\
= \quad & \omega^{15}+\omega^{13}
\end{aligned}
$$

$\square$

Exercise 0.6.

1. $2^\omega = 2^{\omega \cdot 1} = \omega^1 = \omega$
2. $2^{\omega^2} = 2^{\omega \cdot \omega} = \omega^\omega$
3. $2^{\omega^\omega} = 2^{\omega^{1+\omega}} = 2^{\omega^1 \cdot \omega^\omega} = 2^{\omega \cdot \omega^\omega} = \omega^{\omega^\omega}$
4. $(\omega + 1)^\omega = (\omega^1 + 1)^\omega = \omega^{1 \cdot \omega} = \omega^\omega$
5. $(\omega^\omega)^\omega = \omega^{\omega \cdot \omega} = \omega^{\omega^2}$

Exercise 0.7.

$$
\begin{aligned}
& (\omega+1)^n \\
= \quad & (\omega+1)\cdot(\omega+1)\cdot(\omega+1)^{n-2} \\
= \quad & (\omega^2+\omega+1)\cdot(\omega+1)^{n-2} \\
= \quad & (\omega^2+\omega+1)\cdot(\omega+1)\cdot(\omega+1)^{n-3} \\
= \quad & (\omega^3+\omega^2+\omega+1)\cdot(\omega+1)\cdot(\omega+1)^{n-3} \\
= \quad & \ldots \\
= \quad & \omega^n+\omega^{n-1}+\ldots+\omega+1
\end{aligned}
$$

Theorem 0.2. *The set* $W(\alpha)$ *consisting of all ordinals less than* α *is well ordered by relation* $\preceq$*. Moreover, the type of* $W(\alpha)$ *is* α*.*

Theorem 0.3. *Every set of ordinals is well ordered by the relation* $\succeq$*. In other words, in any non-empty set* Z *of ordinals there exists a smallest ordinals.*

Theorem 0.4. *For every set* Z *of ordinals there exists an ordinal greater than all ordinals belongin g to* Z*.*

Corollary 0.1. *There exist no set of all ordinals.*

Corollary 0.2. *There exists a smallest ordinal not belonging to a given set* Z*.*

Definition 0.6 (Transfinite Sequence). *A transfinite sequence ($\alpha-$sequence) is a function ϕ whose domain is $W(\alpha)$ and whose reange is also a set of ordinals.*

If $\beta \prec \gamma \prec \alpha$ implies $\phi(\beta) \prec \phi(\gamma)$, then we say that the $\alpha-$sequence is *increasing.*

Definition 0.7 (Ordinal Limit). *Given an $\alpha-$sequence ϕ, if α is a limit ordinal, there exist ordinals greater than all the ordinals $\phi(\beta)$ where $\beta \prec \alpha$. We call the smallest such ordinal the limit of the $\alpha-$sequence and denote it by $\lim_{\beta<\alpha} \phi(\beta)$.*

Example 0.2.

1. $\lim_{n<\omega} n = \omega$
2. $\lim_{n<\omega} 2^n = \omega$
3. $\lim_{n<\omega} n^n = \omega$

Lemma 0.1. *If* $\alpha \succeq \beta$ *then there exists exactly one ordinal* γ *such that* $\alpha = \beta + \gamma$.

Proof. Let $\overline{A} = \alpha$, let B be a initial segment of A of type β and let $\gamma = \overline{A - B}$. Clearly, $\alpha = \beta + \gamma$. The uniqueness follows from trichotomy and Lemma 0.2. □

Definition 0.8 (Ordinal Subtraction)**.** *The difference of the ordinals* α *and* β *(*$\alpha \succeq \beta$*) is defined to be the unique ordinal* γ *such that* $\alpha = \beta + \gamma$*. The ordinal is denoted by* $\alpha - \beta$*.*

Lemma 0.2 (Monotonic Laws of Addition)**.**

1. $(\alpha \prec \beta) \implies (\gamma + \alpha \prec \gamma + \beta)$.
2. $(0 \prec \beta) \implies (\gamma \prec \gamma + \beta)$.
3. $(\alpha \preceq \beta) \implies (\alpha + \gamma \preceq \beta + \gamma)$.
4. $\gamma \prec \beta + \gamma$.

Lemma 0.3 (Monotonic Laws of Subtraction)**.**

1. $\alpha = \beta + (\alpha - \beta)$ *if* $\alpha \succ \beta$.
2. $(\alpha + \beta) - \alpha = \beta$.
3. $(\alpha \prec \beta) \implies (\alpha - \gamma \prec \beta - \gamma)$.
4. $(\gamma \prec \beta) \implies (\alpha - \beta \preceq \alpha - \gamma)$.

Lemma 0.4 (Monotonic Laws of Multiplication)**.**

1. $(\mathbf{0} \prec \alpha \prec \beta) \implies (\gamma \cdot \alpha \prec \gamma \cdot \beta)$.
2. $(\alpha \preceq \beta) \implies (\alpha \cdot \gamma \preceq \beta \cdot \gamma)$.
3. $(\alpha + \beta) \cdot \gamma \preceq \alpha \cdot \gamma + \beta \cdot \gamma$.

Lemma 0.5 (Monotonic Laws of Exponentiation)**.**

1. $(\mathbf{0} \prec \alpha \prec \beta) \implies \gamma^{\alpha} \prec \gamma^{\beta}$ *if* $\gamma \succ 1$.

Theorem 0.5 (Continuity of Addition). *Assume that λ is a limit ordinal and the ϕ is an increasing λ-sequence. Then we have*

$$\lim_{\xi \prec \lambda} (\alpha + \phi(\xi)) = \alpha + \lim_{\xi \prec \lambda} \phi(\xi).$$

Proof. Note that $\alpha + \phi(\xi)$ is an increasing λ-sequence by Lemma 0.2. Thus the lefthand side is well-defined. Let $\beta = \lim_{\xi \prec \lambda} \phi(\xi)$. If $\xi \prec \lambda$, then $\phi(\xi) \prec \beta$ and therefore $\alpha + \phi(\xi) \prec \alpha + \beta$ by Lemma 0.2 again. Let $\zeta \prec \alpha + \beta$; we need to show that there exists $\xi \prec \lambda$ such that $\zeta \prec \alpha + \phi(\xi)$. If $\zeta \prec \alpha$, then $\zeta \prec \alpha + \phi(0)$. On the other hand, if $\zeta \succeq \alpha$, then $\zeta = \alpha + (\zeta - \alpha)$ and $\zeta - \alpha \prec (\alpha + \beta) - \alpha = \beta$ by Lemma 0.3. It follows that for some $\xi \prec \lambda$ we have $\zeta - \alpha \prec \phi(\xi)$ (since β is the limit of $\phi(\xi)$ where $\xi \prec \lambda$), thus $\zeta \prec \alpha + \phi(\xi)$ by Lemma 0.2 and 0.3. Hence the ordinal $\alpha + \beta$ is the smallest ordinal greater than all ordinals $\alpha + \phi(\xi)$ for $\xi \prec \lambda$. □