The Feferman-Vaught Theory

The first order properties of products of algebraic systems

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Quantifier elimination

- A first-order theory T is said to have quantifier elimination if for any formula $\varphi(\bar{x})$ there is a quantifier free formula $\psi(\bar{x})$ such that

$$T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

- Assume that $\mathscr{L}$ contains at least one constant symbol or $\varphi(\bar{x})$ contains at least one free variable.

- For any first-order theory T there is a conservative extension T' of T such that T' has elimination of quantifiers.

Elimination sets

- A set Φ of formulas is an **elimination set** for a class $\mathbf{K}$ of $\mathscr{L}$-structures if any formula $\varphi(\bar{x})$ is equivalent (in every structure of $\mathbf{K}$) to a formula $\psi(\bar{x})$ which is a boolean combination of formulas in Φ.
- Φ is an **elimination set** for a theory T if Φ is an elimination set for $\mathbf{Mod}(T)$.
- T admits quantifier elimination if and only if the set of quantifier-free formulas forms an elimination set for T.
- An $\mathscr{L}$-structure $\mathfrak{A}$ has elimination of quantifiers if $Th(\mathfrak{A})$ does.

The basic theorem for elimination sets

Let Φ be a set of formulas and let $\Phi^- = \{\neg\varphi \mid \varphi \in \Phi\}$. Suppose that

- every atomic formula of $\mathscr{L}$ is in Φ, and
- for every formula $\theta(\bar{x})$ of $\mathscr{L}$ which is of form

$$\exists y \bigwedge_{i<n} \psi_i(\bar{x}, y)$$

  with ψ_i in $\Phi \cup \Phi^-$, there is a formula $\theta^*(\bar{x})$ of $\mathscr{L}$ which

  - is a boolean combination of formulas in Φ, and
  - is equivalent to θ in every structure in $\boldsymbol{K}$.

Then Φ is an elimination set for $\boldsymbol{K}$.

Criteria of elimination sets

There are several desired properties of an elimination set Φ for a theory T:

1. Φ is reasonably small and non-redundant.
2. Every formula in Φ has a straightforward mathematical meaning.
3. There exists an effective procedure to reduce every formula to a boolean combination of formulas in Φ.
4. There exists an effective procedure to decide whether a formula in Φ is provable or refutable from T.

With (1)-(4) T is a complete and decidable theory.

The point of quantifier elimination

- Classification up to elementary equivalence
- Completeness proofs
- Decidability proofs and decision procedures
- Description of definable relations
- Description of elementary embeddings

Slogan: "..., the method [of elimination of quantifiers] is extremely valuable when we want to beat a particular theory into the ground." [KC90]

Schema of proof-theoretic approaches

- Rewrite formulas to certain normal forms, e.g., the prenex form.
- Let Φ be a candidate for an elimination set and let Ψ be a set of formulas of the form $\exists x \varphi$ where φ is a boolean combination of formulas in Φ.
- Design a ranking function $\rho : \Psi \rightarrow \mathbb{N}$ such that formulas in Ψ with rank 0 reduces to a boolean combination of formulas in Φ.
- Find an effective procedure to reduce every formula φ in Ψ with $\rho(\varphi) > 0$ to a T-equivalent formula ψ in Ψ with $\rho(\psi) < \rho(\varphi)$.

The motivation of the Feferman-Vaught theory

- There are many similar ways of forming products of algebraic systems in modern algebra and set theory.
- The definition of product operations involve an index set I and in some cases take into account the structure on I.
- How to discover the first order properties of a complex system by the properties of its components?
- The theory was developed to reduce the first order properties of a product system to the properties of its factor systems and the properties of some subset algebras on the index set.

Preliminary notations

- Let $\mathfrak{A} = \langle \mathfrak{A}_i \mid i \in I \rangle$ be an indexed family of relational systems $\mathfrak{A}_i = \langle A_i, \ldots \rangle$ of the same signature μ and let the corresponding language be $\mathscr{L}_\mu$.
- Denote by D the standard Cartesian product of $\langle A_i \mid i \in I \rangle$.
- Let $\boldsymbol{f} = \langle f_0, \ldots, f_n, \ldots \rangle \in D^\omega$ and write

$$\boldsymbol{f}[i] = \langle f_0(i), \ldots, f_n(i), \ldots \rangle.$$

- If $g \in D$, $k \in \omega$, write

$$\boldsymbol{f}(k/g) = \langle f_0, \ldots, f_{k-1}, g, f_{k+1}, \ldots \rangle.$$

Preliminary notations

- By a **partitioning formula** we mean a formula $Part_m$ with $m+1$ free variables $V_0, \ldots, V_m$ defined by

$$Part_m = \big[\bigcup_{i \leq m} = \overline{\Lambda} \wedge \bigwedge_{i<j\leq m} (V_i \cup V_j = \Lambda)\big].$$

- Let θ be a formula of $\mathscr{L}_\mu$. Define **value** of θ (under $\boldsymbol{f}$) by

$$K_\theta^{\mathfrak{A}}(\boldsymbol{f}) = \big\{i \mid i \in I \ \textit{ and } \ \mathfrak{A}_i \models \theta\big[\boldsymbol{f}[i]\big]\big\}.$$

- In particular, if θ is a sentence, $K_\theta^{\mathfrak{A}}(\boldsymbol{f})$ does not depend on $\boldsymbol{f}$ and we write

$$K_\theta^{\mathfrak{A}} = K_\theta^{\mathfrak{A}}(\boldsymbol{f}) = \big\{i \mid i \in I \ \textit{ and } \ \mathfrak{A}_i \models \theta\big\}.$$

Subset algebras

- By **subset algebras** we mean the systems

$$\mathfrak{S} = \langle S(I), \Lambda, \cup, \cap, \bar{\ }, \subseteq, M_1, \ldots, M_n \rangle, \text{ \textit{where}}$$

1. I is any non-empty index set,
2. $S(I)$ is the power set of I,
3. $\Lambda, \cup, \cap, \bar{\ }$ and $\subseteq$ are normal set-theoretic operations restricted to $S(I)$, and
4. $M_1, \ldots M_n$ are some relations on subsets of I.

Denote by $\mathscr{L}_\sigma$ the language of systems $\mathfrak{S}$.

- The basic subset algebras (without additional predicates M_i's) are monadic second order structures.

Reduction sequences

- A sequence $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m \rangle$ is called a **reduction sequence** if Φ is a formula of $\mathscr{L}_\sigma$ with at most the free variables $V_0, \ldots, V_m$, and $\theta_0, \ldots, \theta_m$ are formulas of $\mathscr{L}_\mu$.
- A variable v is **free** in ζ if v is free in at least one of $\theta_0, \ldots, \theta_m$.
- A reduction sequence ζ is called a **partitioning sequence** if

$$\bigvee_{i \leq m} \theta_i \qquad \textit{and} \qquad \neg(\theta_i \wedge \theta_j) \quad (\textit{for each } i, j \leq m)$$

are tautologies.

Generalized products

- For each reduction sequence $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m \rangle$ with p free variables, let $Q_\zeta^{\mathfrak{A},\mathfrak{S}}$ be

$$\big\{\langle f_0, \ldots, f_{p-1}\rangle \mid \boldsymbol{f} \in D^\omega \textit{ and } \mathfrak{S} \models \Phi[K_{\theta_0}^{\mathfrak{A}}(\boldsymbol{f}), \ldots, K_{\theta_m}^{\mathfrak{A}}(\boldsymbol{f})]\big\}.$$

- By the **generalized product** $\mathscr{P}(\mathfrak{A}, \mathfrak{S})$ of the algebraic systems $\mathfrak{A} = \langle \mathfrak{A}_i \mid i \in I \rangle$ with respect to the algebra $\mathfrak{S}$ of subsets of I, we mean the system

$$\mathfrak{D} = \Big\langle D, Q_{\zeta_0}^{\mathfrak{A},\mathfrak{S}}, \ldots, Q_{\zeta_n}^{\mathfrak{A},\mathfrak{S}}, \ldots \Big\rangle.$$

- If all the system $\mathfrak{A}_i$ for $i \in I$ are identical to a system $\mathfrak{B}$, then $\mathscr{P}(\mathfrak{A}, \mathfrak{S})$ is called the **generalized power** of $\mathfrak{B}$ with respect to $\mathfrak{S}$.

The basic theorem for generalized products

1. There is an effective procedure to compute, for each formula φ of $\mathscr{L}_\pi$, a partitioning sequence $\zeta = \langle \Phi, \theta_0, \dots, \theta_m \rangle$ such that

   given any non-empty indexed family $\mathfrak{A} = \langle \mathfrak{A}_i \mid i \in I \rangle$ and any algebra $\mathfrak{S} = \langle S(I), \dots \rangle$ with product $\mathscr{P}(\mathfrak{A}, \mathfrak{S}) = \mathfrak{D} = \langle D, \dots \rangle$ and any sequence $\boldsymbol{f} \in D^\omega$, we have:

   $$\mathfrak{D} \models \varphi[\boldsymbol{f}] \quad \textit{iff} \quad \mathfrak{S} \models \Phi\big[K^{\mathfrak{A}}_{\theta_0}(\boldsymbol{f}), \dots, K^{\mathfrak{A}}_{\theta_m}(\boldsymbol{f})\big].$$

2. In particular, if φ is a sentence, so are $\theta_0, \dots, \theta_m$, and

   $$\mathfrak{D} \models \varphi[\boldsymbol{f}] \quad \textit{iff} \quad \mathfrak{S} \models \Phi\big[K^{\mathfrak{A}}_{\theta_0}, \dots, K^{\mathfrak{A}}_{\theta_m}\big].$$

The proof of the basic theorem

- Suppose that φ' and a partitioning sequence $\zeta' = \langle \Phi', \theta'_0, \ldots, \theta'_m \rangle$ satisfy the induction hypothesis.

- Take
$$\varphi = \exists v_k \varphi',$$
$$\zeta = \langle \Phi, \theta_0, \ldots, \theta_m \rangle, \tag{1}$$
where $\theta_i = \exists v_k \theta'_i \quad (i \leq m)$, and
$$\Phi = \exists U_0 \ldots U_m \big[Part_m(U_0, \ldots, U_m) \wedge \bigwedge_{i \leq m} (U_i \subseteq V_i) \wedge \Phi'(U_0, \ldots, U_m) \big]. \tag{2}$$

The proof of the basic theorem

- By IH the statement $\mathfrak{D} \models \varphi[\boldsymbol{f}]$ is equivalent to the statement

$$\mathfrak{S} \models \Phi'\big[K^{\mathfrak{A}}_{\theta'_0}(\boldsymbol{f}(k/g)), \ldots, K^{\mathfrak{A}}_{\theta'_m}(\boldsymbol{f}(k/g))\big], \text{ for some } g \in D. \quad (3)$$

- On the other hand, the condition

$$\mathfrak{S} \models \Phi\big[K^{\mathfrak{A}}_{\theta_0}(\boldsymbol{f}), \ldots, K^{\mathfrak{A}}_{\theta_m}(\boldsymbol{f})\big] \quad (4)$$

can be rephrased to the statement that

$$\begin{cases} \text{there exist sets } Y_0, \ldots, Y_m \text{ such that} \\ (a)\ Y_0, \ldots, Y_m \text{ are partition of } I, \\ (b)\ Y_i \subseteq K^{\mathfrak{A}}_{\exists v_k \theta'_i}(\boldsymbol{f}), \text{ for each } i \leq m, \text{and} \\ (c)\ \mathfrak{S} \models \Phi'[Y_0, \ldots, Y_m]. \end{cases} \quad (5)$$

The proof of the basic theorem

- ((3) $\Rightarrow$ (5)) Take

$$Y_i = K^{\mathfrak{A}}_{\theta'_i}(\boldsymbol{f}(k/g)) \quad (i \leq m). \tag{6}$$

  - Since ζ' is a partitioning sequence, (5a) holds.
  - The condition (5b) follows from the equation (6) and the fact that for each $i \leq m$,

$$K^{\mathfrak{A}}_{\theta'_i}(\boldsymbol{f}(k/g)) \subseteq K^{\mathfrak{A}}_{\exists v_k \theta'_i}(\boldsymbol{f}).$$

  - The condition (5c) follows from the equation (6) and the statement (3).

The proof of the basic theorem

- ((5) $\Rightarrow$ (3)) Since any $i \in I$ belongs Y_j for some unique $j \leq m$, by (5b), we have

$$\mathfrak{A}_i \models \exists v_k \theta'_j \big[\boldsymbol{f}[i]\big]. \tag{7}$$

Choose $g \in D$ such that for each $i \in I$ and the corresponding j,

$$\mathfrak{A}_i \models \theta'_j \big[\boldsymbol{f}(k/g)[i]\big]. \tag{8}$$

  - So if $i \in Y_j$, then $\mathfrak{A}_i \models \theta'_j \big[\boldsymbol{f}(k/g)[i]\big]$. In other words,

$$Y_j \subseteq K^{\mathfrak{A}}_{\theta'_j}(\boldsymbol{f}(k/g)) \quad (j \leq m). \tag{9}$$

  - Since $Y_0, \ldots, Y_m$ is a partition of I, we have

$$Y_j = K^{\mathfrak{A}}_{\theta'_j}(\boldsymbol{f}(k/g)) \quad (j \leq m). \tag{10}$$

Consequences of the basic theorem

- **Theorem 0.1.** *The decision problem for the theory of the generalized product of systems $\langle \mathfrak{A}_i \mid i \in I \rangle$ with respect to $\mathfrak{S}$, in the case that I is finite can be reduced to the decision problem for the theories of the factors. I.e., if each factor has a decidable theory, then so has the (finite) generalized product.*

- **Theorem 0.2.** *The decision problem for the theory of the generalized power $\mathfrak{B}^{\mathfrak{S}}$ reduces to the decision problems for the theories of $\mathfrak{B}$ and of $\mathfrak{S}$. I.e., if theory of $\mathfrak{B}$ and theory of $\mathfrak{S}$ are decidable, so is the theory of $\mathfrak{B}^{\mathfrak{S}}$.*

Consequences of the basic theorem

Theorem 0.3. *Let* $\mathscr{K}$ *be a class of relational systems,* $\mathscr{S}$ *be a class of subset algebras and* $\mathscr{P}_{\mathscr{K},\mathscr{S}}$ *be the class of all generalized product systems* $\mathscr{P}(\mathfrak{A}, \mathfrak{S})$*, where for any non-empty* I*,*

$$\mathfrak{S} = \langle S(I), \ldots \rangle \in \mathscr{S}, \qquad \mathfrak{A} = \langle \mathfrak{A}_i \mid i \in I \rangle,$$

and for each $i \in I$*,* $\mathfrak{A}_i \in \mathscr{K}$*.*

The theory $Th(\mathscr{P}_{\mathscr{K},\mathscr{S}})$ *is decidable if both theories* $Th(\mathscr{K})$ *and* $Th(\mathscr{S})$ *are decidable.*

Moreover, $Th(\mathscr{P}_{\mathscr{K},\mathscr{S}})$ *is uniquely determined by* $Th(\mathscr{K})$ *and* $Th(\mathscr{S})$*.*

The proof of Theorem 0.3

- Let $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m \rangle$ such that for any $\mathfrak{A}$ and $\mathfrak{S}$

$$\mathscr{P}(\mathfrak{A}, \mathfrak{S}) \models \varphi \ \ \textit{iff} \ \ \mathfrak{S} \models \Phi\big[K^{\mathfrak{A}}_{\theta_0}, \ldots, K^{\mathfrak{A}}_{\theta_m}\big].$$

- Let $t = \{j \mid j \leq m \textit{ and } \neg\theta_j \in Th(\mathscr{K})\}$ and let

$$\Psi = \forall V_0 \ldots \forall V_m \big[Part_m(V_0, \ldots, V_m) \\ \wedge \bigwedge_{j \in t} (V_j = \Lambda) \rightarrow \Phi(V_0, \ldots, V_m)\big].$$

- It can be shown that $\Psi \in Th(\mathscr{S})$ if and only if $\varphi \in Th(\mathscr{P}_{\mathscr{K},\mathscr{S}})$.

Examples of generalized products

- **Direct products**
- **Weak direct products**
- Countably weak direct products
- Ordinal products
- Weak ordinal products
- **Cardinal sums**
- Ordinal sums

Direct products

- The **direct product** of a non-empty indexed family $\mathfrak{A} = \langle \mathfrak{A}_i \mid i \in I \rangle$, where for each $i \in I$, $\mathfrak{A}_i = \langle A_i, R_i \rangle$, is the system $\langle D, R \rangle$, where $D = \mathscr{P}(A_i \mid i \in I)$, and for any $f, g \in D$, $\langle f, g \rangle \in R$ if and only if, for every $i \in I$, $\langle f(i), g(i) \rangle \in R_i$. I.e.,

$$\langle f, g \rangle \in R \ \textit{ iff } \ \{i \mid \langle f(i), g(i) \rangle \in R_i\} = I.$$

- A direct product can be viewed as a generalized product by letting
  - $\mathfrak{S} = \langle S(I), \Lambda, \cup, \cap, ^{-}, \subseteq \rangle$,
  - $\theta = Rv_0v_1$, $\Phi \equiv V_0 = \overline{\Lambda}$ *and* $\zeta = \langle \Phi, \theta \rangle$.

Weak direct products

- A **weak direct product** of a non-empty indexed family $\mathfrak{A} = \langle \mathfrak{A}_i \mid i \in I \rangle$, where for each $i \in I$, $\mathfrak{A}_i = \langle A_i, R_i \rangle$, is the system $\langle D, D^*, R^* \rangle$, where $D^* \subseteq D$,

$$f \in D^* \quad \textit{iff} \quad \{\ i \mid i \in I \textit{ and } \mathfrak{A}_i \models \neg\psi[f(i)]\ \} \textit{ is finite},$$

  and where R^* is the relation R restricted to D^*.

- A weak direct product can be viewed as a relativized generalized product by letting
  - $\mathfrak{S} = \langle S(I), \Lambda, \cup, \cap, \bar{\ }, \subseteq, Fin \rangle$,
  - $\theta^* = \neg\psi(v_0)$, $\theta = Rv_0v_1$, $\Phi \equiv V_0 = \overline{\Lambda}$, *and* $\zeta = \langle \Phi, \theta^*, \theta \rangle$.

Cardinal sums

- A **cardinal sum** of a non-empty indexed family $\mathfrak{A} = \langle \mathfrak{A}_i \mid i \in I \rangle$, where A_i and A_j are disjoint for any $i < j$ and $i, j \in I$, is the system

$$\langle \bigcup_{i \in I} A_i, \bigcup_{i \in I} R_i \rangle.$$

- The cardinal sum can be reformulated as the relativized generalized product of the systems $\mathfrak{B} = \langle \mathfrak{B}_i \mid i \in I \rangle$, where

$$\mathfrak{B}_i = \langle A_i \cup \{c_i\}, R_i, \{c_i\} \rangle$$

and for each $i \in I$, $c_i \notin A_i$.

Cardinal sums

- More precisely, let $D = \mathscr{P}(B_i \mid i \in I)$ and let $D^* \subseteq D$ for which

$$f \in D^* \quad \textit{iff} \quad \{i \mid f(i) \neq c_i\} \textit{ is a singleton.}$$

- For $g, h \in D^*$, let

$$\langle g, h\rangle \in R^* \quad \textit{iff} \quad \{i \mid \langle g(i), h(i)\rangle \in R_i\} \neq \Lambda.$$

- A cardinal sum can be viewed as a relativized generalized product by putting
  - $\mathfrak{S} = \langle S(I), \Lambda, \cup, \cap, {}^{-}, \subseteq\rangle$,
  - $\theta^* \equiv v_0 \neq c$, $\theta = Rv_0v_1$, $\Phi \equiv X_0 \neq \Lambda$, *and* $\zeta = \langle \Phi, \theta^*, \theta\rangle$.

The basic subset algebras

- Denote by $\mathfrak{S}_I = \langle S(I), \Lambda, \cup, \cap, ^-, \subseteq \rangle$ the basic subset algebra and let $\mathscr{L}$ be the corresponding language.
- For each $i \in \omega$, introduce a monadic predicate A_i for which $A_i(x)$ means "the set x has at least i elements".
- For each $i \in \omega$, define a monadic predicate E_i for which $E_i(x) = A_i(x) \wedge \neg A_{i+1}(x)$. I.e. $E_i(x)$ means that "the set x has exact i elements."
- Let $\boldsymbol{y} = \langle y_1, \ldots, y_m \rangle$ and a **constituent** $C(x, \boldsymbol{y})$ be the term of the form

$$\pm x \cap \pm y_1 \cap \ldots \cap \pm y_m.$$

The basic subset algebras

- Normalize the formula $\varphi(x, \boldsymbol{y})$ to a disjunction of primitive formulas of the form

$$\exists x \bigwedge \pm A_i \ \Big(\bigcup_{j<m} C_j(x, \boldsymbol{y}) \Big).$$

- Replace $A_i \ \left(\bigcup_{j<m} C_j(x, \boldsymbol{y})\right)$ by

$$\bigvee_{k_0+\ldots+k_{m-1}=i} \ \bigwedge_{j<m} A_{k_j}(C_j\big(x, \boldsymbol{y})\big).$$

- For each j, collect the terms of the form $\pm x \cap C_j(\boldsymbol{y})$ together. Then we can assume that primitive formulas are of the form

$$\exists x \bigwedge_j \bigwedge_i \pm A_i\big(\pm x \cap C_j(\boldsymbol{y})\big). \tag{11}$$

The basic subset algebras

- Since $C_j(x, \boldsymbol{y})$ are pairwise disjoint, (11) is equivalent to

$$\bigwedge_j \exists x \bigwedge_i \pm A_i\big(\pm x \cap C_j(\boldsymbol{y})\big).$$

- Note that $\exists x \bigwedge_i \pm A_i\big(\pm x \cap C_j(\boldsymbol{y})\big)$ is equivalent to

$$A_{i_0}(x \cap C_j(\boldsymbol{y})) \wedge A_{i_1}(\bar{x} \cap C_j(\boldsymbol{y})) \wedge \\ \neg A_{i_2}(x \cap C_j(\boldsymbol{y})) \wedge \neg A_{i_3}(\bar{x} \cap C_j(\boldsymbol{y})),$$

  for some $i_0, i_1, i_2, i_3 \in \omega$.

- Finally any formula $\varphi(\boldsymbol{y})$ reduces to a disjunctive normal form where each literal is in the form

$$E_i(C(\boldsymbol{y})) \quad \textit{or} \quad A_j(C(\boldsymbol{y})).$$

The basic subset algebras

- In particular, a sentence of $\mathscr{L}$ says "how many elements are in the domain."
- - The theory of any one system $\mathfrak{S}_I$ is decidable.
  - The theory of all systems $\mathfrak{S}_I$ is decidable.
  - The theory of all systems $\mathfrak{S}_I$ is the same as the theory of all systems $\mathfrak{S}_I$ where I is finite.
  - Two systems $\mathfrak{S}_I$ and $\mathfrak{S}_{I'}$ are elementarily equivalent if and only if I and I' both have the same finite cardinality or both are infinite.

Consequences of decidability of $\mathfrak{S}_I$

- Suppose that the theory of a system $\mathfrak{A}$ is decidable. Then
  - The theory of any one direct power of $\mathfrak{A}$ is decidable.
  - The theory of all direct powers of $\mathfrak{A}$ is decidable.
- - For any family of systems $\langle \mathfrak{A}_i \mid i \in \omega \rangle$, if a first order sentence is true in each of finite direct products $\mathfrak{A}_0$, $\mathfrak{A}_0 \times \mathfrak{A}_1$, $\ldots$, $\mathfrak{A}_0 \times \ldots \times \mathfrak{A}_n$, $\ldots$, then it is true in the infinite direct product $\mathfrak{A}_0 \times \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_n \times \ldots$.
  - If the class of all models of a set of first order sentences is closed under the operation of taking the direct product of two systems, then it is closed under the operation of taking the arbitrary, infinite, direct products.

Subset algebras with Fin

- Denote by $\mathfrak{S}'_I = \langle S(I), \Lambda, \cup, \cap, ^{-}, \subseteq, Fin \rangle$ the subset algebras with Fin and let $\mathscr{L}'$ be the corresponding language.
- The quantifier elimination procedure for $\mathfrak{S}_I$ carries over with no essential change. Finally any formula $\varphi(\boldsymbol{y})$ reduces to a disjunctive normal form where each literal is in the form

$$\begin{aligned} E_i(C(\boldsymbol{y})) \quad & \textit{or} \quad A_j(C(\boldsymbol{y})) \\ & \textit{or} \quad A_k(C(\boldsymbol{y})) \wedge Fin(C(\boldsymbol{y})) \\ & \textit{or} \quad \neg Fin(C(\boldsymbol{y})). \end{aligned}$$

- A sentence of $\mathscr{L}'$ says “how many elements are in the domain” and/or “whether the domain is finite.”

Multiplication of natural numbers

- Let $\mathfrak{A} = \langle \mathbb{P}, \cdot \rangle$ be the system where $\mathbb{P}$ is the set of positive integers and $\cdot$ is the ordinary multiplication.
- Let $\mathfrak{B} = \langle \omega, + \rangle$ be Presburger arithmetic.
- Let $\mathfrak{D} = \langle D, M \rangle$ be the relational system, where D is the set of sequences $f \in \omega^\omega$ for which

  $$\{i \mid i \in \omega \textit{ and } f(i) \neq 0\} \textit{ is finite},$$

  and for $f, g, h \in D$,

  $$\langle f, g, h \rangle \in M \quad \textit{iff} \quad f(i) + g(i) = h(i) \textit{ for all } i \in \omega.$$

Multiplication of natural numbers

- $\mathfrak{D}$ is a relativized generalized power of $\mathfrak{B}$ with respect to the subset algebra

$$\mathfrak{S}'_\omega = \langle S(\omega), \Lambda, \cup, \cap, ^-, \subseteq, Fin \rangle.$$

- $\mathfrak{D}$ is isomorphic to $\mathfrak{A}$ under the map $f : D \rightarrow \omega$ for which

$$f(x) = f(x_0, \ldots, x_n, \ldots) = p_0^{x_0} p_1^{x_1} \cdots p_n^{x_n} \cdots$$

where $p_0, \ldots, p_n, \ldots$ is the increasing enumeration of primes.

- The theory of $\mathfrak{A}$ is decidable, and so is the theory of multiplication of natural numbers.

Subset algebras with $\prec$

- Denote by $\mathfrak{S}_I^{\prec} = \langle S(I), \Lambda, \cup, \cap, ^{-}, \subseteq, \prec \rangle$ the subset algebras, where I is well-ordered by a relation $<$ and $\prec$ is an order on singleton subsets of I, for which

$$X \prec Y \quad \textit{iff} \quad \textit{there exist } i, j \in I \textit{ s.t. } X = \{i\}, Y = \{j\} \textit{ and } i < j.$$

- Fin can be defined by

$$Fin(X) \;\; \textit{iff} \;\; \forall Y \subseteq X \big(Y \neq \Lambda \rightarrow \exists y \in Y (\forall z \in Y (z < y)) \big).$$

- $\mathfrak{S}_I^{\prec}$ is a version of monadic second order systems of well-ordered structures.

- In particular, when $I = \omega$, $Th(\mathfrak{S}_\omega^{\prec})$ is $S1S$, and hence is decidable.

Multiplication of natural numbers with $<_P$

- Let $\mathfrak{A} = \langle \mathbb{P}, \cdot, <_P \rangle$ be the system of multiplication of positive integers with the order $<_P$ restricted to primes. I.e., for $x, y \in \mathbb{P}$,

$$x <_P y \quad \textit{iff} \quad x, y \textit{ are primes}, \quad \textit{and} \quad x < y.$$

- Let $\mathfrak{B} = \langle \omega, + \rangle$ be Presburger arithmetic.
- Let $\mathfrak{D} = \langle D, M, L \rangle$ be the relational system, where D, M is defined as before, and for $f, g \in D$,

$$\langle f, g \rangle \in L \quad \textit{iff} \quad \textit{there exists exactly one } i, j, k, l \in \omega \textit{ such that}$$
$$f(i) \neq 0,\ g(j) \neq 0,\ f(k) = 1,\ g(l) = 1, \quad \textit{and} \quad k < l.$$

Multiplication of natural numbers with $<_P$

- $\mathfrak{D}$ is a relativized generalized power of $\mathfrak{B}$ with respect to the subset algebra

$$\mathfrak{S}_\omega^\prec = \langle S(\omega), \Lambda, \cup, \cap, ^-, \subseteq, \prec \rangle.$$

- $\mathfrak{D}$ is isomorphic to $\mathfrak{A}$ under the map $f : D \rightarrow \omega$ for which

$$f(x) = f(x_0, \ldots, x_n, \ldots) = p_0^{x_0} p_1^{x_1} \cdots p_n^{x_n} \cdots$$

where $p_0, \ldots, p_n, \ldots$ is the increasing enumeration of primes.

- The theory of $\mathfrak{A}$ is decidable, and so is the theory of multiplication of natural numbers with the standard order restricted to primes. [Mau97]

Ordinal addition

- Let $\mathfrak{A} = \langle \omega, + \rangle$ be Presburger arithmetic.
- Let $\mathfrak{D} = \langle D, R \rangle$ be the relativized generalized power of $\mathfrak{A}$ with respect to the subset algebra $\mathfrak{S}_{\omega}^{\prec}$, where D is the weak power $\omega^{(\rho)}$, and where the relation R is defined such that for $f, g, h \in D$,

$\langle f, g, h \rangle \in R$ *if and only if*

either, *for all* $\xi < \rho$, $g(\xi) = 0$ *and* $f(\xi) = h(\xi)$;

or, *for some* $\xi < \rho$, $g(\xi) \neq 0$ *and* $f(\xi) + g(\xi) = h(\xi)$,

and, for all η *such that* $\xi < \eta < \rho$, $g(\eta) = 0$ *and* $f(\eta) = h(\eta)$,

and, for all $\eta < \xi$, $g(\eta) = h(\eta)$.

Ordinal addition

- Let $\mathfrak{C} = \langle \omega^\rho, S \rangle$ be the relational system of addition of ordinal numbers, where ω^ρ should be read as the ordinal exponentiation, and S is the addition of ordinal numbers restricted to ω^ρ.

- $\mathfrak{C}$ and $\mathfrak{D}$ are isomorphic under the map $f : C \to D$ for which if the Cantor normal form of α ($\alpha < \rho$) is

  $$\alpha = \omega^{\xi_0} \cdot k_0 + \omega^{\xi_1} \cdot k_1 + \ldots + \omega^{\xi_{p-1}} \cdot k_{p-1}$$

  (where $\xi_0 > \xi_1 \ldots > \xi_{p-1}$ and for each $i < p$, $0 \neq k_i \in \omega$), then

  $$f(\xi_i) = k_i \text{ *for each* } i < p, \text{ *and* } f(\eta) = 0 \text{ *otherwise.*}$$

- The theory of $\mathfrak{D}$ is decidable, and so is the theory of $\mathfrak{C}$.

Cardinal addition

- Let $\mathfrak{A} = \langle \omega, + \rangle$ be Presburger arithmetic.
- Let $\mathfrak{B} = \langle \rho, S \rangle$ be the relational system, where ρ is an arbitrary ordinal number different from 0 and S be the relation such that for $\alpha, \beta, \gamma \in \rho$,

$$\langle \alpha, \beta, \gamma \rangle \in S \;\; \textit{iff} \;\; \alpha \leq \beta \textit{ and } \gamma = \beta \textit{ \textbf{or} } \beta \leq \alpha \textit{ and } \gamma = \alpha.$$

- Let $\mathfrak{C} = \langle C_\rho, + \rangle$ be the relational system of addition of cardinal numbers, where C_ρ is the set of all cardinal numbers less than $\aleph_\rho$ and $+$ is the addition of cardinal numbers restricted to $\aleph_\rho$.

Cardinal addition

- Let $\mathfrak{D} = \langle D, R\rangle$ be the relativized generalized product of $\mathfrak{A}$ and $\mathfrak{B}$, where D is the disjoint union of ω and ρ, and where the relation R is defined such that for $\alpha, \beta, \gamma \in D$,

$$
\begin{array}{lll}
\langle \alpha, \beta, \gamma\rangle \in R & \textit{iff} & \alpha, \beta \in \omega \textit{ and } \alpha + \beta = \gamma \\
 & or & \alpha \in \omega \textit{ and } \beta \in \rho \textit{ and } \gamma = \beta \\
 & or & \alpha \in \rho \textit{ and } \beta \in \omega \textit{ and } \gamma = \alpha \\
 & or & \alpha \in \rho \textit{ and } \beta \in \rho \textit{ and } \langle \alpha, \beta, \gamma\rangle \in S.
\end{array}
$$

- $\mathfrak{D}$ and $\mathfrak{C}$ are isomorphic under the map $f : D \to C_\rho$ for which

$$f(\alpha) = \alpha \ \ \textit{if } \alpha \in \omega, \quad \textit{and} \ \ f(\alpha) = \aleph_\alpha \ \ \textit{if } \alpha \in \rho.$$

- The theory of $\mathfrak{D}$ is decidable, and so is the theory of $\mathfrak{C}$.

Subset algebras with equivalence

- Denote by $\mathfrak{S}_I^{\approx} = \langle S(I), \Lambda, \cup, \cap, {}^{-}, \subseteq, \approx \rangle$ the subset algebras, where $\approx$ is the set-theoretic equivalence operator. Let $\mathscr{L}_{\approx}$ be the language of $\mathfrak{S}_I^{\approx}$.

- Define the systems of addition of cardinal numbers by

$$\mathfrak{C}_I = \langle C_I, \mathbf{0}, + \rangle$$

where C_I is the set of all cardinal numbers $\leq \mathfrak{c}(I)$ and $+$ is addition of cardinal numbers restricted to C_I. Let $\mathscr{L}_{\mathfrak{c}}$ be the language of $\mathfrak{C}_I$.

Subset algebras with equivalence

- Let Φ be a formula of $\mathscr{L}_{\approx}$ with n free variables $\{V_j \mid j \in t \subset \omega\}$, and $\boldsymbol{r} = \langle r_0, \ldots, r_{2^n-1} \rangle$ be any one enumeration of subsets of t. Then one can correlate with Φ a formula Ψ of $\mathscr{L}_{\mathfrak{c}}$ with at most 2^n free variables $v_0, \ldots, v_{2^n-1}$ such that

$$\mathfrak{S}_I^{\approx} \models \Phi[X_0, \ldots, X_{n-1}]$$

for any $X_0, \ldots, X_{n-1} \in S(I)$, if and only if

$$\mathfrak{C}_I \models \Psi[\mathfrak{a}_0, \ldots, \mathfrak{a}_{2^n-1}],$$

where for each $i < 2^n$,

$$\mathfrak{a}_i = \mathfrak{c}\Big(\bigcap_{j \in r_i} X_j \cap \bigcap_{j \in t - r_i} \overline{X}_j \Big).$$

Subset algebras with equivalence

- Let $\Phi = \Phi_0 \vee \Phi_1$ and for each $i = 0, 1$, Ψ_i, n_i, t_i and

$$\boldsymbol{r}_i = \left\langle r^i_0, \ldots, r^i_{2^{n_i}-1} \right\rangle$$

satisfy the induction hypothesis.

- Then take $\Psi(v_0, \ldots, v_{2^n-1})$ be

$$\Psi_0\big(\rho^0_0, \ldots, \rho^0_{2^{n_0}-1}\big) \;\vee\; \Psi_1\big(\rho^1_0, \ldots, \rho^1_{2^{n_1}-1}\big)$$

where for each $i = 0, 1$, and $k < 2^{n_i}$,

$$\rho^i_k = \sum_{j \in l^i_k} v_j,$$

and

$$l^i_k = \big\{ j \mid j < 2^n \textit{ and } r^i_k \subseteq r_j \textit{ and } t_i - r^i_k \subseteq t - r_j \big\}.$$

Subset algebras with equivalence

- Let $\Phi = \exists V_k \Phi'$ and suppose that Ψ', $\boldsymbol{r}'$, n' and t' satisfy the induction hypothesis.
- Without loss of generality assume that n, t and $\boldsymbol{r}$ satisfy the conditions: $n' = n + 1$, $t' = t \cup \{k\}$ and

$$\begin{aligned}\boldsymbol{r}' &= \left\langle r'_0, \ldots, r'_{2^{n'}-1} \right\rangle \\ &= \left\langle r_0 \cup \{k\}, \ldots, r_{2^n-1} \cup \{k\}, r_0, \ldots, r_{2^n-1} \right\rangle\end{aligned}$$

- Then take $\Psi(v_0, \ldots, v_{2^n-1})$ be

$$\exists w_0, \ldots, w_{2^{n'}-1} \Big[\prod_{k<2^n} (w_k + w_{2^n+k} = v_k) \wedge \Psi'(w_0, \ldots, w_{2^{n'}-1}) \Big].$$

Subset algebras with equivalence: examples

- If Φ is $V_0 \subseteq V_1$, then take Ψ be $v_{\{0\}} = \mathbf{0}$ by noting

$$V_0 \subseteq V_1 \quad \textit{iff} \quad \mathfrak{c}(V_0 \cap \overline{V}_1) = \mathbf{0}.$$

- If Φ is $V_0 \approx V_1$, then take $\Psi = v_{\{0,1\}} + v_{\{0\}} = v_{\{0,1\}} + v_{\{1\}}$ by noting that

$$V_0 \approx V_1 \quad \textit{iff} \quad \mathfrak{c}(V_0 \cap V_1) + \mathfrak{c}(V_0 \cap \overline{V}_1) = \mathfrak{c}(V_0 \cap V_1) + \mathfrak{c}(\overline{V}_0 \cap V_1).$$

- If Φ is $\exists V_0 \exists V_1 (V_0 \approx V_1)$, then by the construction, we have Ψ be

$$\exists w_{\{0\}} \exists w_{\emptyset} \exists v_{\{0,1\}} \exists v_{\{0\}} \exists v_{\{0,1\}} \exists v_{\{1\}} \big(w_{\{0\}} = v_{\{0,1\}} + v_{\{0\}} \wedge \\ w_{\emptyset} = v_{\{1\}} + v_{\emptyset} \ \wedge \ v_{\{0,1\}} + v_{\{0\}} = v_{\{0,1\}} + v_{\{1\}} \big).$$

Multiplication of natural numbers with $\cong$

- Let $\mathfrak{A} = \langle \mathbb{P}, \cdot, \cong \rangle$ be the system of multiplication of positive integers with a binary relation $\cong$, for which

  $x \cong y$ *iff* x, y *have the same number of distinct prime divisors.*

- Let $\mathfrak{B} = \langle \omega, + \rangle$ be Presburger arithmetic.
- $\mathfrak{A}$ is isomorphic to the relativized generalized product $\mathfrak{D}$ of $\mathfrak{B}$ with respect to the subset algebra $\mathfrak{S}_{\omega}^{\approx}$.
- The theory of $\mathfrak{A}$ is decidable, and so is the theory of multiplication of natural numbers with the relation $\cong$.

Generalized weak powers

- Denote by $\mathfrak{S}_I^* = \langle S^*(I), \Lambda, \cup, \cap, ^-, \subseteq, Fin \rangle$ the relational systems where I is a non-empty set and $S^*(I)$ is the set of all finite subsets of I and their complements.

- 1. There is an effective procedure to compute, for each formula φ of $\mathscr{L}_\pi$, a sequence $\zeta = \langle \Phi, \theta_0, \ldots, \theta_m \rangle$ such that given any $\mathfrak{A} = \langle A, \ldots \rangle$, $\mathfrak{S}_I^* = \langle S^*(I), \ldots \rangle$, $\mathfrak{D} = \mathfrak{A}^{\mathfrak{S}_I^*} = \langle D, \ldots \rangle$, and $\boldsymbol{f} \in D^\omega$, we have:

$$\mathfrak{D} \models \varphi[\boldsymbol{f}] \quad \textit{iff} \quad \mathfrak{S}_I^* \models \Phi\big[K_{\theta_0}^{*\mathfrak{A}}(\boldsymbol{f}), \ldots, K_{\theta_m}^{*\mathfrak{A}}(\boldsymbol{f})\big].$$

  2. In particular, if φ is a sentence, so are $\theta_0, \ldots, \theta_m$, and

$$\mathfrak{D} \models \varphi \quad \textit{iff} \quad \mathfrak{S}_I^* \models \Phi[K_{\theta_0}^{*\mathfrak{A}}, \ldots, K_{\theta_m}^{*\mathfrak{A}}].$$

Generalized weak powers

- The decision problem for $Th(\mathfrak{A}^{\mathfrak{S}_I^*})$ reduces to the decision problems for $Th(\mathfrak{A})$ and $Th(\mathfrak{S}_I^*)$.
- The decidability of systems $\mathfrak{S}_I^* = \langle S^*(I), \Lambda, \cup, \cap, ^-, \subseteq, Fin \rangle$ can be reduced to the decidability of systems

$$\mathfrak{S}_I^+ = \langle S^+(I), \Lambda, \cup, \cap, \subseteq \rangle$$

  where $S^+(I)$ is the set of all finite subsets of I, and in particular, the language of $S^+(I)$ doesn't involve constants $^-$ or Fin.
- The decision problem for $Th(\mathfrak{A}^{\mathfrak{S}_I^*})$ reduces to the decision problems for $Th(\mathfrak{A})$ and $Th(\mathfrak{S}_I^+)$. In particular, if the latter two theories are decidable, so is the former.

References

[FV59] S. Feferman and R.L. Vaught. The first order properties of products of algebraic systems. *Fundamenta Mathematicae*, 47:57–103, 1959.

[Hod97] Wilfrid Hodges. *A Shorter Model Theory*. Cambridge University Press, Cambridge, 1997.

[KC90] H. Jerome Keisler and C. C. Chang. *Model Theory*. Elsevier Science, Netherlands, 1990.

[Mau97] Francoise Maurin. The theory of integer multiplication with order restricted to primes is decidable. *The Journal of Symbolic Logic*, 62(1):123–130, 1997.