Proof Theory 293B: The Ramified Set Theory

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1 Syntax

- The language of $\mathcal{L}_{RS}$: $\in$, $\notin$, **Ad**, $\neg$**Ad** and a constant L_α for every ordinal α.
- The terms of $\mathcal{L}_{RS}$:
  - Every constant L_α is an atomic set term of stage α.
  - If $a_1, \ldots, a_n$ are set terms of stages $< \alpha$, then

    $$\{x \in L_\alpha \mid F(x, a_1, \ldots, a_n)^{L_\alpha}\}$$

    is a set term of stage α.

2 Semantics

The standard interpretation of $\mathcal{L}_{RS}$ is given by

(1) ${L_\alpha}^{\mathbf{L}} = \mathbf{L}_\alpha$,

(2) $\{x \in L_\alpha \mid F(x, a_1, \ldots, a_n)^{L_\alpha}\}^{\mathbf{L}} = \{x \in \mathbf{L}_\alpha \mid \mathbf{L}_\alpha \models F(x, a_1^{\mathbf{L}}, \ldots, a_n^{\mathbf{L}})\}$.

Proposition 2.1. *(1) For every set term s of stage α we have $s^{\mathbf{L}} \in \mathbf{L}_{\alpha+1}$.*

(2) $\mathbf{L} \models s \in L_\alpha$ iff $\mathbf{L} \models s = t$ for some set term t with $\mathbf{stg}(t) < \alpha$.

(3) $\mathbf{L} \models \mathbf{Ad}(t)$ iff $\mathbf{L} \models t = L_\kappa$ for some $\kappa \in \mathbf{Reg}$ with $\kappa \leq \mathbf{stg}(t)$.

(4) $\mathbf{L} \models s \in \{x \in L_\alpha \mid F(x)\}$ iff $\mathbf{L} \models t = s \wedge F(t)$ for some set term t with $\mathbf{stg}(t) < \alpha$.

3 The theory KP1

The theory **KP1** = the theory of $\mathbf{KP}\omega$ + the following axioms.

Ad1 $(\forall u)[\mathbf{Ad}(u) \Rightarrow \omega \in u \wedge Tran(u)]$

Ad2 $(\forall x)(\forall y)[\mathbf{Ad}(x) \wedge \mathbf{Ad}(y) \Rightarrow x \in y \vee y \in x \vee x = y]$

Ad3 $(\forall x)[\mathbf{Ad}(x) \Rightarrow (Pair)^x \wedge (Union)^x \wedge (\Delta_0 - Separation)^x \wedge (\Delta_0 - Collection)^x]$

Lim $(\forall x)(\exists u)[\mathbf{Ad}(u) \wedge x \in u]$

4 Classification of formulas

$\bigvee$-type formulas:

$$s \in r, \quad \mathbf{Ad}(t), \quad A \vee B, \quad (\exists x \in r)G(x)$$

$\bigwedge$-type formulas:

$$s \notin r, \quad \neg\mathbf{Ad}(t), \quad A \wedge B, \quad (\forall x \in r)G(x)$$

5 Characteristic sub-sentences-set

The characteristic sub-sentences-set $\mathcal{C}(F)$ for all sentences F of $\bigvee$-type is defined inductively as follows.

- $\mathcal{C}(s \in r) = \{t = s \mid \mathbf{stg}(t) < \alpha\}$ *if* $r = L_\alpha$
- $\mathcal{C}(s \in r) = \{t = s \wedge F(t) \mid \mathbf{stg}(t) < \alpha\}$ *if* $r = \{x \in L_\alpha \mid F(x)\}$
- $\mathcal{C}(\mathbf{Ad}(t)) = \{L_\kappa = t \mid \kappa \in \mathbf{Reg} \wedge \kappa \leq \mathbf{stg}(t)\}$
- $\mathcal{C}(A \vee B) = \{A, B\}$
- $\mathcal{C}((\exists x \in r)G(x)) = \{G(t) \mid \mathbf{stg}(t) < \alpha\}$ *if* $r = L_\alpha$
- $\mathcal{C}((\exists x \in r)G(x)) = \{G(t) \wedge F(t) \mid \mathbf{stg}(t) < \alpha\}$ *if* $r = \{x \in L_\alpha \mid F(x)\}$

For $\bigwedge$-type sentences F

$$\mathcal{C}(F) := \{\neg G \mid G \in \mathcal{C}(\neg F)\}$$

If $G \in \mathcal{C}(F)$ and F is neither a disjuction nor a conjuction, then G is of the form $H(t)$ where t is a set term. Define

$$o_F(G) = \mathbf{stg}(t).$$

Lemma 5.1. *For every sentence F of $\bigvee$-type we have*

$$\mathbf{L} \models F \quad \textit{iff} \quad \mathbf{L} \models \bigvee_{G \in \mathcal{C}(F)} G$$

and for every sentence F of $\bigwedge$-type we have

$$\mathbf{L} \models F \quad \textit{iff} \quad \mathbf{L} \models \bigwedge_{G \in \mathcal{C}(F)} G$$

6 A Semi-formal system

Definition 6.1. *Define relation $\models^\alpha \Delta$ for finite sets of $\mathcal{L}_{RS}$-sentences Δ inductively as follows.*

($\bigvee$) *F is of $\bigvee$-type, $\models^{\alpha_0} \Delta, G$ for some $G \in \mathcal{C}(F)$ where $\alpha_0 < \alpha$ and $o_F(G) < \alpha$*
$\Rightarrow$

$$\models^\alpha \Delta, F$$

($\bigwedge$) *F is of $\bigwedge$-type, $\models^{\alpha_G} \Delta, G$ for all $G \in \mathcal{C}(F)$ where $\alpha_G < \alpha$*
$\Rightarrow$

$$\models^\alpha \Delta, F$$

Lemma 6.1. *For a $\mathcal{L}_{RS}$-sentence F,*

$$\mathbf{L} \models F \quad \textit{iff} \quad (\exists \alpha) \models^\alpha F.$$

proof sketch. Consider the case where F is of $\bigvee$-type.

$$\begin{aligned}
\mathbf{L} \models F \quad &\Leftrightarrow \quad \mathbf{L} \models \bigvee_{G \in \mathcal{C}(F)} G \\
&\Leftrightarrow \quad \mathbf{L} \models G \textit{ for some } G \in \mathcal{C}(F) \\
&\Leftrightarrow \quad (\exists \beta) \models^\beta G \textit{ for some } G \in \mathcal{C}(F) \\
&\qquad \textit{(let } \alpha = max\{o_F(G), \beta\}\textit{)} \\
&\Rightarrow \quad (\exists \alpha) \models^\alpha F
\end{aligned}$$

□

Lemma 6.2. *For a Σ_1-sentence F and an ordinal γ we have*

$$\models^\alpha F^{L_\gamma} \Rightarrow \mathbf{L}_\alpha \models F.$$

Proof. Let $F \equiv \exists x G(x)$ where $G(x)$ is Δ_0.

$$\begin{aligned}
\models^\alpha F^{L_\gamma} \quad &\Leftrightarrow \quad \models^\alpha (\exists x G(x))^{L_\gamma} \\
&\Leftrightarrow \quad \models^\alpha (\exists x \in L_\gamma) G(x)^{L_\gamma} \\
&\Leftrightarrow \quad \models^\alpha (\exists x \in L_\gamma) G(x) \\
&\Leftrightarrow \quad \models^{\alpha_0} G(t) \textit{ for some } t \textit{ with } \mathbf{stg}(t) < \gamma, \mathbf{stg}(t) < \alpha, \alpha_0 < \alpha \\
&\Rightarrow \quad \mathbf{L}_\alpha \models \exists x G(x)
\end{aligned}$$

□

7 Rank of $\mathcal{L}_{RS}$-expression

Definition 7.1. *Define the rank* $\mathbf{rk}(E)$ *of an* $\mathcal{L}_{RS}$*-expression* E *inductively as follows.*

- $\mathbf{rk}(L_\alpha) := \omega \cdot \alpha$
- $\mathbf{rk}(\{x \in L_\alpha \mid F(x)\}) := max\{\mathbf{rk}(L_\alpha) + 1, \mathbf{rk}(F(L_0)) + 2\}$
- $\mathbf{rk}(\mathbf{Ad}(t)) := \mathbf{rk}(\neg\mathbf{Ad}(t)) := \mathbf{rk}(t) + 5$
- $\mathbf{rk}(s \in t) := \mathbf{rk}(s \notin t) := max\{\mathbf{rk}(s) + 6, \mathbf{rk}(t) + 1\}$
- $\mathbf{rk}(A \vee B) := \mathbf{rk}(A \wedge B) := max\{\mathbf{rk}(A), \mathbf{rk}(B)\} + 1$
- $\mathbf{rk}((\exists x \in s)F(x)) := \mathbf{rk}((\forall x \in s)F(x)) := max\{\mathbf{rk}(s), \mathbf{rk}(F(L_0)) + 2\}$

Example 7.1. *If* $b \neq L_0$*, then* $\mathbf{rk}(L_0 \in b) = \mathbf{rk}(b) + 1$*.*

proof sketch.

$$\begin{aligned}
\mathbf{rk}(L_0 \in b) &= max\{\mathbf{rk}(L_0) + 6, \mathbf{rk}(b) + 1\} \\
&= max\{\omega \cdot 0 + 6, \mathbf{rk}(b) + 1\} \\
&= \mathbf{rk}(b) + 1
\end{aligned}$$

□

Example 7.2. *If* $a \neq L_0, b \neq L_0$*, then* $\mathbf{rk}(a = b) = max\{\mathbf{rk}(a), \mathbf{rk}(b)\} + 4$*.*

proof sketch.

$$\begin{aligned}
\mathbf{rk}(a = b) &= \mathbf{rk}(\forall x \in a[x \in b] \wedge \forall x \in b[x \in a]) \\
&= max\{\mathbf{rk}(\forall x \in a[x \in b]), \mathbf{rk}(\forall x \in b[x \in a])\} + 1 \\
&= max\big\{max\{\mathbf{rk}(a), \mathbf{rk}(L_0 \in b) + 2\}, max\{\mathbf{rk}(b), \mathbf{rk}(L_0 \in a) + 2\}\big\} + 1 \\
&= max\big\{max\{\mathbf{rk}(a), \mathbf{rk}(b) + 3\}, max\{\mathbf{rk}(b), \mathbf{rk}(a) + 3\}\big\} + 1 \\
&= max\{\mathbf{rk}(a), \mathbf{rk}(b)\} + 4.
\end{aligned}$$

□

Lemma 7.1. *Let* b *be an* $\mathcal{L}_{RS}$ *formula and* c *be a set term. We have*

$$\mathbf{stg}(c) < \alpha \Rightarrow \mathbf{rk}(b(c)) < max\{\omega \cdot \alpha, \mathbf{rk}(b(L_0)) + 1\}.$$

proof sketch. By induction on the structure of b. Consider the case where $b \equiv (\exists x \in s)F(x, y)$. So $b(c) \equiv (\exists x \in s)F(x, c)$ and $b(L_0) \equiv (\exists x \in s)F(x, L_0)$. By the definition we have

$$\mathbf{rk}(b(c)) = max\{\mathbf{rk}(s), \mathbf{rk}(F(L_0, c)) + 2\}.$$

If $\mathbf{rk}(b(c)) = \mathbf{rk}(s)$, then

$$\begin{aligned}\mathbf{rk}(b(c)) &< max\{\mathbf{rk}(s), \mathbf{rk}(F(L_0, L_0)) + 2\} + 1 \\ &\leq \mathbf{rk}(b(L_0)) + 1 \\ &\leq max\{\omega \cdot \alpha, \mathbf{rk}(b(L_0)) + 1\}.\end{aligned}$$

Suppose that $\mathbf{rk}(b(c)) = \mathbf{rk}(F(L_0, c)) + 2$. By the induction hypothesis we have

$$\mathbf{rk}(F(L_0, c)) < max\{\omega \cdot \alpha, \mathbf{rk}(F(L_0, L_0)) + 1\}$$

Now consider if $\mathbf{rk}(F(L_0, c)) < \omega \cdot \alpha$ or not. If it is the first case, then

$$\mathbf{rk}(F(L_0, c)) + 2 < \omega \cdot \alpha.$$

and hence

$$\begin{aligned}\mathbf{rk}(b(c)) &= max\{\mathbf{rk}(s), \mathbf{rk}(F(L_0, c)) + 2\} \\ &< max\{\omega \cdot \alpha, \mathbf{rk}(b(L_0)) + 1\}.\end{aligned}$$

If it is the second case, then

$$\begin{aligned}\mathbf{rk}(b(c)) &= max\{\mathbf{rk}(s), \mathbf{rk}(F(L_0, c)) + 2\} \\ &< max\{\mathbf{rk}(s), \mathbf{rk}(F(L_0, L_0)) + 1 + 2\} \\ &\leq max\{\mathbf{rk}(s) + 1, \mathbf{rk}(F(L_0, L_0)) + 1 + 2\} \\ &= \mathbf{rk}(b(L_0)) + 1 \\ &\leq max\{\omega \cdot \alpha, \mathbf{rk}(b(L_0)) + 1\}.\end{aligned}$$

□

Lemma 7.2. $\mathbf{stg}(c) < \alpha \Rightarrow \mathbf{rk}(F(c)) + 1 < \mathbf{rk}(s \in \{x \in L_\alpha \mid F(x)\})$.

proof sketch. If $\mathbf{stg}(c) < \alpha \Rightarrow$, then

$$\begin{aligned}\mathbf{rk}(F(c)) + 1 &< max\{\omega \cdot \alpha + 1, \mathbf{rk}(F(L_0)) + 2\} \\ &= max\{\mathbf{rk}(L_\alpha) + 1, \mathbf{rk}(F(L_0)) + 2\} \\ &= \mathbf{rk}(s \in \{x \in L_\alpha \mid F(x)\})\end{aligned}$$

□

Theorem 7.1. *For $G \in \mathcal{C}(F)$ we have $\mathbf{rk}(G) < \mathbf{rk}(F)$.*

proof sketch. Consider only the case where $F \equiv (\exists x \in \{y \in L_\alpha \mid H(y)\})K(x)$. We have

$$G \equiv H(t) \wedge K(t) \quad \textit{and} \quad \mathbf{stg}(t) < \alpha.$$

$$\begin{aligned}\mathbf{rk}(G) &= max\{\mathbf{rk}(H(t)), \mathbf{rk}(K(t))\} + 1 \\ &< max\big\{max\{\omega \cdot \alpha, \mathbf{rk}(H(L_0)) + 1\}, max\{\omega \cdot \alpha, \mathbf{rk}(K(L_0)) + 1\}\big\} + 1 \\ &= max\{\omega \cdot \alpha + 1, \mathbf{rk}(H(L_0)) + 2, \mathbf{rk}(K(L_0)) + 2\} \\ &= max\{\mathbf{rk}(x \in L_\alpha \mid H(x)\}), \mathbf{rk}(K(L_0)) + 2\} \\ &= \mathbf{rk}(F).\end{aligned}$$

□

Lemma 7.3. *For an $\mathcal{L}_{RS}$ sentence F and all $G \in \mathcal{C}(F)$*

$$o_F(G) < \mathbf{rk}(F).$$

proof sketch. Consider the case where $G = H(t)$ with $t = \{x \in L_\alpha \mid F(x)\}$. Then

$$o_F(G) = \mathbf{stg}(t) < \alpha < \mathbf{rk}(\{x \in L_\alpha \mid F(x)\}) < \mathbf{rk}(G) < \mathbf{rk}(F).$$

□

Theorem 7.2. *For $\mathcal{L}_{RS}$ sentences F*

$$\mathbf{L} \models F \quad \Rightarrow \quad \models^{\mathbf{rk}(F)} F.$$

proof sketch. By Theorem 7.1 and Lemma 7.3. □

8 Another semi-formal system

Definition 8.1. *Define relation $\vdash^\alpha_\rho \Delta$ for finite sets of $\mathcal{L}_{RS}$-sentences Δ inductively as follows.*

($\bigvee$) $F \in \Delta \cap \bigvee$*-type,* $\vdash^{\alpha_0}_\rho \Delta, G$ *for some* $G \in \mathcal{C}(F)$ *with* $\alpha_0 < \alpha$ *and* $o_F(G) < \alpha$ $\Rightarrow$

$$\vdash^\alpha_\rho \Delta$$

($\bigwedge$) $F \in \Delta \cap \bigwedge$*-type,* $\vdash^{\alpha_G}_\rho \Delta, G$ *for all* $G \in \mathcal{C}(F)$ *with* $\alpha_G < \alpha$ $\Rightarrow$

$$\vdash^\alpha_\rho \Delta$$

(cut) $\vdash^{\alpha_0}_\rho \Delta, A$, $\vdash^{\alpha_0}_\rho \Delta, \neg A$ *for some* $\alpha_0 < \alpha$ *and some* A *with* $\mathbf{rk}(A) < \rho$ $\Rightarrow$

$$\vdash^\alpha_\rho \Delta$$

(Ref_κ) $F^{L_\kappa} \in \Pi^\kappa_2$, $(\exists z \in L_\kappa)[z \neq 0 \wedge F^z] \in \Delta$, $\vdash^{\alpha_0}_\rho \Delta, F^{L_\kappa}$, $\kappa \in \mathbf{Reg}$, $\kappa, \alpha_0 + 1 < \alpha$ $\Rightarrow$

$$\vdash^\alpha_\rho \Delta$$

Lemma 8.1 (Soundness)**.** $\vdash^\alpha_\rho \Delta \Rightarrow \mathbf{L} \models \bigvee \Delta$.

Theorem 8.1 (Cut Elimination)**.** *If $\vdash^\alpha_\rho \Delta, \Gamma$ and $\mathbf{L} \not\models F$ for all $F \in \Gamma$, then $\models^\alpha \Delta$.*

proof sketch. Case ($\bigwedge$).
There exists $F \in \Delta \cup \Gamma$ such that

$$\vdash^{\alpha_G}_\rho \Delta, \Gamma, G \quad \textit{for all } G \in \mathcal{C}(F) \textit{ where } \alpha_G < \alpha$$

If $F \in \Gamma$, then $\mathbf{L} \not\models G_0$ for some $G_0 \in \mathcal{C}(F)$. By induction hypothesis we have

$$\models^{\alpha_{G_0}} \Delta$$

and since $\alpha_{G_0} < \alpha$ we have

$$\models^{\alpha} \Delta$$

If $F \notin \Gamma$, then $F \in \Delta$. By induction hypothesis we have

$$\models^{\alpha_G} \Delta, G \;\; \textit{for all} \;\; G \in \mathcal{C}(F) \;\; \textit{with} \; \alpha_G < \alpha$$

By the definition of relation $\models^{\alpha}$, we have

$$\models^{\alpha} \Delta, F$$

which is the same as

$$\models^{\alpha} \Delta$$

Case (*cut*).
In this case we have

$$\vdash^{\alpha_0}_{\rho} \Delta, \Gamma, A \;\; \textit{and} \;\; \vdash^{\alpha_0}_{\rho} \Delta, \Gamma, \neg A \;\; \textit{for some} \;\; \alpha_0 < \alpha \;\; \textit{and some} \;\; A \;\; \textit{with}\, \mathbf{rk}(A) < \rho$$

Without loss of generality, let us assume that $\mathbf{L} \not\models A$. Then by induction hypothesis we have

$$\models^{\alpha} \Delta$$

Case (Ref_{κ})
We have

$$(\exists z \in L_{\kappa})[z \neq 0 \wedge F^z] \in \Delta \cup \Gamma, \quad \vdash^{\alpha_0}_{\rho} \Delta, \Gamma, F^{L_{\kappa}}$$

If $\mathbf{L} \not\models (\exists z \in L_{\kappa})[z \neq 0 \wedge F^z]$, then $\mathbf{L} \not\models F^{L_{\kappa}}$. By the induction hypotheis we have

$$\models^{\alpha_0} \Delta$$

which implies

$$\models^{\alpha} \Delta$$

If $\mathbf{L} \models (\exists z \in L_{\kappa})[z \neq 0 \wedge F^z]$, then $(\exists z \in L_{\kappa})[z \neq 0 \wedge F^z] \in \Delta$. Since

$$\mathbf{rk}((\exists z \in L_{\kappa})[z \neq 0 \wedge F^z]) = \kappa < \alpha$$

we have

$$\models^{\alpha} (\exists z \in L_{\kappa})[z \neq 0 \wedge F^z]$$

and hence

$$\models^{\alpha} \Delta$$

□