A proof of topological completeness for S4 in (0,1)

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Outline

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Background

1. Completeness of modal logic $S4$ for $\mathbb{R}$, real interval $(0,1)$ and every dense-in-itself separable metric space, [MT44].

2. Simplified proofs:

   (a) completeness of $S4$ for Cantor spaces, [Min99].

   (b) completeness of Int for $(0,1)$, [Min00] Chapter 9.

   (c) completeness of $S4$ for $(0,1)$, [AvBB01].

   However, both simplified proofs for $(0,1)$ contain gaps.

3. New proof: combines the ideas in [Min99], [Min00] and [AvBB01] and provides a further simplification.

The idea of the new proof

1. $S4$ is complete for finite rooted Kripke models. I.e., for any wff α, $S4 \vdash \alpha$ if and only if α is valid in every finite rooted Kripke structures.
2. For each finite rooted Kripke model $\mathcal{K}$, there is a continuous and open mapping from the standard topology on $(0, 1)$ onto $\mathcal{K}$.

Corollary: $S4$ is complete for the real segment $(0, 1)$.

$(0,1)$ v.s. a Cantor set

- $\mathscr{B}$: the set of all infinite binary sequences $\vec{b} = b_1 b_2 \ldots$ ($b_i \in \{0,1\}$) except identical 0^ω and sequences ending in a tail of 1's,
- $\mathscr{B}_1 = \{\vec{b} \in \mathscr{B} \mid (\exists i)(\forall j > i)\vec{b}(j) = 0\}$,
- $\mathscr{B}_2 = \mathscr{B} \setminus \mathscr{B}_1$.
- One-to-one correspondence between $\mathscr{B}$ and real interval $(0,1)$ is given by

$$\mathbf{real}(\vec{b}) = \sum_{i=1}^{\infty} \vec{b}(i) 2^{-i}$$

$$B(x) = \text{ the unique } \vec{b} \in \mathscr{B} \text{ such that } \mathbf{real}(\vec{b}) = x$$

Kripke space

- **K** : $\langle W, R, w_0 \rangle$: a finite rooted Kripke $S4$-model where W is a set of worlds, $w_0 \in W$, R is a reflexive and transitive relation on W and Rw_0w holds for all $w \in W$.
- $\mathcal{K}$: the topological space with the carrier W and the base of open sets $\mathcal{O}_w = \{w' \in W \mid Rww'\}$ for all $w \in W$.

Unwinding of $\mathcal{K}$ to cover $\mathscr{B}$

Label finite binary sequences $\bar{b} = b_1 b_2 \dots b_n$, $n \geq 1$ by worlds $w \in W$ as follows.

1. $\mathcal{W}(\emptyset) = w_0$,
2. If $\mathcal{W}(\bar{b}) = w$, no extension of $\bar{b}$ is yet labeled and $w, w_1, \dots, w_m$ are all R-successors of $w \in W$, then let
   - $\mathcal{W}(\bar{b}0^i) = w$ for all $0 < i \leq 2m$
   - $\mathcal{W}(\bar{b}0^{2i-1}1) = w_i$ for $0 < i \leq m$
   - $\mathcal{W}(\bar{b}0^{2i}1) = w$ for $0 \leq i < m$.

Mapping π from $(0,1)$ onto $\mathcal{K}$

Let $\vec{b}\restriction n = \vec{b}(1)\vec{b}(2)\ldots\vec{b}(n)$, for $\vec{b} \in \mathscr{B}$

- Stabilization point.

$$\lambda(\vec{b}) = \text{ the least } n \geq 1 \; [(\forall i, j \geq n) R\mathcal{W}(\vec{b}\restriction i)\mathcal{W}(\vec{b}\restriction j)]$$

- Choice point.

$$\rho(\vec{b}) = \begin{cases} max(1,n) & \text{if } \vec{b} = \vec{b}(1)\vec{b}(2)\ldots\vec{b}(n)10^{\omega} \in \mathscr{B}_1 \\ \lambda(\vec{b}) & \text{if } \vec{b} \in \mathscr{B}_2 \end{cases}$$

- Mapping of $(0,1)$ onto $\mathcal{K}$.

$$\pi(x) = \mathcal{W}(B(x)\restriction\rho(B(x))) \text{ for } x \in (0,1)$$

Two Lemmas

- Closeness measure δ:

$$\delta(\vec{b}) = max(1, n) \text{ if } \vec{b} = \vec{b}(1)\vec{b}(2)\ldots\vec{b}(n)10^{\omega} \in \mathscr{B}_1$$

$$\delta(\vec{b}) = \text{ the least } n > \lambda(\vec{b})(\vec{b}(n) = 1 \text{ and } \vec{b}(n-1) = 0) \text{ if } \vec{b} \in \mathscr{B}_2.$$

- **Lemma 0.1.** *For any $x, y \in (0,1)$, if $|y - x| < 2^{-(\delta(x)+2)}$, then $R\pi(x)\pi(y)$.*

  This means $\delta(x) + 2$ is the modulus of continuity for π.

- **Lemma 0.2.** *For any $x \in (0,1)$, $\epsilon > 0$, $w \in W$ with $R\pi(x)w$, there exists $y \in (0,1)$ such that $|y - x| < \epsilon$ and $\pi(y) = w$.*

  This means π is open.

π is continuous

- Let $W_0 \subseteq W$ be an arbitrary open set in the Kripke space $\mathcal{K}$. (That is, W_0 is closed under R.)
- For $w \in W_0$, let $x \in \pi^{-1}(w)$, that is, $\pi(x) = w$.
- The set $O_x = \{y \mid |x - y| < 2^{-(\delta(x)+2)}\}$ is open in the topology space $(0, 1)$.
- By Lemma 0.1 $\pi(O_x) \subseteq W_0$ and hence π is continuous.

π is open

- Let $\mathcal{O}_x$ be the collection of sets

$$O_{x,i} = \{y \mid |x - y| < 2^{-(i+\delta(x)+2)}\}$$

- $\bigcup_x \mathcal{O}_x$ is a base of the standard topology on $(0, 1)$.
- By Lemma 0.1 for any $w \in \pi(O_{x,i})$ we have $R\pi(x)w$.
- By Lemma 0.2 for any w with $R\pi(x)w$, there exists $y \in O_{x,i}$ such that $\pi(y) = w$, that is, $w \in \pi(O_{x,i})$.
- Hence $\pi(O_{x,i}) = \{w \in W \mid R\pi(x)w\}$ which is obviously closed under R, and so $\pi(O_{x,i})$ is open in $\mathcal{K}$.

Valuation

Lemma 0.3. *Let X_1, X_2 be two topological spaces and $f : X_1 \to X_2$ a continuous and open map. Let V_2 be a valuation on X_2 and define*

$$V_1(p) = f^{-1}(V_2(p))$$

for each proposition p. Then

$$V_1(\alpha) = f^{-1}(V_2(\alpha))$$

for any wff of $S4$.

Proof of Lemma 0.3

The base case and induction steps for connectives $\vee, \wedge, \neg$ are straightforward. Now suppose $\alpha = \Box\beta$. By induction hypothesis,

$$V_1(\beta) = f^{-1}(V_2(\beta))$$

And it follows from openness and continuity that

$$Int(f^{-1}(V_2(\beta))) = f^{-1}(Int(V_2(\beta)))$$

So finally we have,

$$\begin{aligned} V_1(\alpha) &= V_1(\Box\beta) = Int(V_1(\beta)) = Int(f^{-1}(V_2(\beta))) \\ &= f^{-1}(Int(V_2(\beta))) = f^{-1}(V_2(\Box\beta)) = f^{-1}(V_2(\alpha)) \end{aligned}$$

Validity

Lemma 0.4. *Let* $M_1 = \langle X_1, V_1 \rangle$, $M_2 = \langle X_2, V_2 \rangle$ *be two topological models and* $f : X_1 \to X_2$ *a continuous and open mapping. If* V_1 *is induced by* V_2 *as above, then for any* $S4$ *wff* α,

$$M_2 \models \alpha \text{ implies } M_1 \models \alpha$$

Moreover if f *is onto, then*

$$M_2 \models \alpha \text{ iff } M_1 \models \alpha$$

Proof of Lemma 0.4

- Let $M_2 \models \alpha$, but $M_1 \not\models \alpha$.
  Let V_1 be a valuation such that $V_1(\alpha) \neq X_1$.
  By Lemma 0.3 $V_1(\alpha) = f^{-1}(V_2(\alpha))$, so we have $V_2(\alpha) \neq X_2$, that is, $M_2 \not\models \alpha$.

- Let f be onto, $M_1 \models \alpha$, but $M_2 \not\models \alpha$.
  Let V_2 be a valuation such that $V_2(\alpha) \neq X_2$.
  Since f is onto and $V_1(\alpha) = f^{-1}(V_2(\alpha))$, $V_1(\alpha) \neq X_1$, that is, $M_1 \not\models \alpha$.

References

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