A proof of topological completeness for S4 in (0,1)

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Topological interpretation of $S4$

A topological model is an ordered pair $M = \langle X, V\rangle$, where X is a topological space and V is a function assigning a subset of X to each propositional variable. The valuation V is extended to all $S4$ formulas as follows:

$$\begin{aligned} V(\alpha \vee \beta) &= V(\alpha) \cup V(\beta), & V(\neg\alpha) &= X \setminus V(\alpha), \\ V(\alpha \wedge \beta) &= V(\alpha) \cap V(\beta), & V(\Box\alpha) &= \mathbf{Int}(V(\alpha)). \end{aligned}$$

where $\mathbf{Int}$ is the interior operator.

We say that α is valid in a topological model M and write $M \models \alpha$ if and only if $V(\alpha) = X$.

Completeness results

- $S4$ is complete for $\mathbb{R}$, $\mathbb{R}^n$, Cantor spaces, the real interval $(0, 1)$ and in general any dense-in-itself separable metric space [1].
- Several simplified proofs were published later:
  - The completeness of $S4$ for Cantor spaces [2].
  - The completeness of Int for $(0, 1)$ [3] Chapter 9.
  - The completeness of $S4$ for $(0, 1)$ [4].

  However, the last two completeness proofs contain gaps.
- A new proof presented here combines the ideas in [2], [3] and [4] and provides a further simplification.

The idea of the new proof

- $S4$ is complete for finite rooted Kripke models:

$$S4 \vdash \alpha \text{ iff } \models \alpha \text{ for all finite rooted Kripke structures.}$$

- For each finite rooted Kripke model $\mathbf{K}$, there is a continuous and open mapping from the standard topology on $(0,1)$ onto the corresponding Kripke space $\mathcal{K}$.

$$f : (0,1) \stackrel{onto}{\longrightarrow} \mathcal{K} \text{ is continuous and open.}$$

It follows that $S4$ is complete for the real interval $(0,1)$.

$(0, 1)$ and a Cantor set

- Let $\mathscr{B}$ be the set of all infinite binary sequences $\vec{b} = b_1 b_2 \ldots$ ($b_i \in \{0, 1\}$) except the identical 0^ω and the sequences ending with a tail of 1's,
- Define $\mathscr{B}_1$ and $\mathscr{B}_2$ by

$$\mathscr{B}_1 = \{\vec{b} \in \mathscr{B} \mid (\exists i)(\forall j > i)\vec{b}(j) = 0\} \text{ and } \mathscr{B}_2 = \mathscr{B} \setminus \mathscr{B}_1.$$

- A familiar one-to-one correspondence between $\mathscr{B}$ and the real interval $(0, 1)$ is given by

$$\mathbf{real}(\vec{b}) = \sum_{i=1}^{\infty} \vec{b}(i) 2^{-i}$$

$$B(x) = \text{ the unique } \vec{b} \in \mathscr{B} \text{ such that } \mathbf{real}(\vec{b}) = x$$

$(0,1)$ and a Cantor set

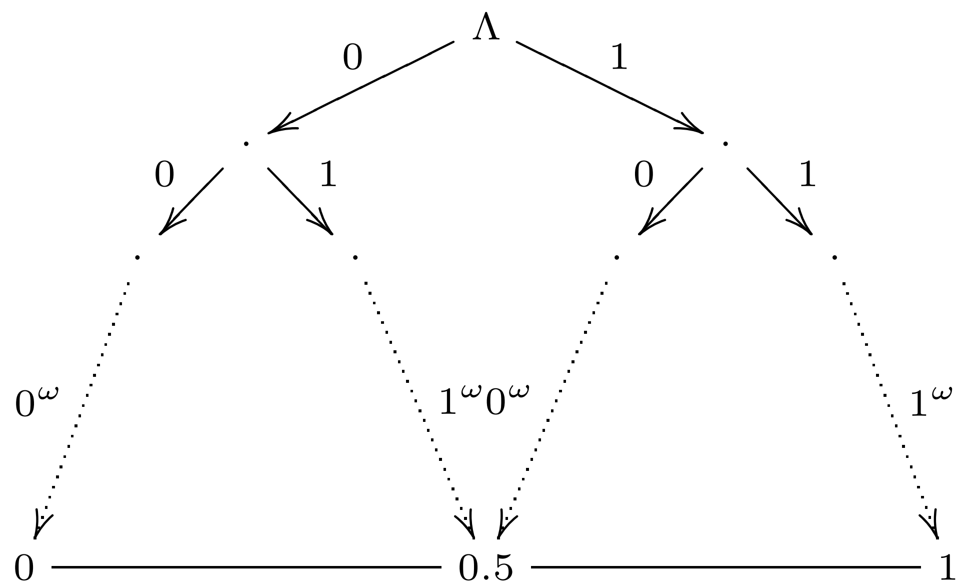

Figure 1: The full infinite binary tree

Kripke frame and Kripke space

- A **Kripke frame** (for $S4$) is an ordered pair $F = \langle W, R\rangle$ where W is a non-empty set and R is a reflexive and transitive relation on W. The elements in W are called **worlds**.

  A Kripke frame is **rooted** if there exists a world w_0 such that any world w in W is an R-successor of w_0.

- A **Kripke space** $\mathcal{K}$ is a topological space with the carrier W and the base of open sets $\mathcal{O}_w = \{w' \in W \mid Rww'\}$ for all $w \in W$.

We identify a Kripke frame $\langle W, R\rangle$ with a topological space $\mathcal{K}$.

Unwind K to cover $\mathscr{B}$

Label finite binary sequences $\bar{b} = b_1 b_2 \ldots b_n$, $n \geq 1$ by worlds $w \in W$ as follows.

- $\mathcal{W}(\Lambda) = w_0$.
- Let $\bar{b} \in \Sigma^*$ be a node in $\mathscr{B}$. Suppose $\bar{b}$ is already labeled by a world w (i.e., $\mathcal{W}(\bar{b}) = w$) , while none of its children has yet been labeled. Let $w, w_1, \ldots, w_m$ be all R-successors of w. Then

$$\begin{aligned} \mathcal{W}(\bar{b}0^i) &= w \text{ for } 0 < i \leq 2m, \\ \mathcal{W}(\bar{b}0^{2i-1}1) &= w_i \text{ for } 0 < i \leq m, \\ \mathcal{W}(\bar{b}0^{2i}1) &= w \text{ for } 0 \leq i < m. \end{aligned}$$

Unwind K to cover $\mathscr{B}$

In placing R-successors of w at right branches $\bar{b}0^{2i-1}1$ ($i > 0$), we interleave w with each of its other successors. This is the main distinction from the construction in [4].

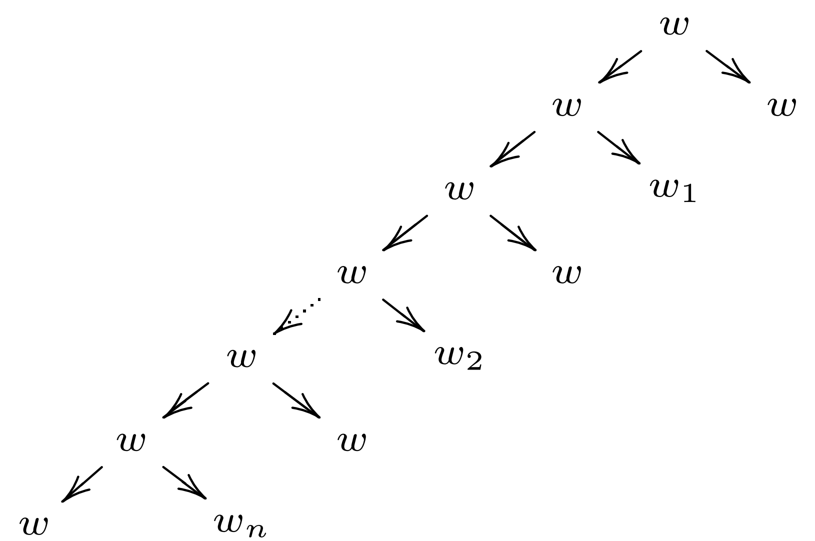

Figure 2: Unwinding and labeling

Map $(0,1)$ onto $\mathcal{K}$

Let $\vec{b}\restriction n = \vec{b}(1)\vec{b}(2)\ldots\vec{b}(n)$.

- Define **stabilization point** by

$$\lambda(\vec{b}) = \text{ the least } n \geq 1 \ \big[(\forall i,j \geq n) R\mathcal{W}(\vec{b}\restriction i)\mathcal{W}(\vec{b}\restriction j)\big].$$

- Define **selection point** by

$$\rho(\vec{b}) = \begin{cases} max(1,n) & \text{if } \vec{b} = \vec{b}(1)\vec{b}(2)\ldots\vec{b}(n)10^{\omega} \in \mathscr{B}_1, \\ \lambda(\vec{b}) & \text{if } \vec{b} \in \mathscr{B}_2. \end{cases}$$

Map $(0,1)$ onto $\mathcal{K}$

- Define a map π from $(0,1)$ onto $\mathcal{K}$ by

$$\pi(x) = \mathcal{W}(B(x) \restriction \rho(B(x))) \quad \text{for} \quad x \in (0,1).$$

- Define the **modulus of continuity** δ:
  if $\vec{b} = \vec{b}(1)\vec{b}(2)\dots\vec{b}(n)10^{\omega} \in \mathscr{B}_1$, then

$$\delta(\vec{b}) = max(1,n),$$

and if $\vec{b} \in \mathscr{B}_2$, then

$$\delta(\vec{b}) = \mu n[n > \lambda(\vec{b}) \;\&\; \vec{b}(n) = 1 \;\&\; \vec{b}(n-1) = 0].$$

Map $(0,1)$ onto $\mathcal{K}$

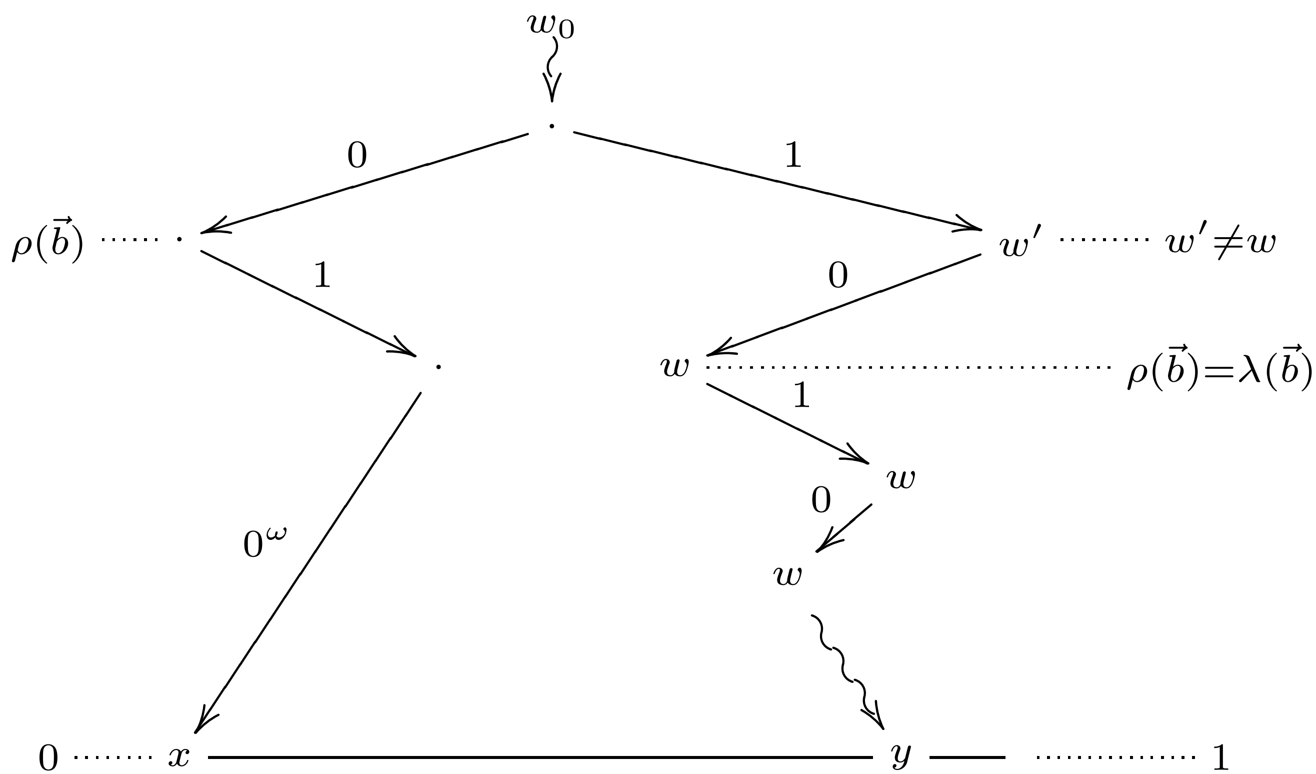

Figure 3: Selection point $\vec{b} \in \mathscr{B}_1$ (left) or $\vec{b} \in \mathscr{B}_2$ (right)

Four Lemmas

- **Lemma 0.1.** *For any* $x, y \in (0,1)$, *if* $|y-x| < 2^{-(\delta(x)+2)}$, *then* $R\pi(x)\pi(y)$.

- **Lemma 0.2.** *For any* $x \in (0,1)$, $\epsilon > 0$, $w \in W$ *with* $R\pi(x)w$, *there exists* $y \in (0,1)$ *such that* $|y-x| < \epsilon$ *and* $\pi(y) = w$.

- **Lemma 0.3.** *Let* X_1, X_2 *be two topological spaces and* $f : X_1 \to X_2$ *a continuous and open map. Let* V_2 *be a valuation on* X_2 *and define*

$$V_1(p) = f^{-1}(V_2(p)) \tag{1}$$

for every propositional variable p*. Then*

$$V_1(\alpha) = f^{-1}(V_2(\alpha))$$

for any $S4$*-formula* α*.*

Four Lemmas

- **Lemma 0.4.** *Let X_1, X_2 be two topological spaces and $f : X_1 \to X_2$ a continuous and open map. Let V_2 be a valuation for topological semantics on X_2 and define V_1 by the equation* (1)*. Then for any $S4$-formula α,*

$$\langle X_2, V_2\rangle \models \alpha \quad \textit{implies} \quad \langle X_1, V_1\rangle \models \alpha.$$

Moreover, if f is onto, then

$$\langle X_2, V_2\rangle \models \alpha \textit{ iff } \langle X_1, V_1\rangle \models \alpha.$$

The proof of Lemma 0.1

- Let $W_0 \subseteq W$ be an open subset of the Kripke space $\mathcal{K}$, i.e., W_0 is closed under R.
- For $w \in W_0$, let $x \in \pi^{-1}(w)$, so $\pi(x) = w$.
- The set $O_x = \{y \mid |x - y| < 2^{-(\delta(x)+2)}\}$ is open in $(0, 1)$.
- By Lemma 0.1 $\pi(O_x) \subseteq W_0$ and hence π is continuous.

The proof of Lemma 0.2

- Let $\mathcal{O}_x$ be the collection of sets

$$O_{x,i} = \{y \mid |x - y| < 2^{-(i+\delta(x)+2)}\}.$$

- $\bigcup_x \mathcal{O}_x$ is a base of the standard topology on $(0, 1)$.
- By Lemma 0.1 for any $w \in \pi(O_{x,i})$ we have $R\pi(x)w$.
- By Lemma 0.2 for any w with $R\pi(x)w$, there exists $y \in O_{x,i}$ such that $\pi(y) = w$, i.e., $w \in \pi(O_{x,i})$.
- Hence $\pi(O_{x,i}) = \{w \in W \mid R\pi(x)w\}$, which is obviously closed under R, and so $\pi(O_{x,i})$ is open in $\mathcal{K}$.

Proof of Lemma 0.3

The base case and induction steps for connectives $\vee, \wedge, \neg$ are straightforward. Now suppose that $\alpha = \Box\beta$. By induction hypothesis,

$$V_1(\beta) = f^{-1}(V_2(\beta)).$$

It follows from openness and continuity that

$$Int(f^{-1}(V_2(\beta))) = f^{-1}(Int(V_2(\beta))).$$

So we have,

$$\begin{aligned} V_1(\alpha) &= V_1(\Box\beta) = Int(V_1(\beta)) = Int(f^{-1}(V_2(\beta))) \\ &= f^{-1}(Int(V_2(\beta))) = f^{-1}(V_2(\Box\beta)) = f^{-1}(V_2(\alpha)). \end{aligned}$$

Proof of Lemma 0.4

- Suppose that $\langle X_2, V_2 \rangle \models \alpha$, i.e., $V_2(\alpha) = X_2$. By Lemma 0.3 $V_1(\alpha) = f^{-1}(V_2(\alpha))$, and so $V_1(\alpha) = X_1$ as required.
- Suppose that f is onto and $\langle X_1, V_1 \rangle \models \alpha$, but $\langle X_2, V_2 \rangle \not\models \alpha$, i.e., $V_2(\alpha) \neq X_2$. Since f is onto and $V_1(\alpha) = f^{-1}(V_2(\alpha))$, we have $V_1(\alpha) \neq X_1$, i.e., $\langle X_1, V_1 \rangle \not\models \alpha$, a contradiction.

$S4$ is complete

- Let α be an non-theorem of $S4$. So there exists a finite rooted Kripke model $\mathbf{K} = \langle X, V' \rangle$ such that $\mathbf{K} \not\models \alpha$.
- Let V be the valuation on $(0,1)$ such that

$$V(p) = \pi^{-1}(V'(p))$$

  for every propositional variable p.
- By Lemma 0.4 $V'(\beta) = X$ if and only if $V(\beta) = (0,1)$ for any $S4$-formula β.
- In particular since $V'(\alpha) \neq X$, we have $V(\alpha) \neq (0,1)$, i.e., $S4$ is refuted on $(0,1)$. It follows that $S4$ is complete for $(0,1)$.

Future works

- The completeness of **Dynamic Topological Logic** (DTL).
  - DTL is a $S4$-based trimodal logic ($\Box, \bigcirc, *$) for dynamical topological systems $\langle X, T\rangle$, where T is a continuous map on the topological space X.
  - The action of T on X is represented by the temporal operator $\bigcirc$ ("tomorrow") interpreted as a pre-image under T.
  - Properties of an orbit of T is described using the operator $*$ ("forever in the future").
- The completeness of **S4C** for $\mathbb{R}$.
  - $S4C$ is a $\Box, \bigcirc$-fragment of DTL.

References

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[3] Grigori Mints. *A short introduction to intuitionistic logic*. Kluwer Academic-Plenum Publishers, New York, 2000.

[4] Marco Aiello, Johan van Benthem, and Guram Bezhanishvili. Reasoning about space: the modal way. Technical Report PP-2001-18, ILLC, Amsterdam, 2001.