ACYCLIC HOMOMORPHISMS AND CIRCULAR COLORINGS OF DIGRAPHS

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Abstract. An acyclic homomorphism of a digraph D into a digraph F is a mapping $\phi\colon V(D) \to V(F)$ such that for every arc $uv \in E(D)$, either $\phi(u) = \phi(v)$ or $\phi(u)\phi(v)$ is an arc of F, and for every vertex $v \in V(F)$, the subgraph of D induced on $\phi^{-1}(v)$ is acyclic. For each fixed digraph F we consider the following decision problem: Does a given input digraph D admit an acyclic homomorphism to F? We prove that this problem is NP-complete unless F is acyclic, in which case it is polynomial time solvable. From this we conclude that it is NP-complete to decide if the circular chromatic number of a given digraph is at most q, for any rational number $q > 1$. We discuss the complexity of the problems restricted to planar graphs. We also refine the proof to deduce that certain F-coloring problems are NP-complete.

Key words. digraph, graph homomorphism, acyclic homomorphism, circular coloring, circular chromatic number, NP-completeness

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1. Introduction. Let H be a fixed graph. An *H-coloring* of a graph G is a graph homomorphism $G \to H$, i.e., a mapping $\phi\colon V(G) \to V(H)$ such that $\phi(u)\phi(v)$ is an edge of H whenever uv is an edge of G. This notion generalizes k-coloring since a K_k-coloring of G is precisely a standard k-coloring of G. For a fixed integer $k \geq 3$, to decide the existence of a k-coloring for a given graph G is one of the basic NP-complete problems. This result has been generalized to H-colorings by Hell and Nešetřil [10], who proved that, for a fixed graph H, to decide the existence of an H-coloring for a given graph G is NP-complete if H is not bipartite (and is polynomially solvable if H is bipartite).

Let $C(k,d)$ be the graph with vertex set $\{0,\dots,k-1\}$ in which distinct vertices i,j are adjacent if and only if $d \leq |i-j| \leq k-d$. The *circular chromatic number* $\chi_c(G)$ of a graph G is the minimum of all rational numbers k/d (where k and $d \leq k$ are positive integers) such that there exists a homomorphism $G \to C(k,d)$ (the minimum must exist [5]). Thus $\chi_c(G) \leq \frac{k}{d}$ if and only if there exists a homomorphism $G \to C(k,d)$, and the result of [10] implies that for every given rational number $q > 2$, it is also NP-complete to decide if a given graph G has $\chi_c(G) \leq q$. (See also [5, 8, 9]; in particular, it is known to be hard to decide if a graph G with $\chi(G) = n$ has $\chi_c(G) \leq n - \frac{1}{k}$ for any integers $k \geq 2, n \geq 3$ [9].)

The theory of circular colorings of graphs has become an important branch of chromatic graph theory with many exciting results and new techniques. We refer to the survey article by Zhu [15]. Recently, one of the authors [14] has extended the notion of circular colorings to graphs with weighted edges, which can be specialized to also yield a notion of the circular chromatic number $\chi_c(D)$ of a digraph [4].

Let D be a digraph. (All digraphs will be assumed to have no loops.) A vertex set $A \subseteq V(D)$ is *acyclic* if the induced subgraph $D[A]$ is acyclic. A partition of $V(D)$

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into k acyclic sets is called a *k-coloring* of D. The minimum integer k for which there exists a k-coloring of D is called the *chromatic number* $\chi(D)$ of the digraph D. (Note that $\chi(D) \leq |V(D)|$ since D has no loops.) Bokal et al. [4] proved that (in contrast with the undirected case) it is NP-complete to decide whether an input digraph D has $\chi(D) \leq 2$.

Let F be a fixed digraph. An *F-coloring* of a digraph D is a digraph homomorphism $D \to F$, i.e., a mapping $\phi\colon V(D) \to V(F)$ such that $\phi(u)\phi(v)$ is an arc of F whenever uv is an arc of D. The *F-coloring problem* asks whether or not an input digraph D admits an F-coloring [1, 2, 3, 12, 13]. In contrast to the case of graphs, no complexity classification of F-coloring problems is known, or conjectured. In fact, it is not even known if each F-coloring problem is polynomial time solvable or NP-complete, and if such a *dichotomy* result were true, then a much more general dichotomy for all constraint satisfaction problems would also hold [6]. There is, however, a conjecture [2] proposing a classification of the complexity of F-coloring problems when each vertex of F has a positive indegree as well as a positive outdegree. Our last result, Theorem 3.1, verifies a special case of this conjecture.

A graph G defines a natural digraph $D(G)$ with the same vertices as G, in which uv is an arc if and only if u and v are adjacent in G. Note that $D(G)$ is a *symmetric* digraph, i.e., the reversal of each arc is an arc. It is easy to see that a mapping $f : V(G) \to V(H)$ is a homomorphism of the graph G to the graph H if and only if it is a homomorphism of the digraph $D(G)$ to the digraph $D(H)$. (We say that the definition of digraph homomorphisms is *consistent* with the definition of graph homomorphisms.)

We introduce a different kind of digraph homomorphisms and obtain a complete classification of the corresponding F-coloring problems.

An *acyclic homomorphism* of a digraph D into a digraph F is a mapping $\phi\colon V(D) \to V(F)$ such that:

(i) for every arc $uv \in E(D)$, either $\phi(u) = \phi(v)$ or $\phi(u)\phi(v)$ is an arc of F, and
(ii) for every vertex $v \in V(F)$, the subgraph of D induced on $\phi^{-1}(v)$ is acyclic.

It is easy to check that the composition of acyclic homomorphisms is again an acyclic homomorphism. It is also easy to see that this definition is also consistent with the definition of graph homomorphims, i.e., that a mapping f is a graph homomorphism of G to H if and only if it is an acyclic digraph homomorphism of $D(G)$ to $D(H)$.

An acyclic homomorphism of D to F will also be called an *acyclic F-coloring* of D. For a fixed digraph F, the *acyclic F-coloring problem* asks whether or not an input digraph D admits an acyclic F-coloring.

We now define a digraph analogue of $C(k,d)$: The digraph $\vec{C}(k,d)$ has the vertex set $V(\vec{C}(k,d)) = \{0,\ldots,k-1\}$, and from each vertex $i \in V(\vec{C}(k,d))$ there are arcs to $i+d, i+d+1, \ldots, i+k-1$, with arithmetic modulo k. Notice that $\vec{C}(n,n-1) \simeq \vec{C}_n$ is the directed n-cycle.

One can again define the *circular chromatic number* $\chi_c(D)$ of the digraph D [4] as the minimum of all rational numbers k/d (where k and $d \leq k$ are positive integers) such that there exists an acyclic homomorphism $D \to \vec{C}(k,d)$. If k and d are positive integers with $k \geq d$, then $\chi_c(\vec{C}(k,d)) = \frac{k}{d}$.

It is not difficult to see [4] that

$$\chi(D) - 1 < \chi_c(D) \leq \chi(D).$$

It follows from [4] that it is NP-complete to decide if $\chi_c(D) \leq 2$. This suggests that to decide if $\chi_c(D) \leq q$ should be NP-complete for every $q \geq 2$, but it gives no insight on what may hold for $q < 2$.

In this paper we verify that the acyclic F-coloring problem is NP-complete, unless F is acyclic, in which case it is polynomial time solvable. This implies, in particular, that to decide if $\chi_c(D) \leq q$ is also NP-complete, for every fixed rational number $q > 1$. Refining this proof, we also conclude that certain F-coloring problems are NP-complete, verifying special cases of two conjectures from [1, 2].

2. Acyclic homomorphisms and colorings. We begin by disposing of the easy positive direction.

PROPOSITION 2.1. *Suppose F is an acyclic digraph. Then a digraph D admits an acyclic F-coloring if and only if D is itself acyclic.*

Proof. If D is acyclic, any constant mapping (all vertices of D map to one vertex of F) is an acyclic homomorphism. Conversely, if D contains a directed cycle C, then any acyclic homomorphism of D to a digraph G takes C to a directed cycle in G. □

For the negative results, we observe that all acyclic F-coloring problems are in the class NP, with the mapping ϕ itself being a concise certificate.

Recall that $\vec{C}_2 = \vec{C}(2,1)$ denotes the directed two-cycle. Note that D admits an acyclic homomorphism to $\vec{C}_2$ if and only if $\chi(D) \leq 2$. Therefore our first negative result follows from [4].

PROPOSITION 2.2. *The acyclic $\vec{C}_2$-coloring problem is NP-complete.*

Proof. We shall present a brief proof, slightly adapting the proof in [4], because we shall need to refer to the details of it in the next section. We shall give a polynomial time reduction from the NP-complete problem of 2-colorability of 3-uniform hypergraphs (also known as the not-all-equal 3-satisfiability problem without negated variables). For such a hypergraph X we construct a digraph D consisting of one vertex x for each vertex x of X, and three vertices a_e, b_e, c_e for each hyperedge $e = abc$ of X. The arcs of D are xx_e and x_ex for each vertex x of X and each hyperedge e containing x, and a_eb_e, b_ec_e, c_ea_e for each hyperedge e of X. We claim that X is 2-colorable if and only if D admits an acyclic $\vec{C}_2$-coloring, i.e., can be colored with two colors so that each color class is acyclic. Given a 2-coloring of X, we can apply the same colors to the vertices x of D, and the opposite color to all vertices x_e for edges e containing x. There will be no monochromatic directed cycle. Moreover, whenever D is colored with two colors without a monochromatic directed cycle, the colouring of the vertices x yields a 2-coloring of X. □

Recall that $D(G)$ is the symmetric digraph associated with the graph G. On the other hand, each digraph D is also associated with a natural graph $H(D)$ which has the same vertices as D, and in which two vertices u, v are adjacent if and only if both uv and vu are arcs of D. Note that the symmetric digraph $D(H(F))$ is obtained from F by removing all arcs uv for which the reversal vu is not an arc. We call this digraph the *symmetric part* of F. It is again easy to see that if a mapping is a digraph homomorphism of D to F, then it is also a graph homomorphism of the symmetric part of D to the symmetric part of G.

Our second negative result follows from [10].

PROPOSITION 2.3. *If the symmetric part of F contains an odd cycle, then the acyclic F-coloring problem is NP-complete.*

Proof. If the symmetric part of F contains an odd cycle, then $H(F)$ is nonbipartite, and hence it is NP-complete to decide if an input graph G admits a homomorphism to $H(F)$ [10]. On the other hand, we claim that G admits a homomorphism to $H(F)$ if and only if $D(G)$ admits an acyclic homomorphism to F. Any homomorphism of G to $H(F)$ is clearly also an acyclic homomorphism of $D(G)$ to F. Thus consider an acyclic homomorphism ϕ of $D(G)$ to F. Since $D(G)$ is a symmetric digraph, ϕ is in fact an acyclic homomorphism of $D(G)$ to the symmetric part of F, i.e., to $D(H(F))$. Therefore ϕ is a homomorphism of G to $H(F)$. □

We are now ready for our first main result.

THEOREM 2.4. *If F contains a directed cycle, then the acyclic F-coloring problem is NP-complete.*

Proof. Let k be the minimum length of a directed cycle in F. We first assume that $k \geq 3$, i.e., that the symmetric part of F is empty. Let F' be the digraph obtained from F by adding an arc uv whenever there is in F a directed path from u to v of length at most $k-1$. Let D' be the digraph obtained from D by replacing each arc xy by a directed path of length $k-1$ from x to y. We claim that there exists an acyclic homomorphism of D to F' if and only if there exists an acyclic homomorphism of D' to F.

Suppose first that ϕ is an acyclic homomorphism of D' to F. Each arc xy of D corresponds to a path of length $k-1$ from x to y in D', which is taken by the acyclic homomorphism ϕ to a path of length at most $k-1$ in F. (This follows from the definition of an acyclic homomorphism and the fact that there are no directed cycles of length less than k in F.) Thus $\phi(x) = \phi(y)$ or $\phi(x)\phi(y)$ is an arc of F'. Moreover, for every $v \in V(F')$, the set $\phi^{-1}(v) \cap V(D)$ is a subset of $\phi^{-1}(v)$ in D', and hence is acyclic in D'. Observe that if $\phi(x) = \phi(y) = v$, then ϕ maps to v all vertices of D' on the $(k-1)$-path from x to y. Therefore, the set $\phi^{-1}(v) \cap V(D)$ induces an acyclic subgraph of D. Thus ϕ restricted to $V(D)$ is an acyclic homomorphism of D to F'. Conversely, suppose that ϕ is an acyclic homomorphism of D to F'. Then it is easy to see that the mapping ϕ can be extended to the all vertices $v \in V(D') \setminus V(D)$ (on the added directed paths of length $k-1$) so that the resulting mapping is an acyclic homomorphism of D' to F.

This argument is a polynomial reduction from the problem of acyclic F'-coloring to the problem of acyclic F-coloring. Since F contains a directed cycle of length k, the digraph F' contains $k \geq 3$ vertices in a complete directed digraph, i.e., the symmetric part of F' contains a triangle. By Proposition 2.3 the acyclic F'-coloring problem, and hence also the acyclic F-coloring problem, is NP-complete.

It remains to deal with the case when the symmetric part of F is bipartite, but not empty. Suppose $H(F)$ has $\ell \geq 1$ edges. For any digraph D we construct, in polynomial time, a digraph $D^{(\ell)}$ consisting of disjoint copies $D(i,j)$ of D, for all pairs $i < j$, with $i, j = 0, 1, \ldots, \ell$, and of special vertices $a_0, a_1, \ldots, a_\ell, b_0, b_1, \ldots, b_\ell$. Moreover, each vertex of $D(i,j)$ has an arc from a_i and b_i, and to a_j and b_j, and there are also arcs a_ib_i, b_ia_i for all $i = 0, 1, \ldots, \ell$ amongst the special vertices (see Fig. 2.1).

We claim that D has an acyclic $\vec{C}_2$-coloring if and only if $D^{(\ell)}$ has an acyclic F-coloring. Indeed, if D has an acyclic $\vec{C}_2$-coloring, then all $D(i,j)$ can be acyclically $\vec{C}_2$-colored by the same $\vec{C}_2$, and this coloring extends to the special vertices as well by coloring all a_i with one color and all b_i with the other. Conversely, if $D^{(\ell)}$ has an acyclic F-coloring, then two pairs a_i, b_i and a_j, b_j must map to the same two vertices u, v belonging to an edge of $H(F)$, by the pigeon-hole principle. This means that each vertex c of $D(i,j)$ must also map to u or v, otherwise u, v and $\phi(c)$ would form a symmetric triangle, contrary to the assumption that $H(F)$ is bipartite. Thus we obtain a $\vec{C}_2$-coloring of D. This amounts to a polynomial reduction of the problem of acyclic $\vec{C}_2$-coloring (which is NP-complete by Proposition 2.2) to the problem of acyclic F-coloring, and hence the latter problem is also NP-complete. □

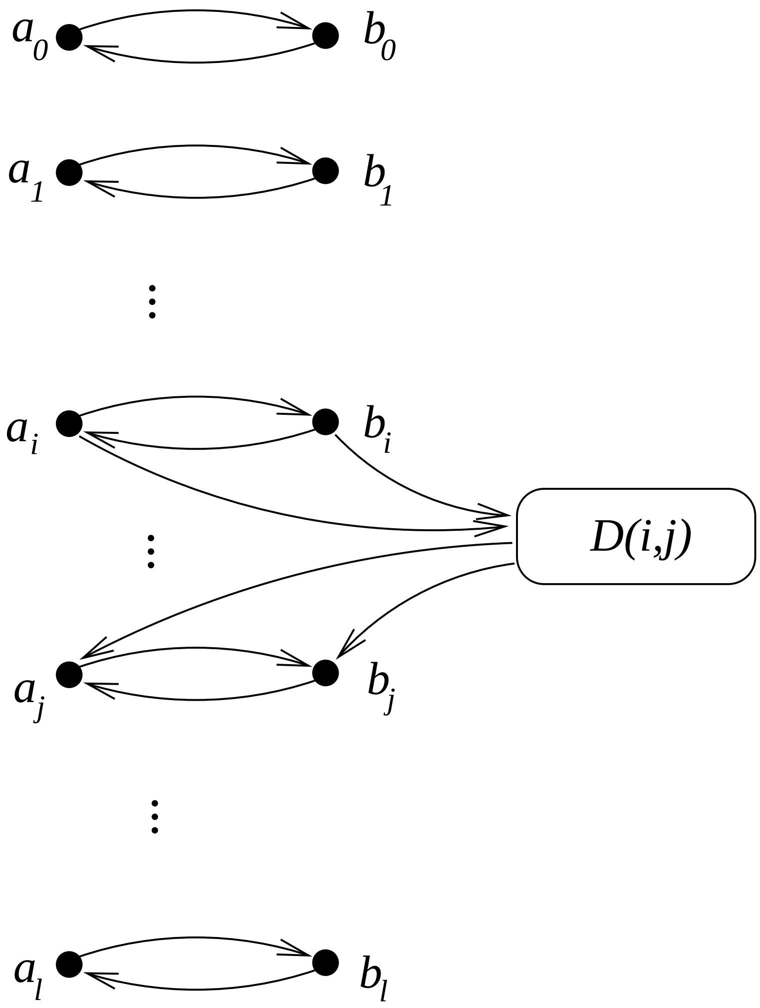

FIG. 2.1. *The digraph* $D^{(\ell)}$

COROLLARY 2.5. *For every fixed rational number* $q > 1$*, it is NP-complete to decide if* $\chi_c(D) \leq q$.

Proof. We have $\chi_c(D) \leq \frac{k}{d}$ if and only if D admits an acyclic homomorphism to $\vec{C}(k,d)$, and as long as $d < k$, $\vec{C}(k,d)$ is not acyclic. ▯

For graphs, it has been shown in [9], that it is NP-hard to decide whether a graph G of chromatic number n satisfies $\chi_c(G) \leq n - \frac{1}{k}$, for any positive integers $k \geq 2$ and $n \geq 3$. One can ask similar questions for circular chromatic numbers of digraphs. We only remark that it is NP-hard to decide if $\chi_c(D') \leq \frac{3}{2}$ even knowing that $\chi(D') = 2$: Consider the digraph $F = \vec{C}_3 = \vec{C}(3,2)$, and apply the proof of the Theorem 2.4, with $k = 3$. The digraph F' will be the symmetric triangle, and acyclic F'-colorability is NP-complete. From that proof we know that an input digraph D has an acyclic F'-coloring if and only if the digraph D' (which has chromatic number 2) has an acyclic F-coloring, i.e., has $\chi_c(D') \leq \frac{3}{2}$.

It would also be interesting to know how the complexity of the acyclic F-coloring problem changes when the inputs are restricted. Typical restrictions may involve maximum degree, or planarity, etc. (For undirected graphs we direct the reader to [11] for a survey.) We first make the following observation.

COROLLARY 2.6. *The acyclic $\vec{C}_3$-coloring problem is NP-complete even when restricted to planar digraphs.*

Proof. Let $F = \vec{C}_3$. Since the shortest directed cycle in F has length three, we can apply the above reduction from the problem of acyclic F'-coloring to the problem of acyclic F-coloring. In this case F' is the symmetric triangle; since 3-coloring is NP-complete for planar graphs [7], the corollary follows. □

We also have a similar result for acyclic $\vec{C}_2$-coloring:

THEOREM 2.7. *The acyclic $\vec{C}_2$-coloring problem is NP-complete even when restricted to planar digraphs.*

Proof. We reduce the problem of planar 3-satisfiability. An instance of 3-satisfiability is *planar* if its associated graph is planar. (The associated graph has a vertex C for each clause, and a vertex x for each variable; there is an edge joining x and C if variable x occurs in clause C, positively or negatively.) It is well known that 3-satisfiability is NP-complete even when restricted to connected planar instances [7].

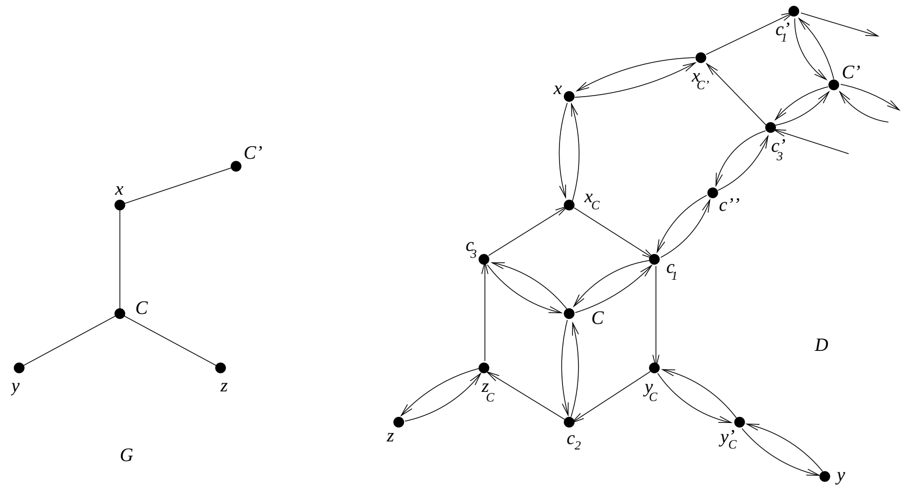

FIG. 2.2. *Construction of D around the clause $C = \neg x \vee y \vee \neg z$*

Thus suppose we have an instance of planar 3-satisfiability, and consider the planar embedding of its (connected) associated graph G. We shall transform G to a digraph D which is $\vec{C}_2$-colorable if and only if the instance was satisfiable. The digraph D will contain all the vertices (C and x) of G, in the same position in the plane as in G. If C was joined to x, y, z (in this clockwise order) in G, we surround it with a directed six-cycle $x_C c_1 y_C c_2 z_C c_3 x_C$, joined to C by the symmetric set of arcs Cc_1, $c_1C, Cc_2, c_2C, Cc_3, c_3C$. The new vertices c_i, called *dummy vertices*, are distinct for each clause C. Further, we replace each edge xC of G by the symmetric arcs xx_C, x_Cx if x occurs negatively in C, or the symmetric path of length two xx'_C, x'_Cx, x'_Cx_C, $x_Cx'_C$ if x occurs positively in C. It is clear that the digraph constructed so far is planar. Now consider, for each vertex x corresponding to a variable, the six-cycles corresponding to the clauses C in which x occurs (positively or negatively), in their circular order of the planar embedding. For any two consecutive six-cycles there exist two dummy vertices c, c' which can be joined without destroying the planar embedding; we add the symmetric path of length two $cc'', c''c, c''c', c'c''$. This is our planar digraph D. This construction around the clause $C = \neg x \vee y \vee \neg z$ and with consecutive neighbors C and C' around x is represented in Figure 2.2.

We now claim that D admits an acyclic $\vec{C}_2$-coloring if and only if the original instance was satisfiable. Indeed, given a satisfying truth assignment, color each vertex corresponding to a variable x by 0 if x is false, and by 1 if x is true, and do the same for all vertices x_C. Furthermore color all dummy vertices by 0, and color all clause vertices C by 1. It is easy to see that all the auxiliary vertices x'_C and c'' can be colored as well so that the result is an acyclic $\vec{C}_2$-coloring of D. Conversely, suppose we have an acyclic $\vec{C}_2$-coloring of D. Because of the two-cycles Cc_iC, all dummy vertices in any one six-cycle must obtain the same color; because of the symmetric paths of length two between dummy vertices of consecutive six-cycles, all dummy vertices must obtain the same color, say color 0. (Recall that we have assumed that G is connected.) It is now easy to see that the coloring defines a satisfying truth assignment. (Because of the 6-cycles $x_Cc_1y_Cc_2z_Cc_3x_C$, at least one of the vertices x_C, y_C, or z_C has color 1.) □

3. A refinement for F-colorings. For a digraph F, let F^p denote the digraph obtained by replacing each vertex of F by the transitive tournament T on $1, 2, \ldots, p$. (The arcs of T are all pairs ij with $i < j$.) There is an arc in F^p from a vertex in the copy of T corresponding to u to a vertex in the copy of T corresponding to v if and only if there is an arc from u to v in F. Then it follows from the definitions that a digraph D admits an acyclic homomorphism to F if and only if it admits a homomorphism to F^p with $p = |V(D)|$.

We let similarly F^ω be obtained from F by replacing each vertex by the countable transitive tournament on $1, 2, \ldots$. Theorem 2.4 shows that if F is not acyclic then the F^ω-coloring problem (appropriately defined for mappings of finite digraphs to a finitely described fixed infinite graph) is intractable. We now refine the result to prove that already the F^2-coloring problem is intractable. More precisely, assume for each vertex v of F, we have an integer $p_v \geq 2$. Let F^* be a digraph obtained the same way from F by replacing each v with the transitive tournament on p_v vertices and defining the arcs between these tournaments as above.

THEOREM 3.1. *If F is not acyclic, then the F^*-coloring problem is NP-complete.*

Proof. If the symmetric part of F contains an odd cycle, then the symmetric part of F^* also contains an odd cycle (and we only need each $p_v \geq 1$ here), and the F^*-coloring problem is NP-complete by exactly the same proof as in Proposition 2.3. (Just substitute F^* for F and omit all the occurrences of the word 'acyclic'.)

If the symmetric part of F is empty, then assume as above that the length of the shortest directed cycle in F is k, where $k \geq 3$. Suppose first that k is odd. Let F' be the digraph on the same vertex set as F^* and with an arc uv whenever there is in F^* a directed path from u to v of length $\frac{k+1}{2}$. A proof similar to the proof of Theorem 2.4 shows that there is a polynomial time reduction from the F'-coloring problem to the F^*-coloring problem. (Take D' to be the digraph obtained from D by replacing each arc xy by a directed path of length $\frac{k+1}{2}$ from x to y. We are using the 'indicator construction', Lemma 1 from [10].) We now note that F' contains symmetric pairs of arcs joining vertices at distance $\frac{k+1}{2}$ and $\frac{k-1}{2}$ in the original directed k-cycle in F, and hence the symmetric part of F' contains an odd cycle.

If k is even, we proceed in exactly the same way using directed paths of length $\frac{k}{2} + 1$. In this case the symmetric part of F' also contains a nonbipartite graph when $k \geq 6$. (There are symmetric pairs of arcs joining vertices at distance $\frac{k}{2} - 1$, $\frac{k}{2}$, and

$\frac{k}{2}+1$.) For $k=4$ we extend our attention to the eight vertices of F^2, a subgraph of F^*, on which the symmetric part of F' is easily seen to have a nonbipartite subgraph. (Indeed, suppose the original 4-cycle in F is $1,2,3,4$ and let $a_1,b_1,a_2,b_2,a_3,b_3,a_4,b_4$ be the corresponding vertices of F^2. Then, using directed paths of length 3, F' contains the symmetric five-cycle $a_1b_2b_4a_2b_3$.)

It remains to prove that

- if F has a nonempty and bipartite symmetric part, then F^*-coloring is NP-complete.

We proceed by contradiction, assuming that F has a nonempty bipartite symmetric part, and F^*-coloring is not NP-complete. We may assume that S^*-coloring is NP-complete for any proper subgraph S of F which has a nonempty bipartite symmetric part.

This part of the proof uses the 'sub-indicator construction', Lemma 2 of [10]. To review it briefly, in the special case we shall need, we define a digraph to be a *core* if it does not admit a homomorphism to a proper subgraph. If a digraph F is not a core, then it contains a unique, up to isomorphism, subgraph S which is a core; this subgraph S is called the *core of F*. It is clear that a digraph admits an F-coloring if and only if it admits an S-coloring. (Thus the F-coloring and S-coloring problems are equivalent.) Let J be a fixed digraph with specified vertices v and w. The *sub-indicator construction*, with respect to J, transforms a given core digraph S, with a specified vertex u, to the subgraph S^- induced by the vertex set V^- defined as follows: Let R be the digraph obtained from the disjoint union of J and S by identifying vertices u and v. Then a vertex x of S belongs to V^- just if there is a homomorphism f of R to S such that $f(y)=y$ for all vertices y of S, and $f(w)=x$. Lemma 2 of [10] gives, for a core S, a polynomial time reduction of the S^--coloring problem to the S-coloring problem.

It follows from our assumptions that F^* is a core, otherwise the core of F^* would be some digraph S^* of a proper subgraph S of F which has a nonempty bipartite symmetric part, and hence both the S^*-coloring and the F^*-coloring problems would be NP-complete.

We first claim that every vertex of F is incident with an edge of $H(F)$. Otherwise, consider the sub-indicator J consisting of three vertices v,w, and z, and two arcs wz,zw. If F contained a vertex x which is not incident with an edge of $H(F)$, then all the p_x vertices of F^* in the transitive tournament replacing x would be missing from $(F^*)^-$ (the vertex u of F^* can be chosen arbitrarily). Thus $(F^*)^-$ would be some S^* where S is a proper subgraph of F which has a nonempty bipartite symmetric part, and hence again both the S^*-coloring and the F^*-coloring problems would be NP-complete.

Next we claim that F cannot have a vertex a and arcs ax,ay such that xy is an edge of $H(F)$. If this were the case, then consider the sub-indicator J consisting of two vertices v and w and the arc vw, and let u be the last vertex in the transitive tournament of F^* replacing the vertex a of F. Then the digraph $(F^*)^-$ is missing all the p_a vertices of the tournament replacing a, but contains the symmetric pairs of arcs arising from x and y. Hence $(F^*)^-$ is some S^* where S is a proper subgraph of F which has a nonempty bipartite symmetric part, and we obtain a contradiction as before.

Finally, we claim that F is a symmetric digraph. Otherwise there would be an arc ab in F, such that ba is not an arc of F. Consider the sub-indicator J consisting of three vertices v,s,w and three arcs vs,sw,ws, and let u be the first vertex in the transitive tournament of F^* replacing the vertex a of F. We first observe that all the p_b vertices of F^* replacing b are missing from $(F^*)^-$: Indeed, since there are no arcs from these p_b vertices to the p_a vertices of F^* replacing a, the only way the vertex w of R can map to one of these p_b vertices, say vertex y, is if there are in F some arcs ux, xy, yx, contradicting the preceding claim. Now we recall that each vertex of F is incident with an edge of $H(F)$; thus there are in F some arcs ac, ca. It follows that $(F^*)^-$ contains the symmetric pairs of arcs arising from the tournaments replacing a and c. This once again contradicts the minimality of F.

Since F is a bipartite symmetric digraph, the core of F must be $\vec{C}_2$, and we only need to consider $F = \vec{C}_2$. In this case F^*-coloring is NP-complete by the same argument as given in the proof of Proposition 2.2. One only needs to note that in any coloring of the digraph D with two colors, each monochromatic set of vertices is not only acyclic, it is a disjoint union of isolated arcs. This means that F^*, with its at least two vertices in a transitive tournament replacing each vertex, has the property that the hypergraph X admits a 2-coloring if and only if the digraph D has an F^*-coloring. Therefore F^*-coloring is NP-complete. □

This result verifies a special case of Conjecture 5.1 in [1], and of Conjecture 6.1 in [2]. In particular, Conjecture 6.1 of [2] states that, for connected digraphs F which have all indegrees and all outdegrees at least one, F-coloring is NP-complete unless the core of F is $\vec{C}_k$ for some integer k (in which case it is known to be polynomial time solvable).

Note that we do not know what the complexity of F^*-coloring is when F is acyclic. Certainly, the problem can be polynomial time solvable: For instance, if F is a transitive tournament, then F^* is also a transitive tournament, and so D is F^*-colorable if and only if it is acyclic and has height no greater than $|V(F^*)|$ (the height of F^*.) Similarly, the problem can be NP-complete: For instance, there are acyclic triangle-free digraphs F (even oriented trees F [12]) such that F-coloring is NP-complete. Then F^p-coloring is also NP-complete, since an input digraph D is F-colorable if and only if D^p is F^p-colorable. One only needs to notice that the fact that F is triangle-free implies that the $2p$-vertex tournaments of D^p corresponding to edges of D must map to the $2p$-vertex tournaments of F^p corresponding to the edges of F.

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