Interval Graphs, Adjusted Interval Digraphs, and Reflexive List Homomorphisms

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Abstract

Interval graphs admit linear time recognition algorithms and have several elegant forbidden structure characterizations. Interval digraphs can also be recognized in polynomial time and admit a characterization in terms of incidence matrices. Nevertheless, they do not have a known forbidden structure characterization or low-degree polynomial time recognition algorithm.

We introduce a new class of 'adjusted interval digraphs'. By contrast, for these digraphs we exhibit a natural forbidden structure characterization, in terms of a novel structure we call an 'invertible pair'. Our characterization yields an easy recognition algorithm of adjusted interval digraphs.

It turns out invertible pairs are also useful for undirected interval graphs, and our result yields a new forbidden structure characterization of interval graphs. In fact, it can be shown to be a natural link proving the equivalence of some known characterizations of interval graphs - the theorems of Lekkerkerker and Boland, and of Fulkerson and Gross. As a consequence, we derive both of these theorems from our results.

In addition, adjusted interval digraphs naturally arise in the context of list homomorphism problems LHOM(H). If H is a reflexive undirected graph, the problem LHOM(H) is polynomial if H is an interval graph, and NP-complete otherwise. If H is a reflexive digraph, LHOM(H) is polynomial if H is an adjusted interval graph, and we conjecture it is also NP-complete otherwise. We show that our results imply the conjecture in two important cases.

1 Introduction

This is a the full journal version of the conference note [11]. We include all proofs and provide additional connections and applications.

An *interval graph* [13] is a graph H which admits an *interval representation*, i.e., a family of intervals $I_v, v \in V(H)$, such that $uv \in E(H)$ if and only if I_u and I_v intersect. A digraph analogue has been defined in [27] - an *interval digraph* is a digraph H which admits an *interval pair representation*, i.e., a family of pairs of intervals $I_v, J_v, v \in V(H)$, such that $uv \in E(H)$ if and only if I_u intersects J_v. Interval graphs admit elegant characterizations [24, 12], cf. [13], and linear time recognition algorithms [1, 15, 4]. By contrast, the class of interval digraphs so far lacks comparable simple forbidden structure characterizations, and the best algorithm for their recognition to date is a dynamic programming algorithm of complexity $O(nm^6(n+m)\log n)$ [25]. Motivated by the study of list homomorphisms (as explained below), we introduce a new digraph analogue of interval graphs, and argue that it has much nicer properties than the usual interval digraphs. Indeed, we will prove a simple forbidden structure characterization, which yields a polynomial time recognition algorithm.

An *adjusted interval digraph* is an interval digraph H that admits an interval pair representation $I_v, J_v, v \in V(H)$, in which the intervals I_v and J_v have the same left endpoint. Note that the definition of an interval graph implies that an interval graph is *reflexive* (each vertex has a loop). Interval digraphs in the classical sense may lack loops. (If the intervals I_v, J_v are disjoint there is no loop at v.) However, an adjusted interval digraph must again be reflexive. In [5] we studied the special case of adjusted interval digraphs H representable by intervals $I_v, J_v, v \in V(H)$, in which each interval J_v is just one point. These are called *chronological interval digraphs* [5], and we have shown that they can be characterized by the absence of certain special forbidden structures. In [26], a related class of *interval catch digraphs* has been characterized by the absence of certain other forbidden structures.

Here we provide a forbidden structure characterization of adjusted interval digraphs. The forbidden structure is described in terms of a novel mechanism of "invertible pairs". Although invertible pairs may appear technical at first, we demonstrate they are a natural technique for describing obstructions to interval graphs and digraphs. In particular, we derive a characterization of undirected interval graphs in terms of invertible pairs, and exhibit its equivalence with other well known characterizations of interval graphs, in terms of induced cycles and asteroidal triples [24], or in terms of a consecutive clique enumerations [12]. As a consequence, we note that our results imply both the theorem of Lekkerkerker and Boland [24] and the theorem of Fulkerson and Gross [12].

The presence of invertible pairs can be detected by an obvious simple algorithm implied by the definition. Thus our characterization directly implies a simple polynomial time recognition algorithm for the class of adjusted interval digraphs.

Each digraph H is associated with two related undirected graphs. We denote by $U(H)$ the *underlying graph* of H, which has an edge uv whenever $u \neq v$ and $uv \in E(H)$ or $vu \in E(H)$, and by $S(H)$ the *symmetric graph* of H, which has an edge uv whenever $u \neq v$ and $uv \in E(H)$ and $vu \in E(H)$. Note that the loops of H, if any, are removed from both $U(H)$ and $S(H)$.

Adjusted interval digraphs are also motivated by the study of list homomorphisms. A *homomorphism* f of a digraph G to a digraph H is a mapping $f : V(G) \to V(H)$ in which $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$ [21]. If $L(v), v \in V(G)$, are *lists* (subsets of $V(H)$), then a *list homomorphism* of G to H (with respect to the lists L) is a homomorphism satisfying $f(v) \in L(v)$ for all $v \in V(G)$. The *list homomorphism problem* LHOM(H) asks whether or not an input digraph G equipped with lists L admits a list homomorphism $f : G \to H$ with respect to L. The complexity of the list homomorphism problem LHOM(H) for undirected graphs H has been classified in [6, 7, 8].

Of particular interest for this paper is the classification in the special case of reflexive graphs.

Theorem 1.1 *[6] Let H be a reflexive graph.*

If H is an interval graph, then the problem LHOM(H) is polynomial time solvable.

Otherwise, the problem LHOM(H) is NP-complete.

The complexity of LHOM(H) for general relational structures (including digraphs) H has been classified in [2]. In the special case of digraphs a new forbidden structure characterization is given in [19]. This result also yields a simplified useful form of Bulatov's characterization for digraphs, Theorem 5.1.

For reflexive digraphs H, we believe LHOM(H) is polynomial precisely when H is an adjusted interval digraph. Specifically, we observe that each adjusted interval digraph H has polynomial time solvable LHOM(H), and conjecture that for any other reflexive digraph H the problem LHOM(H) is NP-complete. (This is an equivalent form of a conjecture from [10, 17].)

We observe that it suffices to verify the conjecture for digraphs whose underlying graphs are interval graphs. Then we proceed to verify it for digraphs whose underlying graphs are complete graphs and trees; these graphs can be viewed as the building blocks of interval graphs.

Thus it appears that in the context of list homomorphisms, adjusted interval digraphs H play the same role for reflexive digraphs as interval graphs H play for reflexive graphs - namely they exactly identify the tractable cases of LHOM(H).

2 Invertible Pairs

If u, v form an edge of the digraph H, we call uv is a *forward edge* if $uv \in E(H)$; a *backward edge* if $vu \in E(H)$; and a *double edge* if it is both a forward edge and a backward edge. We also say that a forward edge which is not double is a *single forward edge*, and similarly for a *single backward edge*. Since a loop is both a forward edge and a backward edge, we consider it a double edge. If $uv \in E(H)$, we say that u *dominates* v (and that v *is dominated by* u) in H, regardless of whether the forward edge uv is single or double.

We define two walks $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ in H to be *congruent*, if they follow the same pattern of forward and backward edges, i.e., if $x_i x_{i+1}$ is a forward (backward) edge if and only if $y_i y_{i+1}$ is a forward (backward) edge, respectively. If P and Q as above are fcongruent walks, we say that P *avoids* Q, if there is no edge $x_i y_{i+1}$ in the same direction (forward or backward) as $x_i x_{i+1}$.

An *invertible pair* in H is a pair of vertices u, v such that

- there exist congruent walks P from u to v and Q from v to u such that P avoids Q,
- there exist congruent walks P' from v to u and Q' from u to v such that P' avoids Q'.

Note that it is possible that P' is the inverse of P and Q' is the inverse of Q, as long as both P avoids Q and Q avoids P. (The *inverse* of a walk $P = x_0, x_1, \ldots, x_n$ is the walk $x_n, x_{n-1}, \ldots, x_0$.)

It will turn out to be useful to reformulate these definitions in terms of an auxiliary digraph. The *pair digraph* H^+ associated with H has vertices $V(H^+) = \{(u, v) : u \neq v\}$, and edges $(u, v)(u', v')$, where

$$uu', vv' \in E(H) \text{ and } uv' \notin E(H), \text{ or}$$

$$u'u, v'v \in E(H) \text{ and } v'u \notin E(H).$$

We note that a directed walk in H^+ from (u, v) to (v, u) yields two congruent walks P, from u to v, and Q, from v to u, such that P avoids Q; and conversely such walks P and Q yield a directed walk from (u, v) to (v, u) in H^+.

Lemma 2.1 *If u, v is an invertible pair in H, then (u, v) and (v, u) belong to the same strong component C of the pair digraph H^+. Moreover, for any (x, y) in C, the reversed pair (y, x) also belongs to C, thus each pair (x, y) in C is invertible.*

If H has no invertible pair, then for each strong component C of H^+ there exists a reversed strong component *C' such that $(x, y) \in C$ if and only if $(y, x) \in C'$.*

Proof. These properties follow from the definition of a strong component and the observation that $(u, v)(u'v') \in E(H^+)$ implies $(v', u')(v, u) \in E(H^+)$. For instance, if $(u, v), (v, u), (x, y) \in C$, then the directed closed walk containing $(u, v), (x, y)$ yields by reversal a directed closed walk containing $(v, u), (y, x)$, and by concatenation with the directed closed walk containing $(u, v), (v, u)$, we obtain a directed closed walk containing $(x, y), (y, x)$. □

We now illustrate the concept of invertible pairs. Consider first the directed four-cycle $01, 12, 23, 30$. Here $0, 2$ is an invertible pair, as we have congruent walks 012 and 230 which avoid each other. They correspond to the closed directed walk $(0,2), (1,3), (2,0), (3,1), (0,2)$ in H^+.

For a more complex example, consider the reflexive tree T_2 from Figure 7. (We denote the middle vertex c, so the edges of T_2 are $aa, a'a', bb, b'b', cc, ac, a'c, cb, cb'$.) The pair a, a' is an invertible pair. Indeed, in H^+, (a, a') dominates (c, a'), which in turn dominates (b, a'), which dominates (b, c), which finally dominates (b, b'). By the same token, according to the definition of H^+, we also have that (b, b') dominates (c, b'), since $cb, b'b'$ are edges of H but $b'b$ is not. Similarly, (c, b') dominates (a', b'), which dominates (a', c), which dominates (a', a). We have obtained a directed walk $(a, a'), (c, a'), (b, a'), (b, c), (b, b'), (c, b'), (a', b'), (a', c), (a', a)$ in H^+, which corresponds to the two congruent walks $a, c, b, b, b, c, a', a', a'$ and $a', a', a', c, b', b', b', c, a$ in H, where the first walk avoids the second one. By symmetry, we also have the walk $(a', a), (c, a), (b', a), (b', c), (b', b), (c, b), (a, b), (a, c), (a, a')$ in H^+.

As a last example, consider the undirected reflexive four-cycle $0, 1, 2, 3$. We view an undirected graph as a symmetric digraph, with each undirected edge $xy, x \neq y$, replaced by the double edge xy, yx. Thus the reflexive four-cycle has the edges $00, 11, 22, 33, 01, 10, 12, 21, 23, 32, 30, 03$. In this example, the pair $0, 2$ is again invertible, but the closed walk $(0,2), (1,3), (2,0), (3,1), (0,2)$ used for the directed version does not qualify, since now, e.g., $03 \in E(H)$. Nevertheless, H^+ contains the closed walk $(0,2), (1,2), (1,3), (2,3), (2,0), (3,0), (3,1), (0,1), (0,2)$.

A linear ordering $<$ of the vertices of H is a *min ordering* of H if it satisfies the following property: if $uv \in E(H)$ and $u'v' \in E(H)$, then $\min(u, u') \min(v, v') \in E(H)$. (A min ordering was also called an *X-underbar enumeration* [14, 21]).

In the case of reflexive digraphs, there is an equivalent simpler definition of a min ordering.

Lemma 2.2 *Let H be a reflexive digraph. Then a linear ordering $<$ of $V(H)$ is a min ordering if and only if for any three vertices $i < j < k$ we have*

- *$ik \in E(H)$ implies $ij \in E(H)$, and*
- *$ki \in E(H)$ implies $ji \in E(H)$.*

Proof. The necessity of the two properties follows by taking the edge ik (respectively ki) and the loop at j. To see the sufficiency, consider edges $xy, x'y'$ of H and assume without loss of generality that $x < x', y' < y$; thus $\min(x, x') \min(y, y') = xy'$. If $x = y'$, then xy' is an edge since H is reflexive. If $x < y'$, then xy' is an edge because of the triple $x < y' < y$. If $y' < x$, then xy' is an edge because of the triple $y' < x < x'$. □

Corollary 2.3 *Let H be a reflexive digraph. An linear ordering of the vertices of H is a min ordering if and only if for each vertex v the vertices that follow v in the ordering consist of*

1. *first, all vertices that are adjacent to v by double edges,*
2. *second, all vertices that are adjacent to v by single edges, either all forward or all backward, and*
3. *last, all vertices that have no edges to or from v.*

Of course, any of the three groups could be empty. Note that, in particular, in a min ordering of H it cannot be the case that a vertex v has both single forward and single backward edges towards vertices that follow it in the ordering.

The following result relates min orderings to adjusted interval digraphs.

Theorem 2.4 *A reflexive digraph is an adjusted interval digraph if and only if it admits a min ordering.*

It is interesting to note that a reflexive undirected graph H has a min ordering if and only if it is an interval graph [6]. Thus Theorem 2.4 provides additional motivation in favour of adjusted interval digraphs.

Proof. Given a min ordering of a reflexive digraph H, we can arrange the common starting points of I_v, J_v in the same order as the vertices v of H appear in the min ordering, and define intervals I_v and J_v as follows. The interval I_v ends at the last vertex w such that vw is a forward edge, and the interval J_v ends at the last vertex such that vw is a backward edge (i.e., wv is an edge of H). It is clear that H is the interval digraph corresponding to the adjusted interval representation $I_v, J_v, v \in V(H)$. Conversely, given an adjusted interval pair representation $I_v, J_v, v \in V(H)$ we obtain a min ordering of H according to the left to right order of the common left endpoints of the intervals. □

According to Corollary 2.3, if v has no single forward edges towards later vertices, the interval I_v ends at the last vertex w such that vw is a double edge, and end the interval J_v at the last vertex w such that vw is a backward edge. (Similarly, if v has no backward edges towards later vertices.)

Min orderings also play an important role for list homomorphism problems, cf. [21].

Theorem 2.5 *[14] If H admits a min ordering, then the problem LHOM(H) is polynomial time solvable.*

Finally, we observe that an invertible pair is an obstruction to the existence of a min ordering.

Lemma 2.6 *If H has an invertible pair, then H does not admit a min ordering.*

Proof. Suppose $(u, v)(u', v')$ is an edge of the pair digraph H^+. Suppose $<$ is a min ordering of H, and suppose $u < v$. The we must also have $u' < v'$. Following the directed closed walk in H^+ which contains (u, v) and (v, u), we obtain a contradiction. □

3 Adjusted Interval Digraphs

In this section we give our new forbidden structure characterization of adjusted interval digraphs. This is the main result of our paper.

Theorem 3.1 *A reflexive digraph H is an adjusted interval digraph if and only if it has no invertible pair.*

In fact, we shall prove the following stronger result.

Theorem 3.2 *The following statements are equivalent for a reflexive digraph H:*

1. *H is an adjusted interval digraph*
2. *H has a min ordering*
3. *H has no invertible pairs*
4. *The vertices of H^+ can be partitioned into sets D, D' such that*
   - *$(x, y) \in D$ if and only if $(y, x) \in D'$*
   - *$(x, y) \in D$ and (x, y) dominates (x', y') in H^+ implies $(x', y') \in D$*
   - *$(x, y), (y, z) \in D$ implies $(x, z) \in D$.*

Proof. It suffices to assume that $U(H)$ is a connected graph.

The equivalence of 1 and 2 is proved in Theorem 2.4. Furthermore, Lemma 2.6 shows that 2 implies 3. It is also quite straightforward to see that 4 implies 2; it suffices to define $x < y$ if $(x, y) \in D$. Thus it remains to show that 3 implies 4.

Therefore, we assume that H has no invertible pair. Note that we may assume that H is weakly connected, otherwise we can order each weak component separately. Recall that for each strong component C of H^+, there is a corresponding reversed strong component C' whose pairs are precisely the reversed pairs of the pairs in C; we shall say that C, C' are *coupled* strong components. Note that a strong component C may be coupled with itself - Lemma 2.1 implies that invertible pairs lie in *self-coupled* components.

The partition of $V(H^+)$ into D, D' will correspond to separating each pair of coupled strong components C, C' of H^+. The vertices of one strong components will be placed in the set D, their reversed pairs will go to D'. We wish to make these choices in such a way as to avoid creating a *circular chain* in D, i.e., a sequence of pairs $(x_0, x_1), (x_1, x_2), \ldots,$ $(x_n, x_0) \in D$.

We shall proceed as follows. Initially the sets D and D' are empty. We say that a strong component C of H^+ is *ripe* when it has no edge *to* another strong component in $H^+ - D$. In the general step, we shall take a ripe component C and place it in D, and simultaneously place C' in D'. (Note that C' need not be ripe, but has no edge *from* another strong component.) We will show that there is always at least one ripe strong component which can be added to D without creating a circular chain.

The sets D, D' will always have the following properties (which are true initially). There is no circular chain in D; each strong component of H^+ belongs entirely to D, D', or to $V(H^+) - D - D'$; the pairs in D' are precisely the reversed pairs of the pairs in D; there is no edge of H^+ from D to a vertex outside of D; and there is no edge of H^+ from a vertex outside of D' to a vertex in D'. At the end of the algorithm each pair (x, y) with $x \neq y$ will belong either to D or to D', and hence the final D will have no circular chain and hence satisfy the transitivity property of 4.

We now prove that the algorithm maintains these properties.

Suppose, for a contradiction, that the current D has no circular chain but the addition of C to D creates a circular chain in $C \cup D$. Suppose $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ is a circular chain that has occurred for the first time during the execution of the algorithm, and also suppose that at that time no shorter circular chain has occurred. Since there are no invertible pairs, and since we never place both a pair and its reverse in D, we must have $n \geq 2$. We may assume without loss of generality that $(x_n, x_0) \in C$; note that other pairs of the circular chain could also be in C.

Case 1. Assume that in H, there is at least one edge between the vertices $x_0, x_1, \ldots, x_n$, say an edge $x_a x_b$.

We claim that this implies that H is complete on $x_0, x_1, \ldots, x_n$. We make the following elementary observations, assuming $j \neq i$.

1. If x_j dominates x_i then x_{j-1} dominates x_i in H.
2. If x_j dominates x_i then x_j dominates x_{i-1} in H.

To prove the first observation, we note that if x_j dominates x_i but x_{j-1} not dominate x_i in H, then (x_{j-1}, x_j) dominates (x_{j-1}, x_i) in H^+. Since (x_{j-1}, x_j) is in $C \cup D$, the pair (x_{j-1}, x_i) must belong to $C \cup D$, implying a shorter circular chain in $C \cup D$.

To prove the second observation, we similarly note that if x_j dominates x_i but x_j does not dominate x_{i-1} in H, then (x_{i-1}, x_i) dominates (x_{i-1}, x_j) in H^+, also implying a shorter circular chain.

Consider now the fact that x_a dominates x_b in H. Property 1 implies that x_{a-1}, $x_{a-2}, \ldots,$ x_{b+1} all dominate x_b. Since x_{b+1} dominates x_b, property 2 implies that x_{b+1} dominates x_{b-1}, $x_{b-2}, \ldots,$ x_{b+2}, i.e., dominates all other vertices. At this point we use 1 again to derive that x_b dominates x_{b-1}, and repeated application of 2 as before implies that x_b dominates all other vertices. Continuing this way, we see that each x_j dominates all other vertices, i.e., the vertices x_0, $x_1, \ldots,$ x_n induce a complete graph in H.

We conclude the proof of Case 1 by showing that C is a trivial component (with a single vertex). If C has more than one vertex, then so does its corresponding coupled component C', which contains the vertex (x_0, x_n). Hence we assume for contradiction that (x_0, x_n) dominates some (a, b) not in $C \cup D$.

Up to symmetry, we may assume that x_0 dominates a in H, x_n dominates b in H and x_0 does not dominate b in H. Since (a, b) is not in $C \cup D$, the pair (x_0, x_1), which is in $C \cup D$, cannot dominate (a, b), which implies that x_1 does not dominate b in H. If x_2 dominates b in H, then (x_1, x_2) dominates (x_0, b) which dominates (a, b) in H^+; this is impossible, as this is a directed path starting in C and ending outside of $C \cup D$, so some edge would exit from $C \cup D$ against the rules we maintain. Therefore x_2 does not dominate b in H; if x_3 dominates b in H, then (x_2, x_3) dominates (x_1, b) which dominates (x_0, b) which dominates (a, b), yielding the same contradiction. Therefore x_3 does not dominate b in H, and continuing this way we would derive that x_n does not dominate b, which is false.

Thus we have $C = \{(x_n, x_0)\}, C' = \{(x_0, x_n)\}$. The same proof also shows that C' is ripe, as no (a, b) dominated by (x_0, x_n) can exist outside of $C \cup D$. It is now easy to see that if both (x_n, x_0) and (x_0, x_n) complete a circular chain with D, then D already had a circular chain.

Case 2. Assume that vertices $x_0, x_1, \ldots, x_n$ are independent in H.

Lemma 3.3 *Let $x_0, x_1, \ldots, x_n$ be independent in H.*

Suppose p is a vertex of H, distinct from $x_0, x_1, \ldots, x_n$, which dominates x_{i+1} but not x_i (or which is dominated by x_{i+1} but not by x_i).

Then $(x_0, x_1), \ldots, (x_i, p), (p, x_{i+2}), \ldots, (x_n, x_0)$ is also a circular chain created at the same time.

Proof. We conclude from the assumption that (x_i, x_{i+1}) dominates (x_i, p) in H^+, and since (x_i, x_{i+1}) is in $C \cup D$, we must also have (x_i, p) in $C \cup D$. Furthermore, since x_{i+1} does not dominate or is dominated by x_{i+2} in H, we also have (x_{i+1}, x_{i+2}) dominating (p, x_{i+2}), whence (p, x_{i+2}) is in $C \cup D$. In conclusion, we see that any such vertex p can replace x_{i+1} in the circular chain $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$. □

Lemma 3.4 *Let vertices $x_0, x_1, \ldots, x_n$ be independent in H.*

- *If p is a vertex of H, distinct from $x_0, x_1, \ldots, x_n$, which dominates x_j and x_k with $j \neq k$, then p dominates each x_i.*
- *If p is a vertex of H, distinct from $x_0, x_1, \ldots, x_n$, which is dominated by x_j and x_k with $j \neq k$, then p is dominated by each x_i.*
- *If p, distinct from $x_0, x_1, \ldots, x_n$, dominates x_j and is dominated by x_k with $j \neq k$, then p both dominates and is dominated by each $x_i, i \neq j, k$.*

Proof. If p dominates x_{i+1} but not x_i, then Lemma 3.3 implies that p can replace x_{i+1} in the circular chain; however at least one of x_j, x_k is not equal to x_{i+1}, whence the vertices of the chain are not independent and we conclude by Case 1. The other items are proved similarly. □

We now claim that the circular chain $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ has at most one pair, say (x_n, x_0), in C (with all other pairs in D). Otherwise, assume some $(x_i, x_{i+1}), i \neq n$ is also in the strong component C, and let P be a directed path in C from (x_n, x_0) to (x_i, x_{i+1}). Let the penultimate pair on this path be (p, q), and, without loss of generality, assume that $px_i, qx_{i+1} \in E(H), px_{i+1} \notin E(H)$. (In the case $x_i p, x_{i+1} q \in E(H), x_{i+1} p \notin E(H)$, the argument is symmetric.) By Lemma 3.4, p does not dominate any x_j with $j \neq i$. Next we claim that q does not dominate x_i. Indeed, if q dominates x_i, then Lemma 3.4 implies that q dominates all x_j. This is a contradiction, since it would mean that (p, q) dominates (x_i, x_{i+2}) in H^+, implying that (x_i, x_{i+2}) is in $C \cup D$ and thus there is a shorter circular chain in H. Therefore q does not dominate x_i. By a double application of Lemma 3.3, we conclude that we can replace x_i and x_{i+1} by p and q in the circular chain in H. Continuing this way, we replace (p, q) by the previous pair on the path P, until we obtain the pair (p', q') which is the first pair after (x_n, x_0). Since x_0 is adjacent to q', we are back in Case 1.

Thus the circular chain $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ has only the pair (x_n, x_0) in C, and any circular chain in $C \cup D$ has exactly one pair in C. We now suppose, in addition to the previous assumptions, that our circular chain minimizes the sum of the lengths of all distances amongst the vertices $x_0, x_1, \ldots, x_n$, in the underlying graph of H .

The digraph H turns out to have a very special structure. We claim that in this situation there exists a non-empty set K of vertices of H such that $H \setminus K$ has weak components $C_1, C_2, \ldots C_m$, where $x_i \in C_i, i = 1, 2, \ldots, n$, and such that if $p \in K$ dominates (respectively is dominated by) a vertex in C_i, then p dominates (respectively is dominated by) all vertices in C_i; moreover, if $x'_0,\ x'_1,\ \ldots, x'_n$ are any vertices with $x'_i \in C_i$, then $(x'_0, x'_1),\ (x'_1, x'_2), \ldots, (x'_n, x'_0)$ is also a circular chain.

Indeed, we let K consist of all vertices of H that dominate each x_i, or are dominated by each x_i. It is easy to see that K must be non-empty, as Lemma 3.4 implies that any p dominated by $x_j, x_k, j \neq k$ belongs to K. Such a p must exist by our new minimality assumption, as otherwise we could replace x_j by its neighbour p on a path joining x_j to x_k by Lemma 3.3.

The same argument shows that two different x_j, x_k cannot lie in the same weak component C_i of $H \setminus K$, as any path joining x_j to x_k was shown to contain a vertex of K. Therefore we can number the components so that C_i contains x_i for $i = 1, 2, \ldots n$. (There may be additional components C_i with $i = n+1, \ldots, m$.) Now Lemma 3.3 implies that each x_i can be replaced by any neighbour in C_i, thus any vertex of C_i can be taken as x_i. Thus each $p \in K$ that dominates a vertex in C_i also dominates all vertices in C_i, and similarly for vertices p dominated by a vertex in C_i.

This creates a situation where any pair (y, y') in the strong component C of H^+ containing (x_n, x_0) must satisfy $y \in C_n, y' \in C_0$. This easily implies that the strong component C does not have any edges entering it from the outside, and hence the strong component C' coupled with C is also ripe. We claim that C' cannot complete a circular chain with D. Otherwise, the pair (x_0, x_n) would also complete a circular chain by the same argument. Thus both (x_0, x_n) and (x_n, x_0) complete a circular chain with D, whence D must already contain a circular chain, a contradiction.

Of course, if the addition of C' does not create a circular chain, then we add C' to D and C to D'. □

This gives us a polynomially verifiable forbidden subgraph characterization of adjusted interval digraphs. As noted above, checking for invertible pairs amounts to computing the strong components of H^+ and checking for the existence of a pair $(u, v), (v, u)$ within one strong component. Thus the recognition of adjusted interval digraphs is polynomial - one can, for instance, explicitly construct the pair digraph H^+ in time $O(m^2 + n^2)$ and test it for invertible pairs in the same time. There may, however, be more efficient ways.

We ask the following questions.

1. Is there a linear time recognition algorithm for adjusted interval digraphs?
2. Are there natural intractable digraph problems that can be solved in polynomial time on the class of adjusted interval digraphs?

In the undirected case of interval graphs, the answer to both questions is yes [13].

4 Interval Graphs

As we noted above, an undirected graph can be viewed as a digraph in which each indirected edge uv is replaced by the directed edges uv, vu. Equivalently, the definitions of invertible pair, min ordering, etc., can be read as written above, but interpreting edges as undirected pairs. Congruent walks become walks of equal length. Note however that H^+ remains a digraph.

Theorem 4.1 *A reflexive graph is an interval graph if and only if it has no invertible pairs.*

Proof. A reflexive graph has a min ordering if and only if it is an interval graph [6]. An invertible pair is an obstruction to having a min ordering, according to Lemma 2.6. For a graph H without invertible pairs, viewing H as a digraph, Theorem 3.2 implies it has a min ordering, whence it must be an interval graph. □

Theorem 4.1 offers a nice link between two of the best known classical characterizations of interval graphs. An *asteroidal triple* in a graph H is a triple of vertices, such that any two are joined by a path that is disjoint from the neighbourhood of the third vertex. An enumeration of the maximal cliques of H is called a *consecutive clique enumeration*, if the cliques containing any particular vertex are consecutive in the enumeration.

Theorem 4.2 *The following statements are equivalent for a reflexive graph H*

1. *H has no asteroidal triple or a chordless cycle $C_k, k > 3$*
2. *H has a consecutive clique enumeration*
3. *H has no invertible pair.*

Proof. An elegant proof of the fact that 1 implies 2 is given in [16], and we do not reprove it here.

To see that 2 implies 3, consider a consecutive clique enumeration $K^1, K^2, \ldots, K^m$ of H, and an invertible pair u, v in H, with its associated walks P, Q, P', Q' (from the definition of invertible pair). Say, P is the walk $u = u_0, u_1, u_2, \ldots, u_k = v$; Q is the walk $v = v_0, v_1, v_2, \ldots, v_k = u$; P' is the walk $v = u_k, u_{k+1}, \ldots, u_{k+k'} = u$; and Q' is the walk $u = v_k, v_{k+1}, \ldots, v_{k+k'} = v$. Let $s = s(x, y)$ denote the superscript of a maximal clique K^s which contains the edge xy. Suppose without loss of generality that $s(u_0, u_1) < s(v_0, v_1)$.

Then we must have $s(u_0, u_1) < s(v_1, v_2)$, since otherwise $s(v_1, v_2) < s(u_0, u_1) < s(v_0, v_1)$, with v_1 in the first and the third clique, but not the second, which is impossible. (Recall that $u_0 v_1 \notin E(H)$.) A similar argument implies that $s(u_1, u_2) < s(v_1, v_2)$, and, continuing in this vein along the paths P, Q, we obtain $s(u_k, u_{k+1}) < s(v_k, v_{k+1})$. Comparing $s(u_0, u_1) < s(v_0, v_1)$ and $s(u_k, u_{k+1}) < s(v_k, v_{k+1})$, we observe that $u = u_0 = v_k$ lies in the cliques with subscripts $s(u_0, u_1), s(v_k, v_{k+1})$, but not $s(v_0, v_1)$, implying that $s(v_k, v_{k+1}) < s(v_0, v_1)$. Thus we were able to derive $s(v_k, v_{k+1}) < s(v_0, v_1)$ from $s(u_0, u_1) < s(v_0, v_1)$ along P, Q. Similarly, $s(u_k, u_{k+1}) < s(v_k, v_{k+1})$ implies that $s(v_0, v_1) < s(v_k, v_{k+1})$ which is a contradiction; therefore, 2 implies 3.

To see that 3 implies 1, suppose that u, v, w is an asteroidal triple. Let, for $\{x, y, z\} = \{u, v, w\}$, $P(x, y)$ denotes a path joining x and y which does not contain a neighbour of z, and let $\ell(x, y)$ be the length of $P(x, y)$. We will show that u, v form an invertible pair. Indeed, let P be the walk consisting of $u, u, \ldots, u$, of length $\ell(v, w)$, concatenated with $P(u, v)$, and concatenated with the walk $v, v, \ldots, v$, of length $\ell(w, u)$, and let Q be the path $P(v, w)$, concatenated with the walk $w, w, \ldots, w$ of length $\ell(u, v)$, and concatenated with the path $P(w, u)$. It is easy to see that P avoids Q and Q avoids P, hence u, v form an invertible pair. Since an induced cycle of length six or more contains an asteroidal triple, it remains to consider only the cycles C_4, C_5, in which case an invertible pair is easily constructed along the same lines. □

Note that the fact that H is reflexive is relevant for testing for invertible pairs, but irrelevant for asteroidal triples, chordless cycles, or consecutive clique enumerations. Thus Theorems 4.1 and 4.2 imply the two best known characterizations of interval graphs, due to Lekkerkerker and Boland [24], and to Fulkerson and Gross [12].

Corollary 4.3 *Let H be a graph.*

- *H is an interval graph if and only if it has no asteroidal triple and no chordless cycle $C_k, k > 3$.*
- *H is an interval graph if and only if it has a consecutive clique enumeration.*

5 Polymorphisms and the List Homomorphism Problem

The min orderings defined above are a particular case of the following general concept. Let k be a positive integer. The *k-th power* of H is the digraph H^k with vertex set $V(H)^k$ in which $(u_1, u_2, \ldots, u_k)(v_1, v_2, \ldots, v_k)$ is an edge just if each $u_i v_i$ is an edge of H. A *polymorphism of order k* is a homomorphism of H^k to H. A polymorphism f is *conservative* if $f(u_1, u_2, \ldots, u_k)$ always is one of $u_1, u_2, \ldots, u_k$. From now on we shall use the word *polymorphism* to mean a *conservative polymorphism*.

A polymorphism f of order two is *commutative* if $f(u,v) = f(v,u)$ for any u,v. If H admits a min ordering $<$, then clearly defining $f(u,v) = \min(u,v)$ is a polymorphism, which is commutative.

A polymorphism $f : H^3 \to H$ is called a *majority polymorphism* if $f(u,u,v) = f(u,v,u) = f(v,u,u) = u$ for any u,v. A ternary polymorphism $f : H^3 \to H$ is *majority over* a,b, if $f(a,a,b) = f(a,b,a) = f(b,a,a) = a$, $f(b,b,a) = f(b,a,b) = f(a,b,b) = b$.

At this point, we can state our classification of LHOM(H) from [19] (a simplification of the result from [2]). Recall that by our definition each polymorphism is conservative.

Theorem 5.1 *[19] Let H be a reflexive digraph.*

If for every pair of vertices a,b of H there exists a polymorphism of H which either is ternary and majority over a,b, or is binary and commutative over a,b, then LHOM(H) is polynomial time solvable.

Otherwise, if some pair of vertices a,b does not admit either of these polymorphisms, then the problem LHOM(H) is NP-complete.

The following fact follows directly from Theorems 2.5 (or Theorem 5.1) and 2.4.

Theorem 5.2 *If H is an adjusted interval digraph, then LHOM(H) is polynomial time solvable.*

We conjecture that the converse also holds. This is an equivalent form of a conjecture from [10, 17].

Conjecture 5.3 *If a reflexive digraph H is not an adjusted interval digraph, then LHOM(H) is NP-complete.*

We also had a similar conjecture for irreflexive digraphs [10, 17]. However, that conjecture has turned out to be false [18, 3], and we shall discuss the case of irreflexive digraphs in [18].

We are currently working, with C. Carvalho, on an algebraic approach to Conjecture 5.3, [3].

6 Intractable cases

Here we collect some available information about known intractable cases of LHOM(H) for reflexive digraphs H.

For the directed reflexive three-cycle $\vec{C}_3$, the problem LHOM($\vec{C}_3$) is shown NP-complete in [10]. (All other problems LHOM(H) for reflexive digraphs with up to three vertices are known to be polynomial time solvable [10]. The NP-completeness of LHOM($\vec{C}_3$) also follows from Theorem 7.1 and the remark following it.) Thus LHOM(H) is NP-complete for any digraph containing an induced reflexive directed three cycle. (LHOM($\vec{C}_3$) can be reduced to such a problem by using the same lists in LHOM(H) as in LHOM($\vec{C}_3$).)

For longer oriented reflexive cycles, we observe that if the underlying graph $U(H)$ contains a chordless cycle $C_k, k > 3$, then even a very special list homomorphism problem (the so-called "retraction problem" RET(H)) is NP-complete, see [9, 23]. Thus when H contains any reflexive cycle of length greater than three, the problem LHOM(H) is also NP-complete.

In [19] we have defined a digraph asteroidal triple (DAT), and shown that a digraph H (not necessarily reflexive) containing a DAT yields an NP-complete problem LHOM(H). It also turns out that DAT-free digraphs have polynomial time solvable problems LHOM(H) [19].

A *digraph asteroidal triple* (DAT) in a digraph H consists of three vertices u, v, w and three invertible pairs of vertices $s(u), b(u)$, and $s(v), b(v)$, and $s(w), b(w)$, so that for any permutation x, y, z of u, v, w there exist walks P from x to $s(x)$, and Q' from y to $b(x)$ and Q'' from z to $b(x)$, such that P avoids both Q' and Q''. Example DATs can be seen at end of this paper. In particular, it was proved in [19] (Theorem 3.4) that in a reflexive digraph H, an (undirected) asteroidal triple in $U(H)$ yields a DAT in H. We conclude that if $U(H)$ is not an interval graph then LHOM(H) is also NP-complete. Furthermore, if $S(H)$ is not an interval graph then the undirected graph problem LHOM($S(H)$) is NP-complete by [6]. Since an undirected instance G of LHOM($S(H)$) can be viewed as a directed graph with each edge symmetric, this implies that LHOM(H) is also NP-complete.

Thus we may restrict our attention to reflexive digraphs H for which both $S(H)$ and $U(H)$ are interval graphs, and moreover such that H does not contain an induced reflexive three-cycle.

7 Trees and Semi-Complete Digraphs

We now verify Conjecture 5.3 in two important cases. Recall that it suffices to focus on digraphs H such that $U(H)$ is an interval graph. All complete graphs and certain trees (that is, caterpilars) are in a sense the most basic interval graphs, and that is where we turn next.

A digraph is *semi-complete* if its underlying graph is complete. A digraph is a *tree* if its underlying graph is a tree in the usual sense.

Theorem 7.1 *Suppose H is a reflexive semi-complete digraph. If H contains an invertible pair, then LHOM(H) is NP-complete.*

Proof. We will show that if there exist invertible pairs in H, then some invertible pair a, b admits no polymorphism as prescribed by Theorem 5.1.

It turns out that some structures in H limit our choices of polymorphisms from the theorem. The first such structure is an invertible pair; this is easy to see from the definition of an invertible pair, and we state it without proof.

Lemma 7.2 *No binary polymorphism of H can be commutative over an invertible pair.*

Let R be the reflexive digraph $V(R) = \{a, b, c\}$ and $E(R) = \{aa, bb, cc, ab, bc, ac, ca\}$.

Lemma 7.3 *There is no polymorphism g on the digraph R that is a majority over a, b.*

Proof. Suppose g is a polymorphism of R which is a majority over a, b, i.e., $g(a, a, b) = g(a, b, a) = g(b, a, a) = a$, and $g(a, b, b) = g(b, a, b) = g(b, b, a) = b$. We claim that g must also be a majority over b, c. Note that $g(c, c, b)g(a, a, b) = g(c, c, b)a \in E(R)$. Hence $g(c, c, b) = c$, as b does not dominate a in R. Similarly, $g(c, b, c) = g(b, c, c) = c$. Also $g(b, b, c)g(b, b, a) = g(b, b, c)b \in E(R)$ thus $g(b, b, c) = b$ and similarly $g(b, c, b) = g(c, b, b) = b$. Now we can conclude that g is also majority over a, c, using the fact that $g(a, a, c)g(b, b, c) \in E(R)$ and $g(b, b, c)g(c, c, a) \in E(R)$.

Now we note that we have $g(a, b, c)g(b, b, c) = g(a, b, c)b \in E(R)$, which implies that $g(a, b, c) \in \{a, b\}$ (since c doesn't dominate b in R); we have $g(a, b, b)g(a, b, c) \in E(R)$, which similarly implies that $g(a, b, c) \in \{b, c\}$; and we have $g(c, a, c)g(a, b, c) \in E(R)$, which similarly implies that $g(a, b, c) \in \{a, c\}$, which is impossible. □

We now proceed with the proof of Theorem 7.1. Thus assume H has an invertible pair. According to out remark at the end of Section 6, we may assume that H does not contain an induced reflexive three-cycle $\vec{C}_3$, and both $S(H)$ and $U(H)$ are interval graphs; in particular, $S(H)$ does not contain an induced four-cycle. Finally, we may assume that in any copy of R induced in H, the pair corresponding to a, b is not invertible. This directly follows from Lemmas 7.2, 7.3, and Theorem 5.1.

Since H has invertible pairs, the pair digraph H^+ has a self-coupled strong component C. According to Lemma 2.1, all pairs in C are invertible. We first note that if (a, b) dominates (c, d) in C, then (a, b) also dominates (a, d), and (a, d) dominates (c, d), thus we also have $(a, d) \in C$, and a, d is an invertible pair as well.

Consider a closed directed walk $W = (x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), (x_0, y_0)$ in C that contains both $(a, b), (b, a)$ for some $a, b \in V(H)$. All pairs x_i, y_i are invertible, and so are all pairs x_i, y_{i+1} (addition modulo n). In fact, the above argument shows that we may assume each (x_i, y_{i+1}) to belong to W. Recall that for each i, the edge from (x_i, y_i) to $(x_{i+1}, y_{i+1}$ in H^+ is due to the edges $x_i x_{i+1}, y_i y_{i+1}$ in H, which could be forward or backward.

We first assume that for some i we have $x_i x_{i+1}$, $y_i y_{i+1}$ forward and $x_{i+1}x_{i+2}, y_{i+1}y_{i+2}$ backward. Without loss of generality, let us assume $i = 0$, i.e., that $x_0x_1, x_2x_1, y_0y_1, y_2y_1 \in E(H)$ and $x_0y_1, y_2x_1 \notin E(H)$. Since H is semi-complete, we must have $y_1x_0, x_1y_2 \in E(H)$. If $y_1x_1 \notin E(H)$, then y_1, x_0, x_1 are all distinct and $y_1x_1 \in E(H)$, whence either $x_1x_0 \notin E(H)$ yielding an induced $\vec{C}_3$ on y_1, x_0, x_1, or $x_1x_0 \in E(H)$ yielding an induced copy of R on the same vertices, with invertible pair x_1, y_1; both contradict our assumptions. Thus $y_1x_1 \in E(H)$, and by a symmetric argument focused on x_1, y_1, y_2, we also deduce $x_1y_1 \in E(H)$. At this point, the absence of R on x_0, x_1, y_1 implies that x_0x_1 is a double edge, and similarly on x_1, y_1, y_2 we conclude that y_1y_2 is also a double edge. Consider now the pair x_0, y_2: if x_0y_2 is not an edge then we find an induced R on y_1, x_0, y_2, and if y_2x_0 is not an edge, we find an induced R on x_1, y_2, x_0. Therefore, x_0y_2 is also a double edge; this means that $S(H)$ contains an induced four-cycle $x_0x_1y_1y_2x_0$, which contradicts our assumptions.

It remains to consider the case when all edges x_ix_{i+1}, y_iy_{i+1} are forward (or all backward). In this situation we claim that all pairs x_i, y_i form a double edge x_iy_i. (Indeed, if $y_ix_i \notin E(H)$, then $x_iy_i \in E(H)$, and we derive an induced $\vec{C}_3$ or R on x_i, y_i, y_{i+1}; and if $x_iy_i \notin E(H)$, then we consider instead x_jy_j where $x_j = y_i, y_j = x_i$ and proceed similarly.) This is impossible, as we have shown that the pairs (x_i, y_{i+1}) may be assumed to be on the cycle, and they cannot form double edges by assumption. □

Thus the conjecture holds for semi-complete digraphs. We now turn to trees. In this case, we will provide a direct proof, as we are able to describe exactly which trees H yield tractable problems LHOM(H).

It is well known [13] that a tree is an interval graph if and only if it is a *caterpillar*, i.e., the removal all leaves yields a path. Thus we want to decide which orientations of caterpillars yield adjusted interval digraphs. Let $S(x)$ denote the set of leaves of H adjacent to the vertex $x \in P$. As usual, we refer to H as a tree, or star, etc., to mean that $U(H)$ (without the loops) is a tree, or star, etc., respectively.

If H is a star, we shall define H to be a *good caterpillar*, if it does not contain, as induced subgraph, the tree T_2 depicted below. If H is not a star, we define it to be a *good caterpillar* if it has a longest path $P = v_0, v_1, \ldots, v_k, v_{k+1}$ satisfying the following conditions for all i. (Note that $v_1, v_2, \ldots, v_k$ is the path P, and that $v_0 \in S(v_1), v_{k+1} \in S(v_k)$.)

1. If $v_iv_{i+1} \in E(H)$, then $v_iv \in E(H)$ for all $v \in S(v_i) - v_{i-1}$.
2. If $v_{i+1}v_i \in E(H)$, then $vv_i \in E(H)$ for all $v \in S(v_i) - v_{i-1}$.

Note that if v_iv_{i+1} is a double edge then so are all $v_iv, v \in S(v_i) - v_{i-1}$. Observe that there are no restrictions on v_0, other than those arising from the restrictions on v_1. Indeed, all edges v_1v for $v \in S(v_1) - v_0$ must follow the direction of the edge v_1v_2 (forward, backward, or double) - with the possible exception of a single vertex v, which must be the vertex v_0. Thus such a v_0 can be chosen if and only if the restrictions on v_1 have at most one exception. Similarly, there are no restrictions on v_{k+1}, other than those arising from the restrictions on v_k. All edges v_kv for $v \in S(v_k)$ must follow the direction of the edge v_kv_{k+1}. It is easy to see that such a v_{k+1} can be chosen if and only if between v_k and $S(v_k)$ there does not exist at the same time a single forward and a single backward edge. Finally, we note that the exceptional case, when H is a star, also conforms to the general definition; we have chosen to state it separately only for convenience.

Theorem 7.4 *Let H be a reflexive digraph that is a tree. Then the following statements are equivalent.*

1. *H is a good caterpillar*
2. *H is an adjusted interval digraph*
3. *H has no invertible pair*
4. *H does not contain, as an induced subgraph, any of the trees $T_1, \ldots T_7$, or their reverses.*

Proof. 1 implies 2 via Theorem 2.4, as a good caterpillar can be ordered starting from v_0 and proceding to $v_1, v_2, \ldots, v_k$, with listing the double edges of $S(v_i) - v_{i-1}$ first, as suggested by Corollary 2.3. The definition of a good caterpillar ensures that the listing for $S(v_i) - v_{i-1}$ can be chosen to end with v_{i+1}.

2 implies 3 by Theorem 3.1, and 3 implies 4 by inspection. (We have only shown an invertible pair in T_2. We leave it to the reader to find invertible pairs in the other trees.)

Theorem 2.4 also allows us to derive 3 from 2: none of the forbidden subtrees allows a min ordering. To see this, in the trees T_1, T_3, T_4 focus on the vertices $0, 1, 2$, and on the trees T_2, T_5, T_6, T_7 focus on the vertices a, a', b, b'.

It remains to show that 3 implies 1. Thus suppose H is a reflexive tree which does not contain any of $T_1 - T_7$ or their reverses. Since H does not contain T_1, $U(H)$ is a caterpillar. If H is a star, the conclusion now follows. Thus assume H is not a star: when all leaves of H are removed we obtain a path P, say $P = p, r, s, \ldots, y, z$. We will prove that one of p, z can be chosen as v_1 and the other as v_k. Suppose first that p cannot be chosen to satisfy the condition for v_1. Then in $S(p)$ there must be two vertices v, v' such that the edges pv, pv' do not follow the direction of the edge pr on P. If pr is a double edge, this means that pv, pv' are single edges. Since H does not contain T_3, both are forward (or

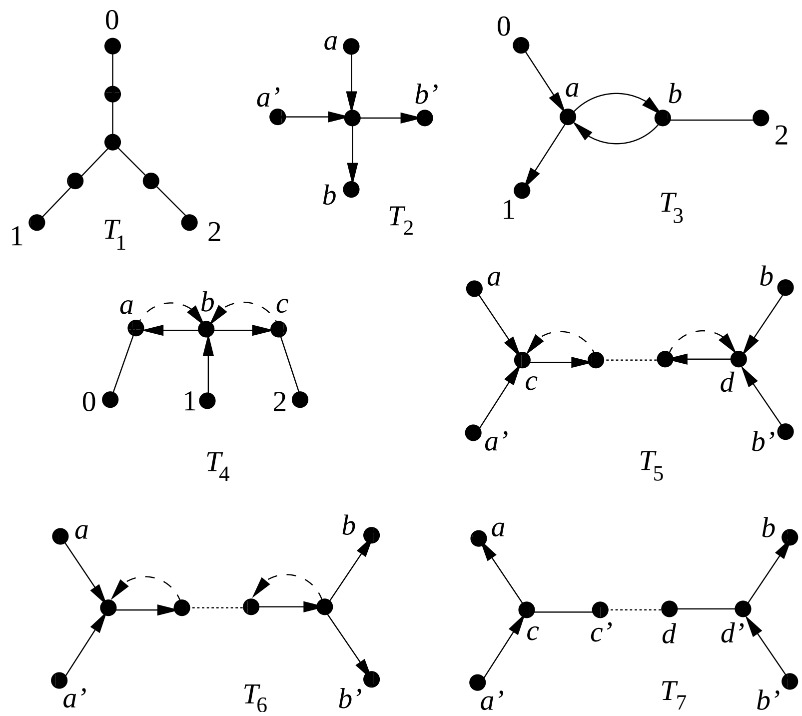

Figure 1: Minimal trees with invertible pairs. (Dashed edges are optional; edges without directions can be forward, backward, or double; dotted lines denote paths; loops are omitted)

both backward) edges. This implies that all edges $pv, v \in S(p)$ follow the direction of pr, and thus p can be chosen to satisfy the condition for v_k. Similarly, if pr is a single (forward or backward edge), p can be chosen as v_k, since H does not contain T_2. Therefore, each of p, z satisfies the condition for v_1 or for v_k. Suppose next that neither p nor z satisfy the condition for v_1. Then each contains two single edges whose direction does not follow the direction of pr; this contradicts the fact that H does not contain T_5 and T_6 or their reverses. Similarly, the absence of T_7 implies that at least one of p, z satisfies the condition for v_k. The absence of T_4 (and its reverse) implies that each intermediate vertex $r, s, \ldots, y$ of P satisfies the condition for v_i if its left or its right neighbour on P plays the role of v_{i+1}. Finally, if one vertex of P requires its left neighbour, while another requires its right neighbour, we again obtain a contradiction as above with the fact that H does not contain the trees T_5, T_6, T_7. □

We now prove the following dichotomy classification.

Corollary 7.5 *Let H be a reflexive digraph that is a tree.*

If H is a good caterpillar, then H has a min ordering and LHOM(H) is polynomial time solvable.

Otherwise, H contains one of the trees $T_1, T_2, \ldots,$ or T_7, or their reverses, as an induced subgraph, and LHOM(H) is NP-complete.

Proof. If H is a good caterpillar, the theorem implies that it has a min ordering and hence LHOM(H) is polynomial time solvable. Otherwise, the theorem implies that H contains $T_1, T_2, \ldots,$ or T_7. We now claim that for each reflexive digraph H containing one of the trees $T_1, T_2, \ldots,$ or T_7, the problem LHOM(H) is NP-complete.

If H contains T_1, then $S(H)$ is not an interval graph and hence LHOM(H) is NP-complete.

If H contains T_2, then H has a DAT on the vertices a, a', b. Indeed, the walk $a \leftarrow a \leftarrow a$ is congruent to and avoids both walks $a' \leftarrow a' \leftarrow a'$ and $b \leftarrow c \leftarrow a'$. (Here we write c for the central vertex of T_3.) Similarly, $a' \leftarrow a' \leftarrow a'$ is congruent to and avoids both $a \leftarrow a \leftarrow a$ and $b \leftarrow c \leftarrow a$; and $b \rightarrow b$ is congruent to and avoids both $a \rightarrow c$ and $a' \rightarrow c$. Finally, we observe that two pairs a, a' and b, c are both invertible: for a, a' we have already noted this in the illustration of invertible pairs below Lemma 2.1. For b, c, we observe that the walk $b \rightarrow b \leftarrow c \leftarrow a \leftarrow a \leftarrow a$ is congruent to and avoids the walk $c \rightarrow b' \leftarrow b' \leftarrow b' \leftarrow c \leftarrow a'$, and so by symmetry, the pair b, c is invertible. Therefore H contains a DAT and LHOM(H) is NP-complete by [19]. Observe that even tough the DAT is defined on the three vertices a, a', b, all the vertices of T_2, including b', are involved in the walks defining the DAT.

If H contains T_3, then it has a DAT on $0, 1, 2$. Specifically, the walk $2 \leftarrow 2 \rightarrow 2$ is congruent to and avoids both walks $0 \leftarrow 0 \rightarrow a$ and $1 \leftarrow a \rightarrow a$; the walk $1 \rightarrow 1 \rightarrow 1 - 1$

is congruent to and avoids both walks $0 \to a \to b - 2$ and $2 \to 2 \to 2 - 2$; and the walk $0 \leftarrow 0 \leftarrow 0 - 0$ is congruent to and avoids both walks $1 \leftarrow a \leftarrow b - 2$ and $2 \leftarrow 2 \leftarrow 2 - 2$. All three pairs $2, a$ and $1, 2$ and $0, 2$ are invertible, as can be seen from Lemma 2.1 using the fact that the walks $a \leftarrow 0 - 0 \leftarrow 0 \to 0 \to a \leftarrow b - 2 \leftarrow 2$ and $2 \leftarrow 2 - b \leftarrow a \to 1 \to 1 \leftarrow 1 - 1 \leftarrow a$ avoid each other. Note that this proof applies regardless of the direction(s) of the arc(s) between b and 2 (as suggested by the notation $b - 2$).

Similar proofs apply to the trees $T_4, \ldots, T_7$. There is always a DAT with vertices $0, 1, 2$ or a, a', b. The details are technical but not difficult to find. □

In particular, Conjecture 5.3 holds for trees.

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