On Barnette's Conjecture

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Abstract

Barnette's conjecture is the statement that every 3-connected cubic planar bipartite graph is Hamiltonian. We develop a framework for the study of such graphs and obtain several infinite families of such graphs for which the conjecture holds. In particular, we show that if the faces of such a graph are 3-colored, two of the three color classes contain only squares and hexagons, and the third color class contains faces surrounded by an even number of squares, then the graph is Hamiltonian. Our main technique is the introduction of a reduced graph that collapses the faces in one of the three color classes, combined with the result that a spanning tree or quasi spanning tree of faces in the reduced graph will give a Hamiltonian circuit in the original graph.

1 Introduction

Let P be the class of cubic planar bipartite graphs. Barnette [2] conjectured that every 3-connected graph in P is Hamiltonian. This conjecture was verified for graphs with up to 64 vertices by Holton, Manvel and McKay [9]. Aldred, Brinkmann and McKay [1] have announced that the conjecture still holds for graphs with up to 84 vertices. The conjecture also holds for the infinite family of graphs where all faces have either four or six sides, as shown by Goodey [6]. Without the assumption of 3-connectivity, it is NP-complete to decide whether a graph in P is Hamiltonian, as shown by Takanori, Takao and Nobuji [12]. The same conjecture without the assumption of bipartiteness was originally formulated by Tait [11] and disproved by Tutte [13].

We begin our study of the class P with the well known fact that the faces of any graph in P can be colored with three colors. In particular, we get a set of independent faces C such that any edge of the graph either joins two faces of C or belongs to a face in C. We use this fact to characterize the possible parities of the distances between vertices incident to a cut for a graph in P.

The independent faces in C for a graph in P can be collapsed to single vertices, thus giving a reduced graph which is planar and has vertices of even degree at least four, and may have parallel edges.

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We observe that an Eulerian tour of the reduced graph without crossings will give a Hamiltonian circuit of the original graph. Such an Eulerian tour corresponds, for a different choice of the independent faces C to be collapsed, to a spanning tree of faces. In fact any spanning tree of faces in the reduced graph will give a Hamiltonian circuit for the original graph.

We show that the problem of deciding whether a reduced graph has a spanning tree of faces that are either digons or triangles can be solved in polynomial time via a reduction to the spanning tree parity problem, which is polynomial as shown by Lovász [10] (see also Gabow and Stellman [4]). This implies that one can decide in polynomial time whether a reduced graph with vertices of degree 4 and 6 has a noncrossing Eulerian tour.

We show that if the subgraph of the reduced graph whose edges are the consecutive parallel edges has at most three components, then the reduced graph has a spanning tree of faces where all but a constant number of these faces are digons.

For the case of a reduced graph which is triangulated, corresponding to the case where all faces not in C are hexagons for the original graph, we find a spanning tree of triangles if the number of vertices is odd, all parallel edges are non-consecutive and contain an even number of vertices inside, and all triangles that are not faces contain an odd number of vertices inside or on such a triangle.

We then introduce the notion of a quasi spanning tree of faces in the reduced graph, which also gives a Hamiltonian circuit in the original graph. We show that every reduced graph which is triangulated has a quasi spanning tree of faces, thus showing that if in the original graph, all faces not in C are hexagons, then the graph has a Hamiltonian circuit. We extend the result to the case of triangulated reduced graphs that may have parallel consecutive edges, provided that every vertex has an even number of pairs of parallel consecutive edges incident to it; this corresponds to the case where all faces not in C are either squares or hexagons, and all faces in C have an even number of squares adjacent to them.

The case of reduced graphs that are triangulated and have arbitrary parallel consecutive edges remains open. One can show that it is not possible to have just a single pair of parallel consecutive edges in such reduced triangulated graphs. For the case of exactly two pairs of parallel consecutive edges, or more generally for the case where if we ignore two special pairs of parallel consecutive edges, then the remaining such pairs occur in even number at each vertex of the reduced triangulated graph, we obtain a quasi spanning tree of faces, and thus a Hamiltonian circuit in the original graph.

A related conjecture of Barnette states that if a 3-connected cubic planar graph only has faces with 3,4,5, or 6 sides, then it is Hamiltonian. As mentioned before, this was verified by Goodey [6] for the case of faces with 4 or 6 sides; it was also verified by Goodey [7] for the case of faces with 3 or 6 sides; and in general by Brinkmann, McKay, and von Nathusius [3] for graphs with up to 250 vertices. The technique we use for interconnecting the two special digons in our last result is reminiscent of the technique used by Goodey for interconnecting two of the four triangles in his result.

We then study the limitations of our approach. We show that deciding whether a reduced graph with vertices of degree 8 has a noncrossing Eulerian tour is NP-complete. We also show that deciding whether an arbitrary reduced graph has a spanning tree of faces is NP-complete. We do not know whether similar NP-completeness results can be obtained for quasi spanning trees of faces in reduced graphs, which generalize both spanning trees of faces and noncrossing Eulerian tours. We finally show that if there is a counter example to Barnette's conjecture, so that some 3-connected graph in P is not Hamiltonian, then deciding whether a 3-connected graph in P has a Hamiltonian cycle is NP-complete.

We have thus exhibited several infinite families of reduced graphs that have a spanning tree of faces or a quasi spanning tree of faces, and thus yield a Hamiltonian cycle for the original graph in the class of cubic planar biparite graphs.

2 Cubic planar bipartite graphs

The following result is well known. We include the proof for completeness.

Theorem 1 *Let G be a cubic planar bipartite graph, and let f be a face in G. Then there exists a collection C of independent faces in G containing f such that every edge of G either belongs to a face in C or joins two faces in C.*

PROOF: The graph G is 2-connected. Otherwise G has a component joined to the rest of the graph by a single edge, so that a single vertex has degree 2 within the component; then the sum of the degrees of vertices in the component would be divisible by 3 in one side of the bipartition and not divisible by 3 in the other side, a contradiction. Therefore no face of G is adjacent to itself.

Say two faces of G are joined if G has an edge not in either face joining the two faces. Starting with $C = \{f\}$, repeatedly add to C faces that are joined to a face in C. When the final collection C is obtained, if all faces of C are independent, then the Theorem is proved.

Suppose C contains two faces f_a, f_b that share an edge e. Then there exists a sequence of distinct faces $f_a = f_1, f_1, \ldots, f_k = f_b$ in C such that f_i and f_{i+1} are joined by an edge e_i. The collection of edges e_i and a choice of a path within each f_i joining e_{i-1} to e_i, or e to e_1 for f_1, or e_{k-1} to e for f_k, form a cycle D that does not contain any of the faces f_i in its interior.

If the interior of the cycle D is a single face, then the chosen path within each f_i consists of a single edge, and the face D has $2k-1$ sides, contrary to the fact that G is bipartite.

Otherwise there is a path $p = e'_1, \ldots, e'_l$ joining two vertices of D through the interior of D. Say p joins a vertex on f_r to a vertex on f_s. Each edge e'_i with i even belongs to a face f'_i not containing e'_{i-1} or e'_{i+1}. See Figure 1.

If l is odd, then the sequence of faces $f_a = f_1, \ldots, f_r = f'_1, \ldots, f'_m = f_s, \ldots, f_k = f_b$ will provide a sequence of faces different from the original sequence of faces f_i, defining a cycle D' on one side of p inside of D; the result follows then by induction since the number of vertices inside D' is smaller than the number of vertices inside D.

If l is even, then the sequence of faces $f_s, f_{s-1}, \ldots, f_r = f'_1, \ldots, f'_l$ will provide a sequence of faces different from the original sequence of faces f_i, defining a cycle D'' on the other side of p inside of D; the result follows then by induction since the number of vertices inside D'' is again smaller than the number of vertices inside D. ■

Let G be a cubic planar bipartite graph. Consider a cut S, T in G with successive edges independent edges $e_i = s_i t_i$ for $i = 1, \ldots, k$ across the cut. Since G is 2-connected as observed in Theorem 1, we have $k \geq 2$. Since G is bipartite, we have $d_i = d(s_i, s_{i+1}) \equiv d(t_i, t_{i+1})$ (modulo 2), with $d_k = d(s_k, s_1) \equiv d(t_k, t_1)$ (modulo 2). Define the parity p_i of d_i to be 0 if

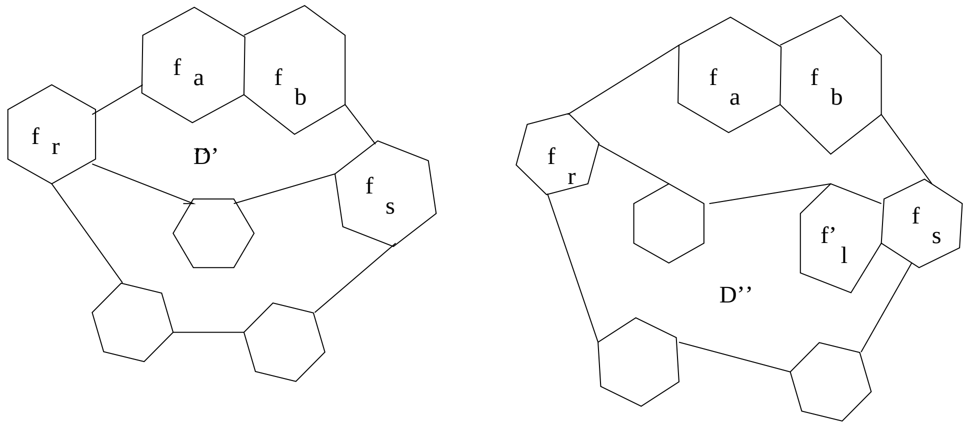

Figure 1: Reducing region D to D' or D''.

d_i is even and to be 1 if d_i is odd. We wish to characterize the possible cyclic sequences of parities $p_1 \dots p_k$. Define two permutations on $\{1, 2, 3\}$ given by $\sigma_0 = (123)$ and $\sigma_1 = (12)(3)$. Consider also the four sequences 11, 000, 0101, and 010010.

Theorem 2 *A sequence of parities p_i can arise for some cut S, T in some cubic planar bipartite graph if and only if the composition of the corresponding σ_{p_i} is the identity permutation on $\{1, 2, 3\}$. Equivalently, if we repeateldly remove one of the four sequences occurring as substrings from the cyclic sequence of p_i, we obtain an empty sequence.*

PROOF: See Figure 2. There are three ways of choosing the collection C of Theorem 1, depending on which face out of three possible faces is chosen for a given vertex. If we consider the three choices for the vertex s_i, we call choice 1 the face having the edges e_i and e_{i+1} in its boundary, choice 2 the face having the edges e_{i-1} and e_i, and choice 3 the face not having the edge e_i.

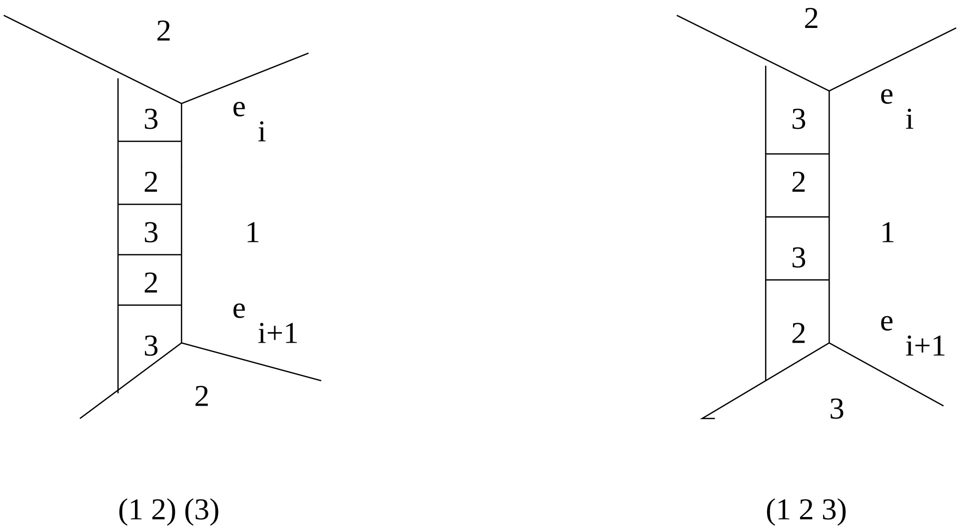

Figure 2: The two permutations for odd and even paths joining e_i and e_{i+1}.

If choice 1 is made for vertex s_i, then choice 2 is made for vertex s_{i+1}, whether d_i is even or odd. This corresponds to both permutations σ_0 and σ_1 taking 1 to 2.

If choice 2 is made for vertex s_i, then if d_i is even, we will have choice 3 made for vertex s_{i+1}, as shown by the faces chosen in an alternating way along the path of length d_i from s_i to s_{i+1}; if d_i is odd, we will have choice 1 made for vertex s_{i+1}, as shown similarly by the choices made in this case on the path of length d_i from s_i to s_{i+1}. This corresponds to permutation σ_0 taking 2 to 3, and permutation σ_1 taking 2 to 1.

If choice 3 is made for vertex s_i, then if d_i is even, we will have choice 1 made for vertex s_{i+1}; if d_i is odd, then we will have choice 3 made for vertex s_{i+1} as well. This corresponds to permutation σ_0 taking 3 to 1, and permutation σ_1 taking 3 to 3.

Thus if a sequence of p_i arises, then the composition of the corresponding σ_{p_i} must be the identity permutation on $\{1,2,3\}$. One can verify that the three sequences 11, 000, 0101, and 010010 give the identity permutation. If the sequence 11 does not arise, then every 1 in the sequence is surrounded by 0's. If two 1's have a single 0 between them, then the sequence 0101 arises, and if they have three 0s between them, then the sequence 000 arises. We are thus left with the possible remaining nonempty sequences being 0, 1, 00, 01, 001, or $(001)^k$. The first five do not give the identity permutation, and the remaining family only gives the identity permutation for k even, where it can be rewritten as $(010010)^{k/2}$.

It remains to construct an instance G with a cut S, T for valid sequences of parities p_i. If $p_i = 1$, then we put an edge $s_i s_{i+1}$ in S (and similarly for T). If $p_i = 0$ then we put a path $s_i v_i s_{i+1}$ of length 2 in S. We must decide how to inter-connect the vertices v_i within S (and similarly within T). For the sequence 000, we add a single vertex w incident to the three corresponding vertices v_i. For the sequence 0101, we add an edge joining the two corresponding vertices v_i. And for the sequence 010010, we join the two v_i corresponding to the first two 0's, and also join the two v_i corresponding to the last two 0's. This completes the construction of the instance G and the proof. ■

3 Families with spanning trees of faces

Let G be a cubic planar bipartite graph. We have seen that there are three choices for the collection C of independent faces from Theorem 1, depending on which choice is made at one specific vertex. We refer to the three collections as the green faces, the red faces, and the blue faces.

We define a reduced graph H by collapsing each face of one color, say green, to a single vertex. See Figure 3. Thus H is an arbitrary graph with vertices of even degree, possibly with parallel edges. The graph G is 3-connected if and only if the graph H is not separated by removing two edges or by removing a single vertex. Any cut in H will have an even number of edges across, because the p_i in Theorem 2 for the corresponding G are all equal to 1, and this is only possible if we have an even number of 1s from Theorem 2.

Finding a Hamiltonian cycle in G corresponds to finding a tour in H going through all the vertices, although not necessarily all the edges, so that each edge of H is visited at most once. The $2k$ edges used by the tour for a vertex v in H corresponding to a green face in G must be paired up consecutively by going around v, into k pairs of edges used for entering and leaving v by the tour in its k visits to v. There must be no edge incident to v between

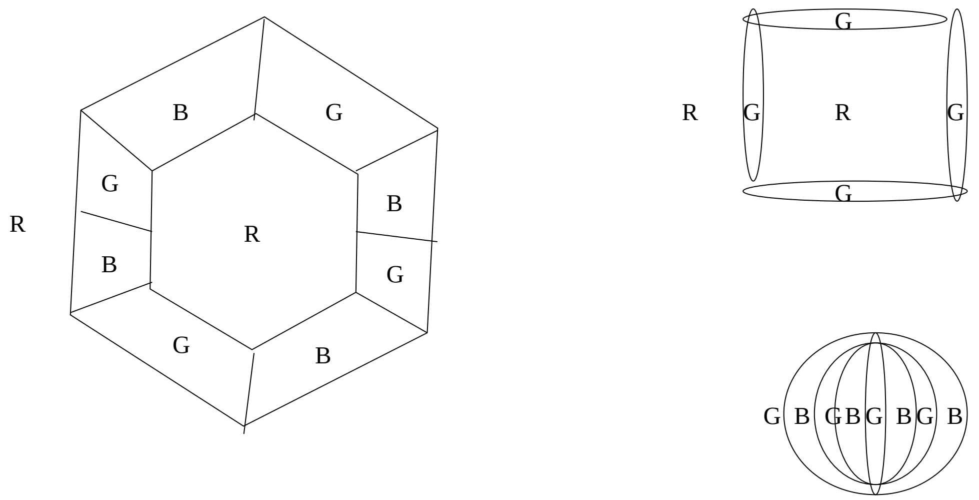

Figure 3: Constructing the reduced graph.

two consecutive such pairs, otherwise some vertex of the corresponding green face of G would not be visited.

If the tour of H visits every edge, then the tour of H is a noncrossing Euler tour, in the sense that a vertex v entered by the Euler tour through some edge e exits the vertex through one of the two edges e', e'' incident to v on either side of e. We shall relate the existence of noncrossing Euler tours in one choice of reduced graph to the existence of a certain spanning tree of faces in a different choice of reduced graph. See Figure 4.

A *tree of faces* in the reduced graph H is a collection F of faces in H such that (1) two faces in F share at most one vertex, and (2) the faces in F do not contain a *cycle*, where a cycle is a cyclic sequence $f_1, f_2, \ldots, f_k$ of distinct faces in F such that each face f_i has two distinct vertices u, v, where u is a vertex in f_{i-1} and v is a vertex in f_{i+1} (where $f_{k+1} = f_1$ and $f_0 = f_k$). A *spanning tree of faces* is a tree of faces F such that every vertex of H is in some face in F.

Proposition 1 *Let G be a cubic planar bipartite graph. Then G has a Hamiltonian cycle with red faces in one side, blue faces in the other side, and green faces in either side, if and only if the reduced graph H obtained by collapsing green faces has a noncrossing Euler tour, if and only if the reduced graph H' obtained by collapsing red faces has a spanning tree of green faces, if and only if the reduced graph H'' obtained by collapsing blue faces has a spanning tree of green faces.*

Proof: Suppose G has a Hamiltonian cycle C with red faces in one side, blue faces in the other side, and green faces in either side. Then in the reduced graph H obtained by collapsing green faces every edge of H is visited by C, because it separates a red face from a blue face, so C gives a noncrossing Euler tour in H. In the reduced graph H' obtained by collapsing red faces (or similarly the reduced graph H'' obtained by collapsing blue faces), if we choose the outer face to be blue (resp. red), then the inside of the cycle contains only green faces in H' (resp. H''), and these green faces cannot contain a cycle because all red faces (resp. blue faces) collapsed are inside the cycle C, so a cycle of green faces would split

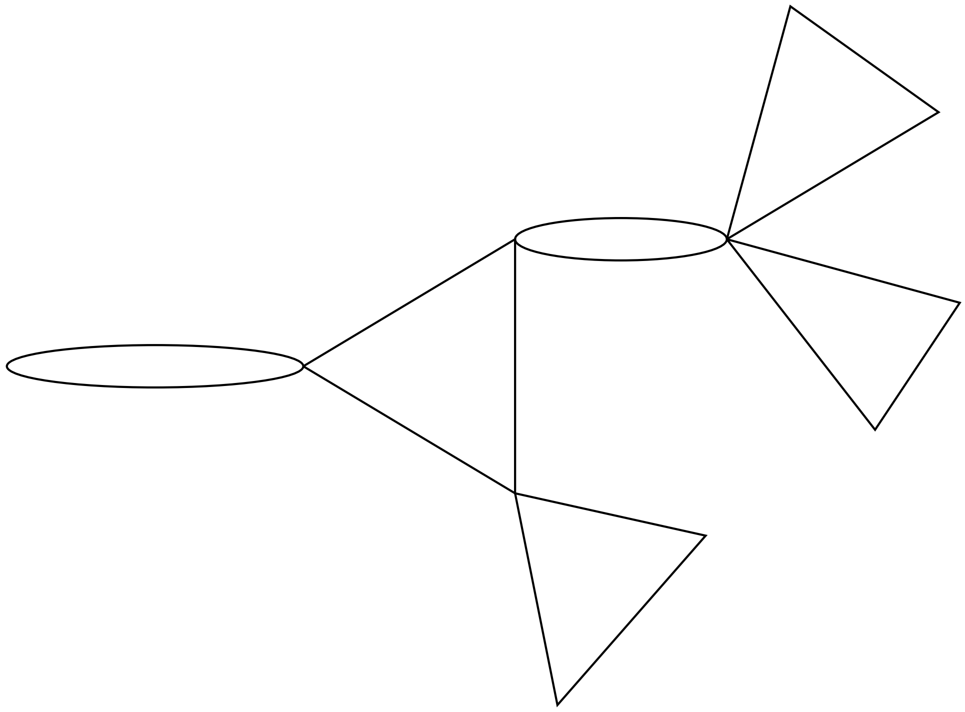

Figure 4: A spanning tree of faces.

the cycle C into two cycles. Therefore the inside of C gives a spanning tree of green faces in H' (resp. H'').

Suppose H obtained by collapsing green faces has a noncrossing Euler tour. This tour separates red and blue faces in H, and thus corresponds to a Hamiltonian cycle in G with red faces in one side and blue faces in the other side. Suppose H' obtained by collapsing red faces (or similarly H'' obtained by collapsing blue faces) has a spanning tree of green faces, and the outer face is not green. Then going around the perimeter of the spanning tree of green faces in H' gives a Hamiltonian cycle that has green faces and all collapsed red faces (resp. collapsed blue faces) inside, that is a Hamiltonian cycle with red faces in one side and blue faces in the other side. ■

More generally, any spanning tree of faces in H, not necessarily all blue or all red, will also give a Hamiltonian cycle, by traversing its perimeter and putting the green vertices inside the tour. The following result concerns the case where most faces in the spanning tree of faces are digons.

Theorem 3 *Let G be a cubic planar bipartite graph. Let H be the reduced graph for G, and let H' be the subgraph of H obtained by removing all edges that do not have consecutive parallel edges. If H' has one, two, or three connected components, assuming also in the case of three connected components that G is 3-connected, then H has a spanning tree of faces, and thus G has a Hamiltonian cycle. The case of a single component for H' includes the case where all red faces are squares.*

PROOF: If H' is a single connected component, then we can choose a spanning tree of H', corresponding to a spanning tree of digons in H.

If H' has two connected components, then we may choose a face f of H that has vertices from both components. Starting with this face f, we also consider two spanning trees of digons for the two components of H', and add these digons one at a time as long as they do not form a cycle containing f. Eventually, the single face f and the added digons will span H.

If H' has three connected components, then it may be that H has a face f touching all three components, and we may proceed from f as for the case for two components, by considering the three spanning trees of digons for the two components. Otherwise some component, say the first, has faces touching it and the second component and also faces touching it and the third component. Both sets of faces have at least four faces, since a cut of H has at least four edges by 3-connectivity and the fact that any cut has an even number of edges, so we may choose a face f touching the first and second component, and a face f' touching the first and third component, so that these two faces do not share any vertices. Starting with these two faces, we may then again add digons from the three spanning trees for the three components so long as they do not form a cycle, until a spanning tree of faces for H is obtained. ■

The proof for three connected components extends to the case of four connected components, but the result does not hold in the case of five connected components.

We show next that one can decide whether the reduced graph H has a spanning tree of faces that are either digons or triangles in polynomial time. The result easily extends to the case of a spanning tree where all but a constant number of faces are either digons or triangles.

Theorem 4 *Let G be a cubic planar bipartite graph, and let H be the corresponding reduced graph. Let C be a collection of faces in H such that all faces in C are either digons or triangles. Then we can decide whether H has a spanning tree of faces in C, giving a Hamiltonian cycle for G, in polynomial time, by a spanning tree parity algorithm [10, 4].*

PROOF: See Figure 5. Construct a graph H' related to H as follows. The vertex set of H' is the same as the vertex set for H. If xy is a digon in C, then put an edge xy in H'. If xyz is a triangle in C, then put edges xy and yz in H'. A spanning tree of faces in C for H then corresponds to a standard spanning tree in H' which must contain either both or none of edges xy and yz corresponding to a triangle xyz in C. These conditions on pairs of edges in H' make the equivalent problem in H' a spanning tree parity problem. ■

If C contains faces with four or more sides, say a face $xyzt$, then we could include three edges linking these four vertices, say xy, yz, and zt, and require that a spanning tree contain either all three or none of these three edges. Such a spanning tree tri-arity problem, as we shall later see, turns out to be NP-complete.

Corollary 1 *Let G be a cubic planar bipartite graph, and let H be the corresponding reduced graph. Suppose all vertices of H have degree four or six, so that all green faces of G are either squares or hexagons. Then one can decide in polynomial time whether H has a noncrossing Euler tour giving a Hamiltonian cycle for G.*

PROOF: Let H' be the reduced graph for G by collapsing the blue or red faces instead of the green faces. Then the green faces give in H' digons or triangles. By Proposition 1, a

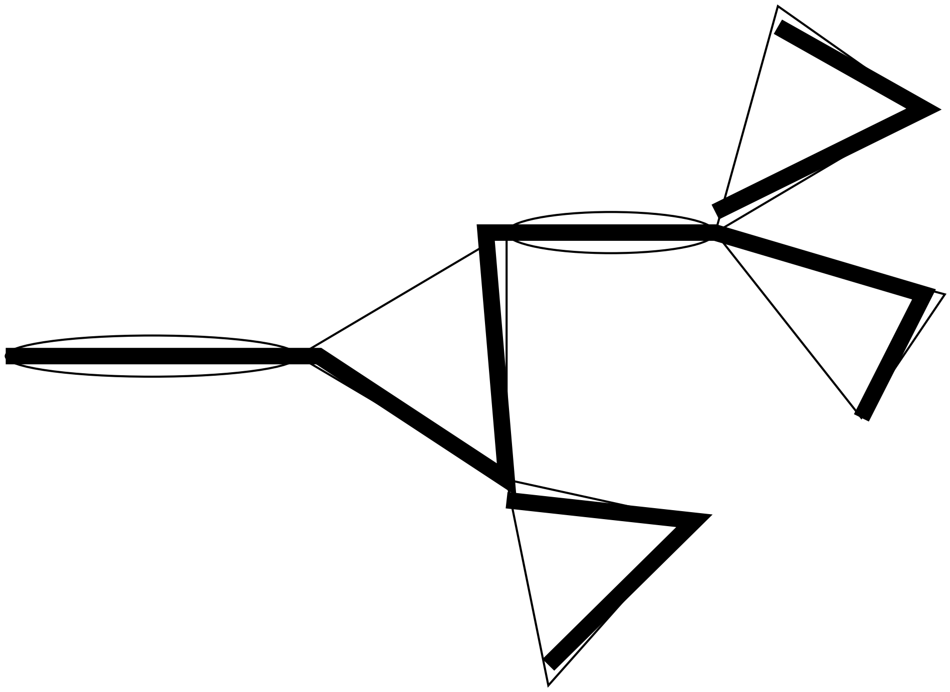

Figure 5: Correspondence between spanning tree of faces and regular spanning tree.

noncrossing Euler tour for H corresponds to a spanning tree of green faces in H'. Since all green faces are either digons or triangles in H', one can decide whether such a spanning tree of green faces in polynomial time by Theorem 4. ■

4 Triangulated reduced graphs

We next consider the case where H is a triangulated graph, which corresponds to the case where all blue and all red faces have six sides, while the green faces, represented by vertices in H, may have any even number of sides. We attempt to characterize when such a graph H has a spanning tree of triangles, so that a Hamiltonian cycle for G may be obtained in this way.

Proposition 2 *Let G be a cubic planar bipartite graph, and suppose that the corresponding reduced graph H is triangulated.*

(1) If H has an even number of vertices, then H does not have a spanning tree of triangles.

(2) There is a triangulated H not in cases (1) or (3) with three parallel non-consecutive edges such that the number of vertices inside the two regions defined by the parallel edges is odd, that does not have a spanning tree of triangles.

(3) There is a triangulated H not in cases (1) or (2), with triangles that are not faces such that the number of vertices inside or on such triangles is even, that does not have a spanning tree of triangles.

Proof: A spanning tree of triangles always starts with a single triangle and successively joins two additional vertices by a triangle. Thus the total number of vertices covered by the spanning tree of triangles is odd, proving (1).

For (2), let J be a triangulated graph with an odd number of vertices, which are of even degree at least four, and not in case (3). Let a, b be two adjacent vertices in the outer face of J. Let J' and J'' be two copies of J with corresponding vertices a', b' and a'', b''. Identify a, a', a'', and identify b, b', b'' to obtain a triangulated graph H with an odd number of vertices and three parallel non-consecutive edges corresponding to the edge ab. If H has a spanning tree of triangles, then this spanning tree must give within J an odd number of spanning trees, since each spanning tree has an odd number of vertices; each of these spanning trees must contain either a or b, so we in fact have a single spanning tree within J, and similarly a single spanning tree within J' and within J''. The two spanning trees joining a and b within J and within J' give then a cycle in the spanning tree of H, a contradiction.

For (3), let J be a triangulated graph with an odd number of vertices, which are of even degree at least four, and not in case (2). Note that the number of faces of J is even and strictly greater than the number of vertices of J, since the vertices are of even degree at least four and all faces are triangles. Define H by adding to J, for each triangle 123 in J, a triangle abc, with additional edges forming triangles $1ab$, $2bc$, and $3ca$. A spanning tree of H must form two trees within the gadget with vertices $1, 2, 3, a, b, c$, one of which contains two out of $1, 2, 3$, say $1, 2$, and the other one which contains the remaining vertex, say 3. We may view 1 and 2 so joined as an edge 12 in J. The number of edges so selected for J thus equals the number of triangles in J, which is strictly greater than the number of vertices of J. These selected edges thus contain a cycle in J, corresponding to a cycle in H, a contradiction. ■

If we exclude the three properties that may give counter examples in the preceeding proposition, we indeed obtain a spanning tree of triangles.

Theorem 5 *Let G be a cubic planar bipartite graph, and suppose that the corresponding reduced graph H is triangulated. Suppose also that (1) H has an odd number of vertices, (2) H does not have parallel consecutive edges, or parallel non-consecutive edges with an inside region having an odd number of vertices, and (3) H has at most one triangle that is not a face such that the number of vertices inside or on the triangle is even. Then H has a spanning tree of triangles, giving a Hamiltonian cycle for G.*

Proof: Suppose H is a triangulated graph with an odd number of vertices, which are of even degree at least four, H does not have parallel edges, and H does not have a triangle that is not a face. Let K be the dual graph of H. Then K is cubic and has faces with an even number of sides at least four, so K is a cubic planar bipartite graph. The green faces of K correspond in H to a vertex surrounded by triangles, i.e., a wheel. The green faces of K thus decompose the faces of H into a collection of wheels.

We shall maintain for H a connected region R delimited by a single cycle which is a union of these wheels. See Figure 6. Initially, R is a single wheel. We repeatedly select a single wheel w not in R that is joined to R by a single sequence of consecutive edges of the boundary of w, which are also consecutive edges of the boundary of R. We then add w to R. We proceed in this fashion until all wheels are added to R.

The boundary of R has an even number of edges, which are labeled alternatively as on- and off-edges. Suppose an edge e is not part of the boundary of R but joins two vertices

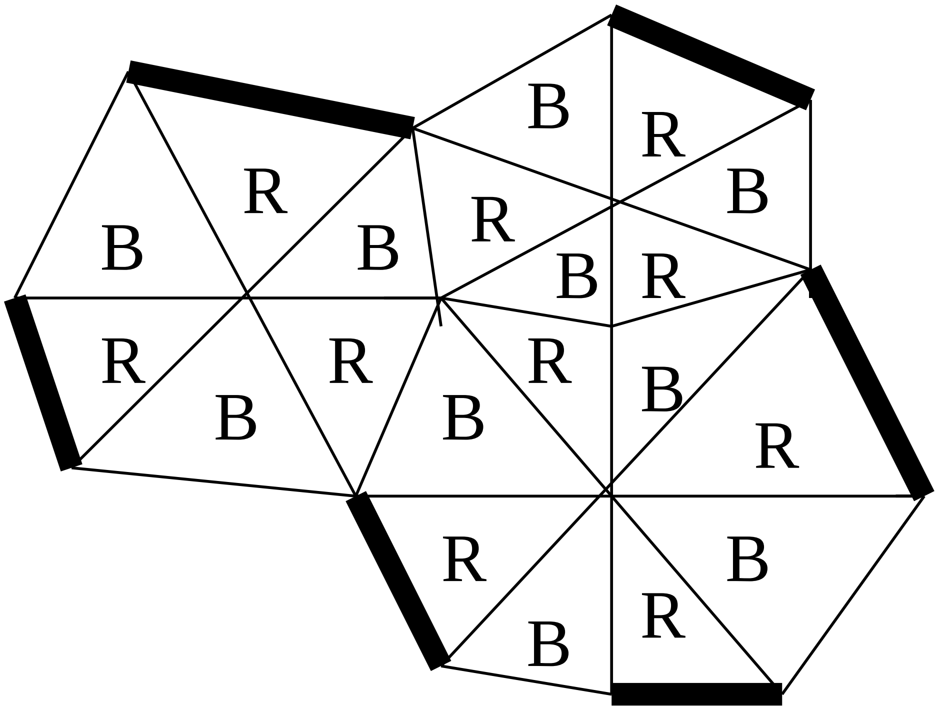

Figure 6: Region R consisting of three wheels, with on-edges.

of the boundary of R. We shall maintain the property that for every such edge e, if the boundary of R were extended to reach e, then e would be an on-edge. Notice that initially, when R is a single wheel, there are no such edges e, since otherwise e and the edges joinining its endpoints to the center of the single wheel of R would form a triangle that is not a face. We shall also guarantee that the single wheel w added to R joins R in at least one off-edge; this assumption only matters if w joins R in a single edge, in which case this edge is an off-edge.

In Figure 7, starting with region R consisting of wheel A, we cannot add wheel B because it is joined by a single on-edge, and we cannot add wheel C because of the cross-edge that would be an off-edge joining it to A next to D, but we can add wheel D because it is joined by an off-edge.

As the region R grows, we shall grow a set of triangles forming several trees within R. Each vertex strictly inside R is joined to a vertex in the boundary of R by some such tree. We shall always have a subset S of the on-edges in the boundary of R, such that two vertices on the boundary of R are in the same tree if and only if they are the endpoints of an edge in S. Initially, when R is a single wheel w, we select a single triangle of w to form a tree, giving a single edge for S.

See Figure 8. Suppose we wish to add a wheel w to R that meets R in k consecutive edges, at least one of which is an off-edge. Suppose that some l of these k edges are on-edges in S. The $k+1$ vertices given by the k consecutive edges can be viewed as $k+1-l \geq 2$ super vertices by treating two vertices joined by an edge in S as a single super vertex. We pair up the super vertices into pairs of consecutive super vertices, starting from the left and

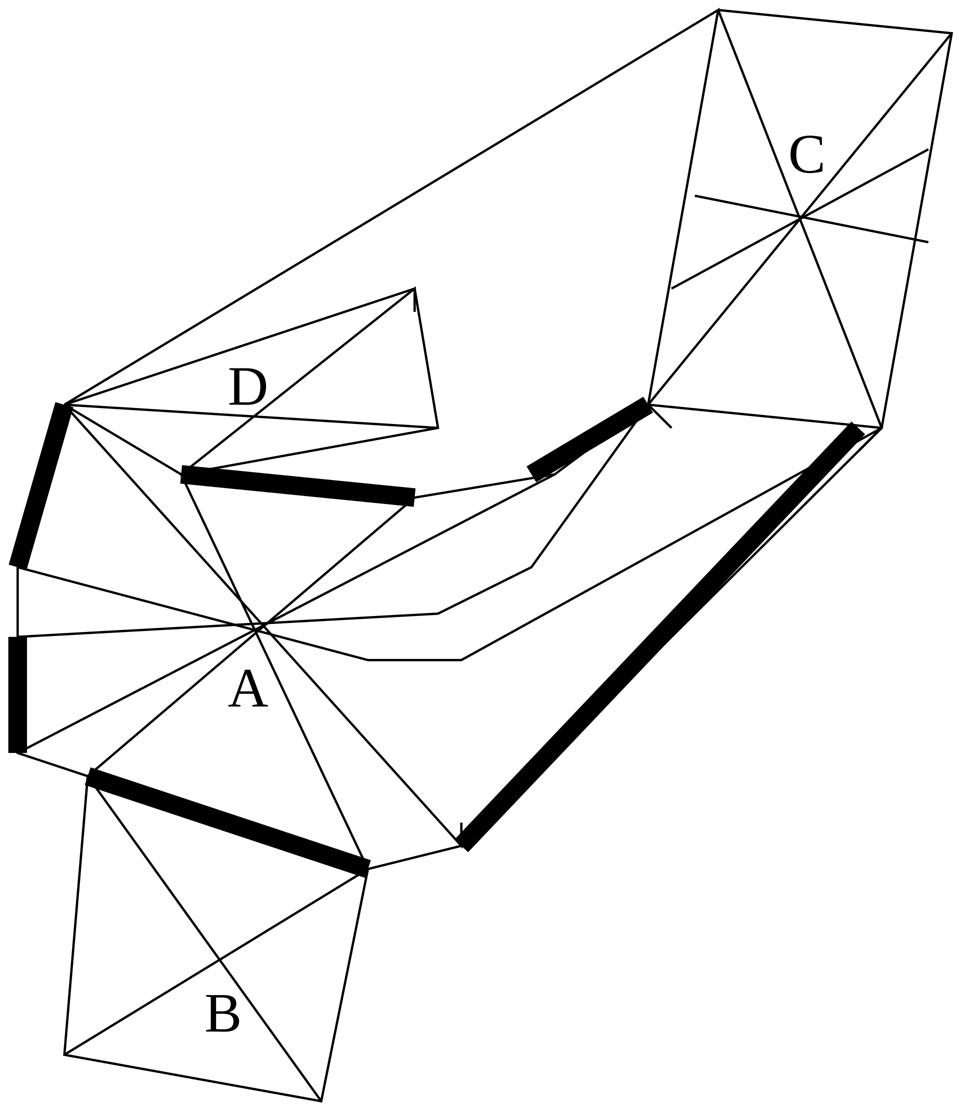

Figure 7: Adding wheels to region R consisting of wheel A.

skipping the first super vertex. Each selected pair of super vertices is joined by a triangle to the center of w. If $k+1-l$ is odd, then this forms a tree that is joined to the last super vertex, which remains in the boundary of R after w is added. If $k+1-l$ is even, then this forms a tree that is not joined to the boundary of R; in this last case, we join the tree to the boundary of R augmented by w by selecting a triangle joining the center of w to an edge e in the boundary of w that is not in the boundary of R. Notice that even in the case where there is only one such edge e, it can be chosen to be an on edge as guaranteed by assumption.

In the end, when the last wheel w is about to be added, there will be an even number of super vertices in the boundary of R, which are all paired up and each pair is joined by a triangle to the center of w, giving a single tree.

It remains to show that we can always choose the wheel w to add so that it has the desired properties. First we must guarantee that the wheel w joins R in consecutive edges with at least one off-edge. If we have selected a w that joins R in at least two non consecutive

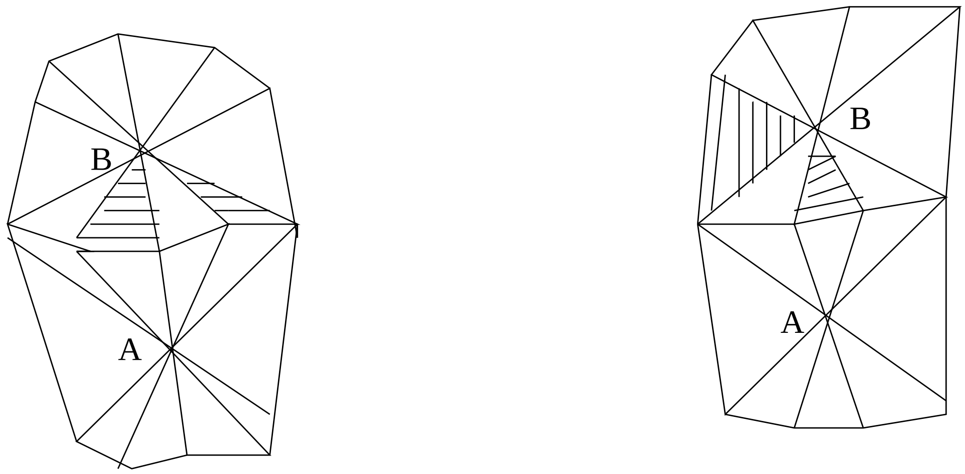

Figure 8: Selected triangles for adding wheel B for odd and even number of super vertices.

segments, then each two such segments are separated by at least two consecutive edges in the boundary of R, because if they are separated by no edge there would be parallel edges coming out of the center of w, and if they are separated by one edge, this edge would form with the center of w a triangle that is not a face. We may then restrict the search for a better w to the segment of at least two consecutive separating the earlier w. At least one of these two edges will be an off-edge e, and we must succeed in finding w by proceeding similarly with the wheel joining R at e, by induction.

So suppose that we have succeeded in finding a wheel w that joins R in consecutive edges, at least one of which is an off-edge. Suppose there is an edge e joining w to R that is not in the boundary of w or of R. We must ensure that such an edge e, if the boundary of R were also extended to reach it, would be an on-edge. Say this edge e is an off-edge and joins a vertex in the boundary of w to a vertex v to the right of w along the boundary of R. Then the edge f on the boundary of R to the left of v is also an off-edge, and we select instead w to be the face meeting R at f. We will thus eventually succeed in finding w such that all such edges e are on-edges, by induction.

This completes the proof for the case where there are no triangles that are not faces. Suppose now there are triangles that are not faces, but the number of vertices inside or on such triangles is odd. Select one such triangle t with the property that there are no such triangles inside of t. Remove the vertices inside of t from H to obtain H' with t as a face. Since H' is smaller than H, we may assume inductively that H' has a spanning tree T' of triangles, which may or may not include t. Let H'' be obtained from H by removing the vertices that are outside of t. Then H'' has t as its outer face and has no triangles that are not faces. Note that the vertices of t have even degree in both H' and H'', since the vertices of H are 3-colored.

If T' did not contain t, then we may view T as selecting the outer face t of H'', and we extend T' by looking for a spanning tree of triangles T'' in H'' containing the outer face t with the above proof; we can always choose t as the selected triangle in the first wheel w forming R above. The tree T for H is obtained by combining T' outside t with T'' inside t.

In the other case, if T' contained t, then we may view T as not selecting the outer face t of H'', and we extend T' by looking for a spanning tree of triangles T'' in H'' not containing the outer face t with the above proof; we can always ensure that t is not a selected triangle by making t part of the first wheel w forming R above, and selecting a different triangle from w. The tree T for H is again obtained by combining T' outside t with T'' inside t, completing the proof for triangles with inside regions having an odd number of faces.

If we allow just a single triangle t that is not a face and with the number of vertices on or inside t. The inside and the outside of t then give two problems with an even number of vertices. In the above proof, when adding the last wheel w to R while working with H' or with H'' having an even number of vertices, there will be an odd number of super vertices in the boundary of R, and we get two trees for each of H' and H''. We may choose t to be a triangle of the last wheel w containing an off-edge e in the boundary of R, so that the two vertices joined by e end up in different trees. Thus if t has vertices x, y, z, then while working within H', then say x, y will be in one tree and z in the other, and then within H'' we ensure that x, y are in different trees by making $e = xy$. The two trees outside t and the two trees inside t then form a single tree T for H.

Similarly, for pairs of non-consecutive parallel edges joining two vertices x, y with an even number of vertices inside, we may solve this inside problem with two trees joined at x and y respectively, and then replace the region defined by the two parallel edges with a single edge for the outside problem. Note that x, y have even degree in the inside region of the two parallel edges with one of the parallel edges removed, since the vertices of H are 3-colored. This completes the proof. ■

We introduce the notion of a quasi spanning tree of faces. This is the same as a spanning tree of faces, except that some vertices of degree $2k$ in the reduced graph, which are surrounded by k faces in the spanning tree, are left outside, so that the k faces are not considered to be inter-connected in the spanning tree.

In the quasi spanning tree of Figure 9, faces A and D are not joined since the vertex they share is left outside of the quasi spanning tree. More formally, a *quasi spanning tree of faces* in the reduced graph H is a collection F of faces in H such that (1) every vertex of H is in some face of F; (2) there exists a collection V of vertices of H such that if a vertex v in V has degree $2k$, then F has exactly k faces containing v, obtained by picking alternating faces around v; (3) two faces of F share at most one vertex not in V; (4) the faces of F do not contain a cycle not involving vertices in V, where such a cycle is a cyclic sequence $f_1, f_2, \ldots, f_k$ of distinct faces in F such that each face f_i has two distinct vertices u, v, not in V where u is a vertex in f_{i-1} and v is a vertex in f_{i+1} (where $f_{k+1} = f_1$ and $f_0 = f_k$); and (5) for every two faces f, f' in F, there exists a sequence of faces $f = f_1, f_2, \ldots, f_k = f'$ such that f_i and f_{i+1} share a vertex not in V.

Theorem 6 *Let G be a cubic planar bipartite graph, and suppose that the corresponding reduced graph H is triangulated (and does not have consecutive parallel edges). Then H has a quasi spanning tree of faces, and therefore G has a Hamiltonian cycle.*

PROOF: Suppose first H does not have non-consecutive parallel edges and does not have triangles that are not faces. If the number of vertices is odd, then the result follows from the previous theorem. If the number of vertices is even, then in the previous theorem, during

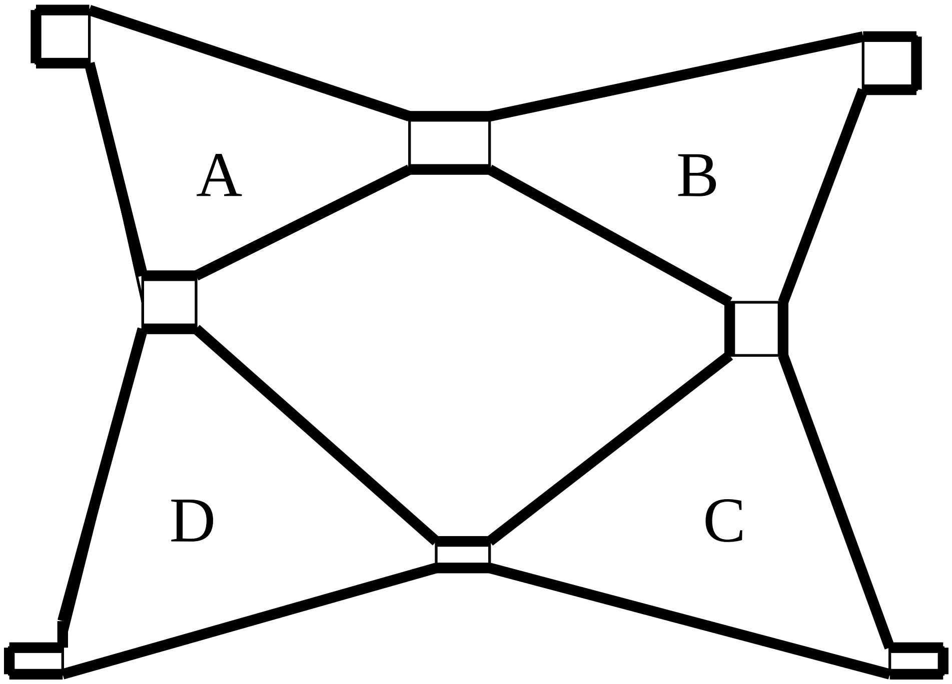

Figure 9: A quasi spanning tree with A and D not joined.

the last step, we would get two trees instead of just one. We can choose the center of the last wheel to be a vertex of degree four. Then there is a single matched edge surrounded the last wheel, which would lead to two trees by selecting just a single triangle at the last vertex. Instead, we select two triangles at the last vertex, but leave the last vertex out to obtain a single quasi spanning tree.

Suppose next that H has either non-consecutive parallel edges or triangles that are not faces. Remove then the region within parallel edges, leaving just one edge. Remove also vertices inside triangles that are not faces. If the resulting number of vertices is odd, then solve the problem with a spanning tree of faces. This will require to later visit regions within parallel faces with two trees joined at the two vertices, and also to visite the inside of triangles that are not faces with a single tree if the triangle face was chosen, without choosing the outer face, or with a single tree if the triangle face was not chosen, by choosing also the outer face.

If the resulting number of vertices is even, we can solve the problem by obtaining two spanning trees of faces. We can ensure that the two trees are the center of the first wheel for one tree, and the rest of the graph for the other tree, by just not connecting the center of the first wheel with any triangle. In this way, we can ensure that any two adjacent vertices x, y are in different trees. We can thus choose x, y to be two adjacent vertices joined by parallel edges, in which case the inside region of the parallel edges will be joined by a single tree; otherwse we can choose x, y so that xyz is a triangle that is not a face, in which case the inside region will be visited by two spanning trees with x, z in different trees if x, z were in the same tree before.

There are thus five cases of inside regions of parallel edges or triangles, which we group into two sets of groups, depending on whether two trees or a single tree is needed for such a region. The case of a single tree could be either a single tree inside parallel edges, or a single tree inside a triangle that is not a face by either including or excluding the outer face. The case of two trees for two vertices x, y could either have x, y as the two adjacent vertices joined by parallel edges, or x, y as two vertices of a triangle xyz with vertices inside.

Suppose first the region inside parallel edges or inside a triangle does not have parallel edges or triangles inside. Again remove inside of such inside regions. It is easy to do a single tree if the resulting number of vertices is odd, or to do two trees if the resulting number of vertices is even. To do just one tree if the number of vertices is even, choose a vertex of degree four not in the special triangle xyz, make it the center of the last wheel so that two triangles that are not joined are chosen for that vertex as before; the outside triangle of xyz that must be chosen or not chosen will be in the first wheel.

To do two trees if the number of vertices is odd, choose a vertex of degree four as center of the first wheel, so that two triangles that do not connect are chosen for that vertex, with x, y belonging to the last wheel. If either x or y is the center of the last wheel, it is easy to ensure that x, y are in different trees. If x, y are not centers of last wheel, then we must ensure that xy is an off-edge, by choosing on- and off-edges appropriately for the first wheel.

Suppose instead that the region inside parallel edges or triangle also has parallel edges or triangles with vertices inside. Again remove inside of such inside regions. The case of one tree is again easy if the number of vertices is odd, and the case of two trees is again easy if the number of vertices is even.

To do one tree with an even number of vertices, we may again use a vertex of degree four if there is such a vertex. Otherwise choose x', y' in two different trees either, joined by parallel edges or in a triangle that has inside region, with x' being the center of the last wheel, and special chosen or not chosen triangle in the first wheel, thus not incident to x'.

To do two trees with an odd number of vertices, we may again use a vertex of degree four if there is such a vertex. Otherwise choose x', y' adjacent to be in two different trees, with x, y adjacent that must be in two different trees from before, for a total of three trees. Make x the center of the first wheel, with x', y' beloning the the last wheel. ■

Corollary 2 *Let G be a cubic planar bipartite graph, and suppose that the corresponding reduced graph H is triangulated, and may also have consecutive parallel edges, provided that every vertex is incident to an even number of pairs of consecutive parallel edges. Then H has a quasi spanning tree of faces, and therefore G has a Hamiltonian cycle.*

PROOF: Insert inside every pair of consecutive parallel edges an octahedron, so that the parallel edges are now non-consecutive with three new edges coming out of the two vertices joined by the parallel edges. Now the reduced graph has no consecutive parallel edges, so we may apply the preceding theorem to obtain a quasi spanning tree of triangles. This will correspond, back in the original reduced graph, to a quasi spanning tree of triangles and digons. ■

It is known that a cubic planar graph cannot have exactly two faces with an odd number of sides which are adjacent, with all other faces having an even number of sides ([8], p. 272). Applying this result to the dual graph, it follows that a triangulated graph cannot have exactly two vertices of odd degree which are adjacent. Consequently a reduced triangulated graph, all of whose vertices have even degree, cannot have just a single pair of consecutive parallel edges, since removing one of these consecutive parallel edges gives two adjacent vertices of odd degree. We consider the case where there are just two pairs of consecutive parallel edges.

Theorem 7 *Let G be a cubic planar bipartite graph, and suppose that the corresponding reduced graph H is triangulated, and has two special pairs of consecutive parallel edges such that each vertex has an even number of pairs of consecutive parallel edges that are not special incident to it. Then H has a quasi spanning tree of faces, and therefore G has a Hamiltonian cycle.*

PROOF: As in Corollary 2, insert inside every pair of consecutrive parallel edges that are not special an octahedron. A quasi spanning tree of faces for this reduced graph will give a quasi spanning tree of faces for the original reduced graph. We may thus assume that there are only two pairs of consecutive parallel edges in H.

If H has a digon that is not a face and does not contain either special digon face, then the region inside the digon is a reduced triangulated graph without digons, with vertices of even degree, since including both parallel edges in this region would give a single digon, which is not possible as observed before the statement of Theorem 7. The region inside this digon can be visited with two trees joined at the two endpoints of the digon as in Theorem 6. The digon region can then be replaced by a single edge and solved inductively.

If H has a triangle that is not a face and does not contain either special digon face, then the region inside the triangle is a reduced triangulated graph without digons, with vertices of even degree, since otherwise two vertices of the outer triangle giving this region would have odd degree which could be made of even degree by adding a parallel edge, giving a single digon which is not possible as observed before the statement of Theorem 7. We may then remove the region inside the triangle, solve the resulting smaller problem inductively, and solve the region inside the triangle with a single tree which includes the outside of the triangle if the triangle was not chosen, and does not include the outside of the triangle if the triangle was chosen.

After these transformations, we show that there are two spanning trees of faces, one containing both digons and the other one containing neither digon if the number of vertices is odd, and one containing exacltly one of the two digons for either digon if the number of vertices is even. The only possibility for a digon that is not a face is a digon that has one special digon inside and the other special digon outside. This digon that is not a face divides the problem into an inside problem and an outside problem, each with two special digons, and a solution that chooses this digon for the inside problem can be combined with a solution that does not choose this digon for the outside problem, and simililarly a solution that chooses this digon for the outside problem can be combined with a solution that does not choose this digon for the inside problem, inductively.

We solve the remaining case with two digon faces which has no digons that are not faces, and only has triangles that are not faces if they have one special digon inside and the other special digon.

See Figure 10. Let a_0a_1 be the two verices joined by the first pair of consecutive parallel edges, and let a_ka_{k+1} be two two vertices joined by the second pair of consecutive parallel

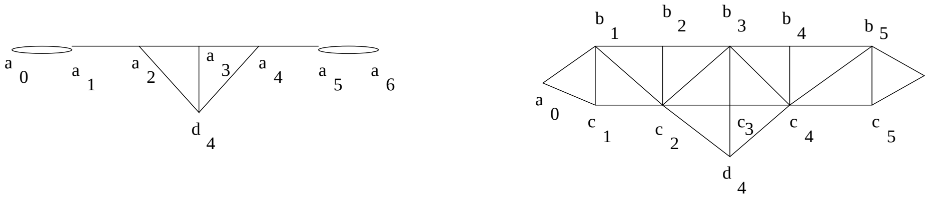

Figure 10: Transforming H with two digons into H' without digons.

edges. Let $a_1, a_2, \ldots, a_k$ be a path in H joining a_1 to a_k. We define an associated triangulated reduced graph H' having no consecutive parallel edges by replacing each a_i with $1 \leq i \leq k$ with two vertices b_i, c_i and edges a_0b_1, a_0c_1, b_ic_i, b_ib_{i+1}, c_ic_{i+1} for $1 \leq i < k$, b_kc_k, b_ka_{k+1}, c_ka_{k+1}. In addition, for each square $c_ib_ib_{i+1}c_{i+1}$, in increasing order of i, we include either the edge b_ic_{i+1} or the edge c_ib_{i+1} to obtain two triangles; the choice of which of these two edges to add is made in order to make the degrees of b_i and c_i even, since b_i and c_i have in addition the neighbors of a_i in either side of the path from a_0 to a_{k+1} respectively. The degrees of b_k and c_k will automatically be even, since otherwise we would have two adjacent vertices of odd degree with all other vertices of even degree in H', which is not possible as mentioned just before the Theorem.

We shall insist that the edge b_ic_{i+1} be chosen for i odd, and the edge c_ib_{i+1} be chosen for i even. This requires each vertex a_i for $1 < i < k$ to have an odd number of neighbors to each side of the path from a_1 to a_k. We shall later show that it is always to choose a path with this property, and with the additional property that no edge joins two vertices a_i, a_j with $j \geq i+2$ on this path.

Note that H' no longer has triangles that are not faces, since such triangles separated the two digons and when the vertices are split up they become squares.

The triangulated graph H' has no parallel edges. To obtain a spanning tree of faces for H, we shall find a spanning tree of faces for H' which in addition contains the edges b_ic_i for each $1 \leq i \leq k$. We may also use the digon a_0a_1 in the spanning tree of faces for H, which corresponds to selecting the triangle $a_0b_1c_1$ instead of the edge b_1c_1, and similary use the digon a_ka_{k+1} in the spanning tree of faces for H, which corresponds to selecting the triangle $b_kc_ka_{k+1}$ instead of the edge b_kc_k.

We can decompose H' into wheels as before, so that c_i is a center of wheel for i odd, and b_i is a center of wheel for i even. The faces of H' have two colors, say red and blue, so that each edge of H' separates a red face from a blue face; this follows from the fact that the faces of the original graph G were colored with three colors, and one of the three classes of faces was contracted. We shall assume that the triangle $a_0b_1c_1$ is red.

Given the region of wheels that is incrementally built by the algorithm, we shall assume that the on-edges delimit red triangles in the boundary of the region. We shall start with the wheel centered at c_1, and end with the wheel centered at b_k or c_k, adding the wheels centered at b_i or c_i at each step in increasing order of i.

When adding the wheel centered say at b_i, the boundary between this wheel and the previous region starts with the edge c_ib_{i-1}, where c_i is a single super-vertex joined to the center b_i by a selected edge. If there are no other super-vertices other than c_i and b_{i-1}, or more generally if there is an even number of super-vertices other than the two boundary super-vertices, then these super-vertices can be joined in pairs by triangles to the center b_i. If there is an odd number of such super-vertices, then we do not join the super-vertex for b_{i-1}, which is connected to the center c_{i-1} of a previous wheel; Instead, we select a red triangle $c_{i-1}c_id_i$ with $d_i \neq b_{i-1}$. We must ensure that the on-edge c_id_i is still in the boundary of the region. This is guaranteed by the fact that c_i is not connected to any center of previous wheels, because otherwise there would be an edge joining a_i, a_j with $j \geq i+2$ in H. Furthermore, any edge joining c_i to another vertex in the selected region is an on-edge, since c_id_i is an on-edge. Thus the addition of intermediate wheels not centered at b_i or c_i that may be required because a wheel touches the region twice or creates a cross-edge joining two vertices with an off-edge will not interfere with c_id_i being an on-edge in the boundary of the region at the time the wheel centered at b_i is added.

It only remains to construct the path $a_1, a_2, \ldots, a_k$ so that each a_i with $1 < i < k$ has an odd number of neighbors to each side of the path, and so that non-consecutive a_i on the path are not adjacent. We start by choosing the path to be a shortest path from a_1 to a_k. The non-adjacency of non-consecutive a_i is satisfied since the path is a shortest path. If some a_i has an even number of neighbors to each side of the path, then select one side of the path and replace the segment $a_ia_{i+1}a_{i+2}$ by the boundary half-wheel centered at a_{i+1} and going from a_i to a_{i+2} along the boundary of the wheel in that side of the path. The half-wheel centered at a_{i+1} may not have any common vertices with the half-wheel centered at a_j for $j \geq i+4$, because this would shorten the path from a_1 to a_k; if the half-wheel centered at a_{i+1} shares a vertex v with the haf-wheel centered at a_{i+3}, then we may modify the path from a_1 to a_k to remain a shortest path and go through v.

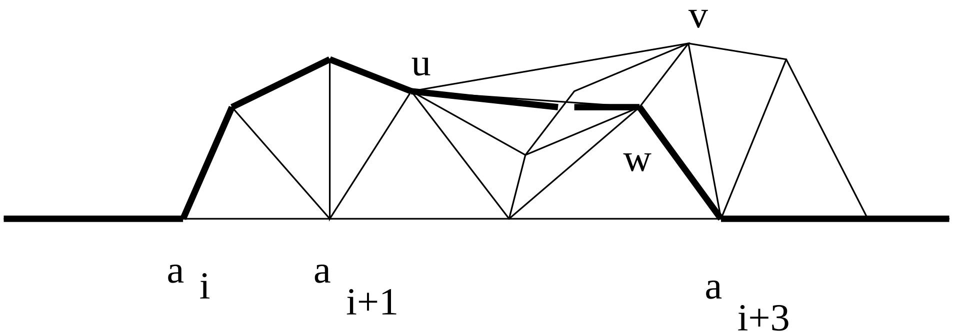

Figure 11: Construction of special path joining a_1 to a_k.

We must also guarantee that there is no edge uv joining two vertices in two such half-wheels centered at a_{i+1} and at a_j. See Figure 11. This is clear for $j \geq i+5$ since otherwise we would have a shorter path going through u and v; for $j = i+4$, we obtain instead a path of the same length going through u and v. For $j = i+3$, we may take the edge uv from u and continue from v either along the half-wheel centered at a_j or go directly to the center a_j; or if it is not possible to take the edge uv, then take from u the half-wheel centered at v in the direction towards the path from a_1 to a_k, until the vertex w preceding v in the half-wheel centered at a_j is reached, at which point we go from w to the center a_j.

This transformation guarantees that non-consecutive vertices on the path will not be adjacent, because vertices on the same wheel cannot be adjacent, since otherwise this adjacency together with the center of the wheel would form a triangle that is not a face.

The only cases remaining to be handled are the cases where a vertex u in a half-wheel centered at a_i on one side of the path is adjacent to some a_j with $j \neq i$ by an edge going to a_j the other side of the path from the side having the half-wheel containing u. If $j \leq i - 3$ or $j \geq i + 3$, then $a_j u a_i$ reduces the length of the shortest path, which is not possible. If $j = i - 2$ or $j = i - 1$, then we may choose the half-wheel centered at a_i on the other side of the shortest path, which will be separated from earlier wheels by the edge ua_j. We thus proceed from there on to select half-wheels on the other side of the shortest path. If $j = i+2$, then we can change the path from $a_i a_{i+1} a_{i+2}$ to $a_i u a_{i+2}$. The last case has $j = i + 1$. We may choose u to be the last vertex of the half-wheel centered at a_i that is adjacent to a_{i+1} by going to the other side of the wheel. There is then at least one edge coming out of u between the edge joining u to its successor in the half-wheel and the edge joining u to a_{i+1}. We may then proceed directly from u to a_{i+1}, skipping the rest of the half-wheel, or if the parity of the number of edges on either side of the path does not so work, choose the next edge coming out of u, and continue along the half-wheel centered at a_{i+1} on the other side of the path from the half-wheel centered at a_i, until a_{i+2} is reached. We thus proceed from there on to select half-wheels on the other side of the shortest path. This completes the proof. ■

5 NP-completeness

In this section we obtain several NP-completeness results.

Theorem 8 *Let G be a 3-connected cubic planar bipartite graph, and let H be the corresponding reduced graph. Suppose that the red faces in H are quadrilaterals and the blue faces in H are digons. Then the question of whether H has a spanning tree of red faces is NP-complete.*

PROOF: The question of whether a 3-connected planar cubic graph R has a Hamiltonian cycle is NP-complete, as shown by Garey et al. [5]. Let $e = (u, v)$ be a particular edge in R. Then the question of whether R has a Hamiltonian cycle going through e is also NP-complete. Let R' be the graph obtained from R by removing the edge e, so that R' has two vertices of degree 2, namely u and v, and all other vertices of R' have degree 3. The question of whether R' has a Hamiltonian path from u to v is also NP-complete.

Given R', construct a reduced graph H of some 3-connected cubic planar bipartite graph G as follows. The graph H has vertices $V = V_1 \cup V_2$, where V_1 is the set of vertices w of R' and V_2 is the set of faces f of R'. A vertex w in V_1 is joined to a vertex f in V_2 by two parallel edges in H if and only if the vertex w in R' is a vertex in the face f of R'. Thus the blue faces are digons joining two vertices w and f, while the red faces are quadrilaterals (w, f, w', f') corresponding to edges (w, w') in R' that separate two faces f and f' in R'.

We show that a set L of edges in R' forms a Hamiltonian path from u to v in R' if and only if the set M of red quadrilaterals in H corresponding to the edges M' in R' that are not in L has the property that M is a spanning tree of red faces in H. Therefore H has a spanning tree of red faces if and only if R' has a Hamiltonian path from u to v, and so the question of whether H has a spanning tree of red faces is NP-complete.

Suppose L is a Hamiltonian path from u to v in R', let M' be the edges in R' not in L, and let M be the corresponding red quadrilaterals in H. Note that for any two edges g and h in M', there is a sequence of edges $g = e_1, e_2, \ldots, e_k = h$ in M' such that each pair of edges e_i, e_{i+1} share a face. Therefore the red faces in M induce a connected subgraph of H. Notice also that every vertex in H belongs to some face in M, since every vertex w in R' is incident to an edge not in L, and every face f in R' has at least one edge not in L. Finally, the red faces in M do not contain a cycle. Otherwise, if we had a cycle $u_1, u_2, \ldots, u_k, u_1$ of red faces in M, then we can observe that every vertex w in V_1 belongs to exactly one red face in M, so u_i and u_{i+1} share a vertex f in V_2 corresponding to a face in R'. Thus the edges $e_1, e_2, \ldots, e_k, e_1$ in M', corresponding to the faces u_i in the cycle of red faces in M, separate the graph R' into two components, so the Hamiltonian path L would have to contain at least one of these edges e_i in M', a contradiction. Therefore the red faces in M form a spanning tree of red faces for H.

In the other direction, suppose the red faces in M form a spanning tree of red faces for H. Let M' be the corresponding edges in R', and let L be the set of edges in R' not in M'. Each vertex w in V_1 belongs to exactly one red face (w, f, w', f') in M, since every other red face in M containing w also contains either f or f', and therefore the two red faces form a cycle and cannot both be in the spanning tree of red faces M. Therefore every vertex in R' is incident to exactly one edge in M', and so the two vertices u and v of degree 2 in R' are incident to exactly one edge in L, while the remaining vertices w of degree 3 in R' are incident to exactly two edges in L. That is, L consists of a path joining u and v in R', plus a collection of cycles in R', such that the path and the cycles are disjoint and cover all the vertices in R'. We show that L cannot contain a cycle in R', and so L is just a path joining u to v containing all vertices of R', that is, L is a Hamiltonian path from u to v in R'. Suppose L contains a cycle $e_1, e_2, \ldots, e_k, e_1$ in R'. Let f and f' be faces of R' inside and outside the cycle of e_i respectively. Since f and f' are vertices in the spanning tree of red faces M, there is a sequence of red faces $u_1, u_2, \ldots, u_l$ in M such that u_1 contains f, u_l contains f' and each pair u_{i-1}, u_i share a vertex f_i in V_2. In particular, if we denote $f_0 = f$ and $f_l = f'$, then for some pair f_i, f_{i+1} we must have the face f_i in R' inside and the face f_{i+1} outside the cycle of e_i. This implies that the red face u_i in M corresponds to one of the edges e_i in L and not in M', a contradiction. This completes the proof. ■

We obtain two Corollaries from this result.

Corollary 3 *Let G be a 3-connected cubic planar bipartite graph, and let H be the corresponding reduced graph. Suppose all vertices of H have degree eight. Then the question of whether H has a noncrossing Euler tour is NP-complete.*

Proof: The reduced graphs H from Theorem 8 have all red faces as quadrilaterals, corresponding to octagons in G. If we collapse these red faces, we obtain a reduced graph H' with vertices of degree 8. By Proposition 1, H' has a noncrossing Euler tour if and only if H has a spanning tree of red faces, and this problem is NP-complete by Theorem 8. ■

Corollary 4 *Let G be a 3-connected cubic planar bipartite graph, and let H be the corresponding reduced graph. Suppose that the red faces in H are octagons and digons and the blue faces in H are triangles. Then the question of whether H has a spanning tree of arbitrary faces is NP-complete.*

PROOF: Let H be the reduced graph of Theorem 8, with red quadrilaterals and blue digons. If e and f are the two parallel edges of a blue digon, insert a vertex w in the middle of e and a vertex x in the middle of f, with w and x joined by two parallel edges. The blue digon splits thus into two blue triangles and a red digon, while the red quadrilaterals become red octagons, in the new reduced graph H'.

Suppose H has a spanning tree of red quadrilaterals M. Select the corresponding red octagons in H'. For a blue digon consisting of two edges e and f in H, if one of the two red quadrilaterals containing e or f is in M, then select the red digon joining the middle vertices w and x; if neither of the two red quadrilaterals containing e or f is in M, then select one of the two blue triangles containing w and x. The red and blue faces in H' thus selected, involving red octagons, red digons, and blue triangles, for a spanning tree of faces in H'.

Conversely, suppose H' has a spanning tree of faces M'. Let M be the set of red quadrilaterals in H such that the corresponding red octagon is in M'. Note that for each digon in H, only one of the corresponding two blue triangles and red digon in H' can be in M'. Thus M is a spanning tree of red faces.

Thus H' has a spanning tree of arbitrary faces if and only if H has a spanning tree of red faces, and NP-completeness follows from Theorem 8. ■

We finally show:

Theorem 9 *Suppose there exists a 3-connected cubic planar bipartite graph G_0 that is not Hamiltonian. Then the question of whether a 3-connected cubic planar bipartite graph G has a Hamiltonian cycle is NP-complete.*

PROOF: Takanori et al. [12]. showed that the question of whether a 2-connected cubic planar bipartite graph G has a Hamiltonian cycle is NP-complete. If such a G has two edges e and f that separate it into two components G' and G'', then their endpoints in either side are at odd distance by Theorem 2, so we may instead join the two endpoints of e and f in G' and G'' separately, and ask whether G' and G'' both contain a Hamiltonian cycle containing the added edge joining the endpoints of e and f. Repeating this decomposition process, we eventually reduce the question of whether G has a Hamiltonian cycle to the question of whether various G_i each contain a Hamiltonian cycle going through certain prespecified edges, with each G_i being 3-connected. Thus the question of whether a 3-connected cubic planar bipartite graph G has a Hamiltonian cycle going through certain prespecified edges is NP-complete.

Suppose G_0 is a minimal 3-connected cubic planar bipartite graph that is not Hamiltonian. We shall construct from G_0 a 3-connected cubic planar bipartite graph G_1 that has a Hamiltonian cycle, such that for some edge $e = (u, v)$ in G_1, every Hamiltonian cycle in G_1 goes through e. Furthermore, If f and g are the two edges other than e incident to u in G_1, then G_1 has a Hamiltonian cycle going through f and e and a Hamiltonian cycle going through g and e. Given a graph G with certain prespecified edges that a Hamiltonian cycle must go through, consider each such prespecified edge e' incident to a vertex u', and replace u' with the complement of u in G_1, so that the three edges incident to u' in G are replaced by the three edges incident to u in G, with e' corresponding to e in G_1. These replacements then force a Hamiltonian cycle for the new graph to correspond to a Hamiltonian cycle going through the prespecified edges e' in G. Therefore whether the resulting 3-connected cubic planar bipartite graph has a Hamiltonian cycle is NP-complete.

We construct G_1. We shall first do so to enforce that every Hamiltonian cycle in G_1 goes through e, and later guarantee that such a cycle can go through either f or g. Let R be a square in G_0. Suppose that if we remove two opposing sides of R from G_0 and replace the two paths of length 3 resulting from the remaining two sides of R by single edges, then we obtain a 3-connected graph G_1. By minimality of G_0, the graph G_1 has a Hamiltonian cycle. Furthermore, no Hamiltonian cycle for G_1 goes through either of the two edges of G_1 corresponding to the edges that were kept for the square R in G_0, since if either edge is in a Hamiltonian cycle for G_1, then we can extend this cycle to a Hamiltonian cycle visiting the four vertices of the square R in G_0, a contradiction. We have thus guaranteed that a Hamiltonian cycle in G_1 does not visit a particular edge e_0, and thus must visit an edge e adjacent to e_0, as desired.

Suppose instead that the two graphs G_1 obtained in the preceding construction by either of the two choices of two opposing edges of the square R to be removed are both only 2-connected. Then each choice of two opposing edges of R belongs to a cut of four edges in G_0 that separate G_0. Removing the two sets of four edges from the two cuts of four edges thus separates G_0 into four components C_1, C_2, C_3, C_4, with the removed edges of G_0 including an edge from C_1 to C_2, an edge from C_2 to C_3, an edge from C_3 to C_4, and an edge from C_4 to C_1, plus four edges from the four C_i going into the four vertices of the square R. That is, each C_i has three incoming edges that can be joined to a single vertex, since their three endpoints in C_i are at even distance by Theorem 2. By minimality of G_0 each such graph resulting from C_i with an additional vertex has a Hamiltonian cycle, yet it is not the case that each of the three choices of two edges going into each C_i yields a Hamiltonian cycle, since otherwise we would obtain a Hamiltonian cycle for G_0. Thus one of the three edges joining some C_i to the additional vertex must belong to every Hamiltonian cycle, thus giving a 3-connected cubic planar bipartite graph G_1 with an edge e that belongs to every Hamiltonian cycle of G_1.

It remains to ensure that a Hamiltonian cycle in G_1, which is forced to take $e = (u, v)$, can take either f or g out of u. Suppose instead that every Hamiltonian cycle is forced to take f as well. Consider the 3-connected cubic planar bipartite graph K of a cube with 8 vertices. Replace as before a vertex of K with the complement of u in G_1. This forces two particular edges e and f of the cube incident to the replaced vertex to be visited by a Hamiltonian cycle, yet the Hamiltonian cycle for the cube can still be chosen in two different ways so that either of the two edges f' and g' sharing the endpoint of e that was not replaced can be visited. This produces the required G'_1 with edge e forced and the choice between f' and g' still available for a Hamiltonian cycle, completing the proof. ■

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