On Barnette's Conjecture

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May 14, 2009

Abstract

Barnette's conjecture is the statement that every 3-connected cubic planar bipartite graph is Hamiltonian. Goodey showed that the conjecture holds when all faces of the graph have either 4 or 6 sides. We generalize Goodey's result by showing that when the faces of such a graph are 3-colored, with adjacent faces having different colors, if two of the three color classes contain only faces with either 4 or 6 sides, then the conjecture holds. More generally, we consider 3-connected cubic planar graphs that are not necessarily bipartite, and show that if the faces of such a graph are 2-colored, with every vertex incident to one blue face and two red faces, and all red faces have either 4 or 6 sides, while the blue faces are arbitrary, provided that blue faces with either 3 or 5 sides are adjacent to a red face with 4 sides (but without any assumption on blue faces with $4, 6, 7, 8, 9, \ldots$ sides), then the graph is Hamiltonian. The approach is to consider the reduced graph obtained by contracting each blue face to a single vertex, so that the reduced graph has faces corresponding to the original red faces and with either 2 or 3 sides, and to show that such a reduced graph always contains a proper quasi spanning tree of faces. In general, for a reduced graph with arbitrary faces, we give a polynomial time algorithm based on spanning tree parity to decide if the reduced graph has a spanning tree of faces having 2 or 3 sides, while to decide if the reduced graph has a spanning tree of faces with 4 sides or of arbitrary faces is NP-complete for reduced graphs of even degree. As a corollary, we show that whether a reduced graph has a noncrossing Euler tour has a polynomial time algorithm if all vertices have degree 4 or 6, but is NP-complete if all vertices have degree 8. Finally, we show that if Barnette's conjecture is false, then the question of whether a graph in the class of the conjecture is Hamiltonian is NP-complete.

1 Introduction

Let P be the class of 3-connected cubic planar bipartite graphs. Barnette [2] conjectured that every graph in P is Hamiltonian. This conjecture was verified for graphs with up to 64 vertices by Holton, Manvel and McKay [8]. Aldred, Brinkmann and McKay [1] have announced that the conjecture still holds for graphs with up to 84 vertices. The conjecture

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also holds for the infinite family of graphs where all faces have either 4 or 6 sides, as shown by Goodey [6]. Without the assumption of 3-connectivity, it is NP-complete to decide whether a graph in P is Hamiltonian, as shown by Takanori, Takao and Nobuji [11]. The same conjecture without the assumption of bipartiteness was originally formulated by Tait [10] and disproved by Tutte [12].

We begin our study of the class P with the well known fact that the faces of any graph in P can be colored with three colors. In particular, we get a set of independent faces C such that any edge of the graph either joins two faces of C or belongs to a face in C. The independent faces in C for a graph in P can be collapsed to single vertices, thus giving a reduced graph which is planar and has vertices of even degree at least four, and may have parallel edges.

We observe that an Eulerian tour of the reduced graph without crossings will give a Hamiltonian circuit of the original graph. Such an Eulerian tour corresponds, for a different choice of the independent faces C to be collapsed, to a spanning tree of faces. In fact any spanning tree of faces in the reduced graph will give a Hamiltonian circuit for the original graph, and the same holds when a quasi spanning tree of faces is found in the reduced graph.

We may extend the class P to include 3-connected cubic planar graphs that are not necessarily bipartite, but which contain a set of independent faces C such that any edge of the graph either joins two faces of C or belongs to a face in C, or equivalently, every vertex of the graph is incident to one face in C and two faces not in C. For this extended class P, we may again collapse the faces in C to single vertices to obtain a corresponding reduced graph, and look for spanning trees or quasi spanning trees of faces to give a Hamiltonian circuit for the original graph.

We refer to the faces in C as blue faces and to the faces not in C as red faces. We show that if a graph in the extended class P has red faces with either 4 or 6 sides, and arbitrary blue faces, provided that a blue face with either 3 or 5 sides is ajacent to at least one red face with 4 sides (but without any assumption for blue faces with $4, 6, 7, 8, 9, \ldots$ sides), then the graph is Hamiltonian. The proof shows that the corresponding reduced graph, with faces corresponding to the original red faces, and thus having either 2 or 3 sides, has a proper quasi spanning tree of faces. As a special case, if the faces of a graph in the original class P of bipartite graphs are 3-colored, so that each vertex is incident to one face of each color, and two of the three color classes contain only faces with 4 or 6 sides, then the graph is Hamiltonian, as we may let C be the third color class, containing faces with $4, 6, 8, 10, \ldots$ sides.

We consider more generally the problem of finding a spanning tree or a quasi spanning tree of faces for any reduced graph either in the original or in the extended class P, as an approach to finding a Hamiltonian cycle for the original graph. We show that the problem of deciding whether a reduced graph has a spanning tree of faces that have either 2 or 3 sides can be solved in polynomial time via a reduction to the spanning tree parity problem, which is polynomial as shown by Lovász [9] (see also Gabow and Stellman [4]). This implies that one can decide in polynomial time whether a reduced graph with vertices of degrees 4 and 6 has a noncrossing Eulerian tour. By contrast, we show that deciding whether a reduced graph has a spanning tree of faces with 4 sides, or a spanning tree of arbitrary faces, is NP-complete, for reduced graphs of even degree (corresponding to graphs in the original class P of bipartite graphs). This implies that it is NP-complete to decide whether a reduced graph with vertices of degree 8 as a noncrossing Euler tour. Finally, we show that if Barnette's conjecture is false, so that a graph in the original class P of bipartite graphs is not Hamiltonian, then deciding whether a graph in the original class P is Hamiltonian is NP-complete.

An open problem is whether one may remove the assumption that every blue face with 3 or 5 is adjacent to at least one red face with 4 sides, so that all red faces having 4 or 6 sides would be enough to obtain a Hamiltonian circuit for graphs in the extended class P. A related conjecture of Barnette states that if a 3-connected cubic planar graph only has faces with 3,4,5, or 6 sides, then it is Hamiltonian. As mentioned before, this was verified by Goodey [6] for the case of faces with 4 or 6 sides; it was also verified by Goodey [7] for the case of faces with 3 or 6 sides; and in general by Brinkmann, McKay, and von Nathusius [3] for graphs with up to 250 vertices.

2 Hamiltonian Cycle from Quasi Spanning Tree of Faces

Let G be a graph in the class of 3-connected cubic planar graphs that has a set C of faces such that every vertex in G is incident to one face in C and to two faces not in C. We refer to the faces in C as blue faces and to the faces not in C as red faces. Let H be the corresponding reduced graph obtained by contracting the faces in C to single vertices. A *spanning tree of faces* in H is a set D of faces of H such that no two faces in D share an edge, and such that if we let T be the graph with vertices corresponding to vertices in H and faces in D, and edges joining the vertices corresponding to faces d in D to the vertices in H incident to d, then T is a tree. A *quasi spanning tree of faces* in H is a set D of faces of H and a set V of vertices in H such that no two faces in D share an edge, every vertex of H not in V has even degree, say $2r$, and is surrounded by r faces in D, and such that if we let T be the graph with vertices corresponding to vertices in V and faces in D, and edges joining the vertices corresponding to faces d in D to the vertices in V incident to d, then T is a tree. The vertices of H not in V are called *quasi vertices*, and a *proper quasi vertex* is a quasi vertex of degree 4 such that none of the 4 faces surrounding it is a digon (i.e., has only two sides). A *proper quasi spanning tree of faces* is a quasi spanning tree of faces all of whose quasi vertices are proper quasi vertices.

Given a spanning tree of faces in H, we may assume the external face is not in D, and traverse the perimeter of the spanning tree of faces, to obtain a Hamiltonian cycle in G that has all faces of the collapsed set C inside. Given a quasi spanning tree of faces in H, we may assume the external face is not in D, and traverse the perimeter of the quasi spanning tree of faces, to obtain a Hamiltonian cycle in G such that the faces of the collaped C are inside the cycle for vertices in V and outside the cycle for vertices not in V (quasi vertices). This gives the following.

Proposition 1 *The reduced graph H has a spanning tree of faces with the external face not in D if and only if G has a Hamiltonian cycle with the external red face outside, with all blue faces inside and such that no two red faces sharing an edge are both inside. The reduced graph H has a quasi spanning tree of faces with the external face not in D if and only if G*

has a Hamiltonian cycle with the external red face outside, with all blue faces corresponding to vertices in V inside, with all blue faces corresponding to vertices not in V (quasi vertices) outside, and such that no two red faces sharing an edge are both inside.

We now prove the main result of this section.

Theorem 1 *Suppose all red faces have either 4 or 6 sides, while the blue faces are arbitrary, and assume also that blue faces with 3 or 5 sides are adjacent to at least one red face with 4 sides (no assumption is made for blue faces with $4, 6, 7, 8, 9, \ldots$ sides). Then the reduced graph H obtained by collapsing blue faces has a proper quasi spanning tree of faces, giving a Hamiltonian cycle for G.*

Suppose all red faces of G have either 4 or 6 sides, while the blue faces are arbitrary. Suppose the reduced graph H has a triangle T that contains at least one vertex inside, such that no triangle inside of T is not a face (i.e., contains at least one vertex inside), and no digon inside of T is not a face (i.e., contains at least one vertex inside).

We shall successively simplify the inside of the triangle T, while preserving the property that there is no digon inside of T that is not a face, but allowing the presence of triangles inside of T that are not faces, but with the following requirement. Treat all sets of parallel edges as a single edge. Suppose T_1 and T_2 are distinct triangles inside of T, with T_1 containing T_2 and possibly T_1 equal to T, where T_2 is not a face, and such that there is no triangle T_3 distinct from T_1 and T_2 such that T_1 contains T_3 and T_3 contains T_2. In that case, we say that T_2 is a *child* of T_1. We shall require that no triangle T_1 has three distinct children T_2, T_2', and T_2'', at all steps in the simplification of the inside of the triangle T.

We define the *invariant property for T* to be that there is no digon inside of T that is not a face, and no triangle inside T has three distinct children.

Lemma 1 *If T has at least two vertices inside and satisfies the invariant property, then it is possible to select a triangle T' that is a face inside of T and collapse T' to a single vertex in such a way that T still satisfies the invariant property.*

Proof: Suppose triangle T_1 inside of T contains at least two vertices inside, and there is no triangle inside of T_1 that is not a face. Writing $T_1 = v_1v_2v_3$, we claim that v_1 has at least two distinct neighbors v_4 and v_5 inside of T_1. Otherwise, if v_1 has no such neighbors, then v_1 belongs to a triangle inside of T_1 that has an edge v_2v_3 parallel to the side of T_1, contrary to the assumption that there is no digon inside of T that is not a face; and if v_1 has only one such neighbor v_4 inside of T_1, then $v_2v_3v_4$ is a triangle inside of T_1 that is not a face, contrary to assumption. We may then choose v_4 and v_5 so that v_2, v_4, v_5 are consecutive neighbors of v_1, and collapse the triangle $v_1v_4v_5$. This will produce no digons that are not faces, since such a digon would come before the collapsing from a triangle that is not a face inside of T_1, contrary to assumption. There may however appear triangles that are not faces inside of T_1. Such triangles come from quadrilaterals $v_1v_4v_6v_7$, $v_1v_5v_8v_9$, and $v_4v_5v_{10}v_{11}$. The quadrilaterals $v_1v_5v_8v_9$ are of two kinds, either containing v_4 or not containing v_4, but may not have diagonal edges v_1v_8 or v_5v_9, otherwise either there was a triangle that is not a face inside of the quadrilateral, or collapsing the side v_1v_5 does not give for the quadrilateral a triangle that is not a face. This implies that all such quadrilaterals containing v_4 are pairwise contained in each other, and all such quadrilaterals not containing v_4 are pairwise contained in each other. The analogous properties hold for the quadrilaterals $v_1v_4v_6v_7$, but these are of only one kind, namely containing v_5, otherwise $v_6 = v_2$ and we have the diagonal edge v_2v_4. The analogous properties also hold for the quadrilaterals $v_4v_5v_{10}v_{11}$, but these are again of only one kind, namely not containing v_1, since they are contained in the triangle $T_1 = v_1v_2v_3$. Furthermore, a quadrilateral $v_1v_4v_6v_7$ containing v_5 must contain any quadrilateral $v_1v_5v_8v_9$ not containing v_4 and must also contain any quadrilateral $v_4v_5v_{10}v_{11}$ not containing v_1, and a quadrilateral $v_1v_5v_8v_9$ containing v_4 must also contain any quadrilateral $v_4v_5v_{10}v_{11}$ not containing v_1. This guarantees that these quadrilaterals will not lead, after collapsing $v_1v_4v_5$, to three triangles that are not faces that do not contain each other inside T_1, thus preserving the property that no triangle has three children.

In the remaining case for collapsing a triangle, there is a triangle T_1 that has either one child T_2 or two children T_2 and T_3, where both T_2 and T_3 have exactly one vertex inside. Suppose T_2 shares no sides with either T_1 or T_3. Writing $T_2 = v_1v_2v_3$, we must again consider quadrilaterals $v_1v_2v_4v_5$, $v_1v_3v_6v_7$, and $v_2v_3v_8v_9$. There may not simulataneously exist quadrilaterals $v_1v_2v_4v_5$ containing v_3, $v_1v_3v_6v_7$ containing v_2, $v_2v_3v_8v_9$ containing v_1, and $v_1v_2v_4'v_5'$ not containing v_3. For if $v_6 = v_5$, then $v_1v_5v_7$ is not a face and thus equals T_1, so v_1 is a vertex of T_1 and the quadrilateral $v_2v_3v_8v_9$ cannot contain v_1; if $v_7 = v_4$ then $v_1v_5v_4$ is again T_1 and the same argument holds; and if $v_6 = v_4$, then $v_8 = v_5$ and $v_9 = v_7$, so the triangle $v_5v_4v_7$ is T_1, and the quadrilateral $v_1v_2v_4'v_5'$ would be inside the triangle $v_1v_2v_7$ which is a face, and this is not possible. Therefore, by symmetry, we may assume that either there is no quadrilateral $v_1v_2v_4v_5$ containing v_3 or no quadrilateral $v_1v_2v_4v_5$ not containing v_3 that will give rise to a new triangle that is not a face after identifying v_1 and v_2. Thus if $v_1v_2v_3$ contains the single vertex v_0, collapsing the triangle $v_1v_2v_0$ identifies v_1 and v_2 and creates only triangles with pairwise containment involving the new vertex $v_1 = v_2$, besides the triangle T_3, thus preserving the property that no triangle has three children. If $T_2 = v_1v_2v_3$ shares one side with T_1, say the side v_2v_3, then one of the other two sides is not shared with T_3, say the side v_1v_2, and the quadrilaterals $v_1v_2v_4v_5$ cannot contain v_3, so again we may collapse the triangle $v_1v_2v_0$ with v_0 inside T_2, creating only triangles with pairwise containment involving the new vertex $v_1 = v_2$, besides the triangle T_3, thus preserving the property that no triangle has three children. And if T_2 shares a side v_1v_3 with T_3, then every quadrilateral $v_1v_2v_4v_5$ containing v_3 also contains T_3, so collapsing $v_1v_2v_0$ with v_0 inside T_2 gives two familes of triangles with pairwise containments involving $v_1 = v_2$, one family containing v_3 and the other family not containing v_3, again preserving the property that no triangle has three children. ∎

The following Proposition generalizes a result of Herbert Fleischner.

Proposition 2 *Suppose all red faces of G have either 4 or 6 sides, while the blue faces are arbitrary. Suppose the reduced graph H has no triangle that is not a face other than the outer triangle, and H has no digon that is not a face either. Suppose H has an odd number of vertices. Then H has a spanning tree of faces that are triangles, and so G is Hamiltonian.*

Proof: Apply Lemma 1 repeatedly to collapse triangle faces to single vertices while preserving the invariant property. Each step reduces the number of vertices by two, so this number remains odd until we are left with just the outer face. The collapsed triangles form a spanning tree of faces. ∎

This result follows from the following main observation.

Lemma 2 *Let G be as in Theorem 1. Suppose the reduced graph H has a triangle T that contains at least one vertex inside, such that no triangle inside of T is not a face (i.e., contains at least one vertex inside), and no digon inside of T is not a face (i.e., contains at least one vertex inside). Then finding a proper quasi spanning tree of faces for H reduces to finding a proper quasi spanning tree of faces for H' obtained from H by removing all vertices inside of T and their incident edges, and adding a parallel edge inside of T to each edge of T.*

Proof: We may proceed with a triangle T as in the preceeding Lemma, and end up with either a single vertex v inside of T or no vertex inside of T by repeatedly collapsing triangle faces. In the case of a single vertex v inside of T, selecting one of the three triangles involving v corresponds to selecting one of the three digons added for the sides of T in H' for a quasi spanning tree of faces, and in the case of no vertex v inside of T, we may either select or not select the triangle T in H' for a quasi spanning tree of faces. The case of a single vertex v inside of T is reached when T initially contains an odd number of vertices inside of T, and the case of no vertex v inside of T is reached when T initially contains an even number of vertices inside of T.

It remains to show the two cases that are used to change the parity inside of T. If there is initially a digon v_1v_2 with at least one endpoint inside of T, then we may select and collapse v_1v_2, creating triangles that are not faces from quadrilaterals $v_1v_2v_4v_5$, and there are again two families of such quadrilaterals, given by the two triangle faces $v_1v_2v_3$ and $v_1v_2v_3'$, namely quadrilaterals containing v_3 and quadrilaterals containing v_3'. Each of the two families of quadrilaterals creates triangles with pairwise containment, thus giving the property that no triangle T_1 either equal to T or inside of T has three children, satisfying the invariant property.

If there is no such digon v_1v_2 , then we shall show that there is inside of T a vertex v of degree 4, belonging to 4 triangles at most one of which shares a side with T. We then make v a proper quasi vertex and select two triangles involving v that do not share an edge with each other or with T, say vv_1v_2 and vv_3v_4, to be included in the proper quasi spanning tree of faces. Collapsing both triangles corresponds to removing v, identifying v_1 with v_2, and identifying v_3 with v_4, thus replacing 5 vertices with just 2 vertices and changing the parity of the number of vertices inside of T. This will thus complete the proof of the Lemma.

If initially there are no such digons inside of T, and we write $T = v_1v_2v_3$, then there must be at least two vertices inside T, otherwise the single vertex v_0 inside T would have degree 3 and would be incident to no digons, contrary to the assumption that every blue face with 3 sides is adjacent to at least one red face with 4 sides. This implies that v_1 must have at least two distinct neighbors inside T, otherwise a single neighbor v_0 would be the only vertex inside of T because there are no triangles that are not faces inside of T. Thus v_1 has degree at least 4, if we also count the edges v_1v_2 and v_2v_3, and similarly both v_2 and v_3 have degree at least 4. Furthermore, there is no vertex inside of T of degree either 3 or 5, since such a vertex would be incident to a digon because all blue faces with 3 or 5 sides are adjacent to at least one red face with 4 sides. Therefore, by Euler's formula, the total number of vertices of degree 4 either on T or inside of T is at least 6, and there are at least 3 vertices of degree

4 inside T. Say v_0 inside of T of degree 4 has consecutive neighbors $v_4v_5v_6v_7$. At most one edge of this quadrilateral can be shared with the triangle $T = v_1v_2v_3$, because if both v_1v_2 and v_1v_3 are shared then v_1 has a single neighbor v_0 inside T and would have only degree 3, not at least 4. Say only v_4v_7 can be shared with T. In that case, we make v_0 a proper quasi vertex and select the two triangles $v_0v_4v_5$ and $v_0v_6v_7$, thus removing v_0, identifying v_4 and v_5, and identifying v_6 and v_7, reducing the number of vertices by 3. The quadrilaterals $v_4v_5v_8v_9$ containing the edge v_6v_7 that yield new triangles must contain the quadrilaterals $v_6v_7v_{10}v_{11}$ that yield new triangles not containing the edge v_4v_5, and in the other direction the quadrilaterals $v_6v_7v'_{10}v'_{11}$ containing the edge v_4v_5 that yield new triangles must contain the quadrilaterals $v_4v_5v'_8v'_9$ that yield new triangles not containing the edge v_6v_7, so again we obtain just two families of new triangles by identification of v_4 and v_5, and identification of v_6 and v_7, with each family giving pairwise containment among its triangles.

This gives again the property that no triangle T_1 either equal to T or inside of T has three children, thus completing the proof that one can change the parity of the number of vertices inside of T before proceeding to reduce the number of vertices inside of T by increments of 2 until either a single vertex or no vertex is inside of T. As argued above, this reduces the problem of finding a proper quasi spanning tree of faces for H to the problem with the vertices inside of H removed and parallel edges added to the sides of T to obtain H'. ■

This Lemma yields Theorem 1 as follows. The outer face is a digon or a triangle that contains vertices inside. There must therefore exist either a triangle that contains vertices inside but contains no triangle or digon with vertices inside, or a digon that contains vertices inside but contains no triangle or digon with vertices inside. A triangle T that contains vertices inside but contains no triangle or digon with vertices inside can be simplified according to Lemma 1 by removing the vertices inside and adding parallel edges to the sides of T. A digon v_1v_2 that contains vertices inside but contains no triangle or digon with vertices inside must contain a single vertex v_0 inside, otherwise the digon v_1v_2 would contain a triangle with vertices inside. Either v_0v_1 or v_0v_2 must be a digon, say v_0v_1 is a digon, otherwise v_0 would have just degree 2. We may thus either select the digon v_0v_1 or the triangle $v_0v_1v_2$, which corresponds after removing the vertex v_0 to either not selecting or selecting the digon v_1v_2 which has become a face. The graph H can thus be simplified until the outer face contains no vertices inside, in which case selecting the face inside involving all vertices completes the proper quasi spanning tree of faces for H and the proof of Theorem 1.

The following Corollary is a special case of Theorem 1 and generalizes Goodey's result for graphs G that only have faces with 4 or 6 sides.

Corollary 1 *Suppose G is a 3-connnected cubic planar bipartite graph, and if the faces of G are 3-colored, with each vertex of G incident to one face of each color, then two of the three color classes contain only faces that have 4 or 6 sides. Then the reduced graph H obtained by collapsing the third color class has a proper quasi spanning tree of faces, and so G has a Hamiltonian cycle.*

3 Polynomial and NP-Complete Problems

The following result concerns the case where most faces in a spanning tree of faces are digons.

Theorem 2 *Let G be a 3-connected cubic planar bipartite graph. Let H be the reduced graph for G, and let H' be the subgraph of H obtained by removing all edges that do not have consecutive parallel edges. If H' has one, two, or three connected components, then H has a spanning tree of faces, and thus G has a Hamiltonian cycle. The case of a single component for H' includes the case where all faces in one of the three color classes are squares.*

Proof: If H' is a single connected component, then we can choose a spanning tree of H', corresponding to a spanning tree of digons in H.

If H' has two connected components, then we may choose a face f of H that has vertices from both components. Starting with this face f, we also consider two spanning trees of digons for the two components of H', and add these digons one at a time as long as they do not form a cycle containing f. Eventually, the single face f and the added digons will span H.

If H' has three connected components, then it may be that H has a face f touching all three components, and we may proceed from f as for the case for two components, by considering the three spanning trees of digons for the two components. Otherwise some component, say the first, has faces touching it and the second component and also faces touching it and the third component. Both sets of faces have at least four faces, since a cut of H has at least four edges by 3-connectivity and the fact that any cut has an even number of edges, so we may choose a face f touching the first and second component, and a face f' touching the first and third component, so that these two faces do not share any vertices. Starting with these two faces, we may then again add digons from the three spanning trees for the three components so long as they do not form a cycle, until a spanning tree of faces for H is obtained. ■

The proof for three connected components extends to the case of four connected components, but the result does not hold in the case of five connected components.

We show next that one can decide whether the reduced graph H has a spanning tree of faces that are either digons or triangles in polynomial time. The result easily extends to the case of a spanning tree of faces where all but a constant number of faces are either digons or triangles.

Theorem 3 *Let G be a 3-connected cubic planar graph that has a corresponding reduced graph H obtained by collapsing a set of faces C. Let D be a collection of faces in H such that all faces in D are either digons or triangles. Then we can decide whether H has a spanning tree of faces in D, giving a Hamiltonian cycle for G, in polynomial time, by a spanning tree parity algorithm [9, 4].*

Proof: Construct a graph H' related to H as follows. The vertex set of H' is the same as the vertex set for H. If xy is a digon in D, then put an edge xy in H'. If xyz is a triangle in D, then put edges xy and yz in H'. A spanning tree of faces in D for H then corresponds to a standard spanning tree in H' which must contain either both or none of edges xy and yz corresponding to a triangle xyz in D. These conditions on pairs of edges in H' make the equivalent problem in H' a spanning tree parity problem. ■

If D contains faces with four or more sides, say a face $xyzt$, then we could include three edges linking these four vertices, say xy, yz, and zt, and require that a spanning tree contain either all three or none of these three edges. Such a spanning tree tri-arity problem, as we shall later see, turns out to be NP-complete.

A noncrossing Euler tour of a reduced graph H is a tour that visits every edge once and has the property that a vertex v entered by the Euler tour through some edge e exists the vertex through one of the two edges e', e'' incident to v on either side of e. A noncrossing Euler tour of H gives a Hamiltonian circuit for G. The following is immediate from Proposition 1.

Proposition 3 *Let G be a 3-connected cubic planar bipartite graph whose faces are 3-colored. Then G has a Hamiltonian cycle with red faces in one side, blue faces in the other side, and green faces in either side, if and only if the reduced graph H obtained by collapsing green faces has a noncrossing Euler tour, if and only if the reduced graph H' obtained by collapsing red faces has a spanning tree of green faces, if and only if the reduced graph H'' obtained by collapsing blue faces has a spanning tree of green faces.*

An application of Theorem 3 and Proposition 2 gives the following.

Corollary 2 *Let G be a cubic planar bipartite graph, and let H be the corresponding reduced graph. Suppose all vertices of H have degree 4 or 6, so that all green faces of G are either squares or hexagons. Then one can decide in polynomial time whether H has a noncrossing Euler tour giving a Hamiltonian cycle for G.*

PROOF: Let H' be the reduced graph for G by collapsing the blue or red faces instead of the green faces. Then the green faces give in H' digons or triangles. By Proposition 2, a noncrossing Euler tour for H corresponds to a spanning tree of green faces in H'. Since all green faces are either digons or triangles in H', one can decide whether such a spanning tree of green faces in polynomial time by Theorem 3. ■

We now obtain several NP-completeness results.

Theorem 4 *Let G be a 3-connected cubic planar bipartite graph, and let H be the corresponding reduced graph. Suppose that the red faces in H are quadrilaterals and the blue faces in H are digons. Then the question of whether H has a spanning tree of red faces is NP-complete.*

PROOF: The question of whether a 3-connected planar cubic graph R has a Hamiltonian cycle is NP-complete, as shown by Garey et al. [5]. Let $e = (u, v)$ be a particular edge in R. Then the question of whether R has a Hamiltonian cycle going through e is also NP-complete. Let R' be the graph obtained from R by removing the edge e, so that R' has two vertices of degree 2, namely u and v, and all other vertices of R' have degree 3. The question of whether R' has a Hamiltonian path from u to v is also NP-complete.

Given R', construct a reduced graph H of some 3-connected cubic planar bipartite graph G as follows. The graph H has vertices $V = V_1 \cup V_2$, where V_1 is the set of vertices w of R' and V_2 is the set of faces f of R'. A vertex w in V_1 is joined to a vertex f in V_2 by two parallel edges in H if and only if the vertex w in R' is a vertex in the face f of R'. Thus the blue faces are digons joining two vertices w and f, while the red faces are quadrilaterals (w, f, w', f') corresponding to edges (w, w') in R' that separate two faces f and f' in R'.

We show that a set L of edges in R' forms a Hamiltonian path from u to v in R' if and only if the set M of red quadrilaterals in H corresponding to the edges M' in R' that are not in L has the property that M is a spanning tree of red faces in H. Therefore H has a spanning tree of red faces if and only if R' has a Hamiltonian path from u to v, and so the question of whether H has a spanning tree of red faces is NP-complete.

Suppose L is a Hamiltonian path from u to v in R', let M' be the edges in R' not in L, and let M be the corresponding red quadrilaterals in H. Note that for any two edges g and h in M', there is a sequence of edges $g = e_1, e_2, \ldots, e_k = h$ in M' such that each pair of edges e_i, e_{i+1} share a face. Therefore the red faces in M induce a connected subgraph of H. Notice also that every vertex in H belongs to some face in M, since every vertex w in R' is incident to an edge not in L, and every face f in R' has at least one edge not in L. Finally, the red faces in M do not contain a cycle. Otherwise, if we had a cycle $u_1, u_2, \ldots, u_k, u_1$ of red faces in M, then we can observe that every vertex w in V_1 belongs to exactly one red face in M, so u_i and u_{i+1} share a vertex f in V_2 corresponding to a face in R'. Thus the edges $e_1, e_2, \ldots, e_k, e_1$ in M', corresponding to the faces u_i in the cycle of red faces in M, separate the graph R' into two components, so the Hamiltonian path L would have to contain at least one of these edges e_i in M', a contradiction. Therefore the red faces in M form a spanning tree of red faces for H.

In the other direction, suppose the red faces in M form a spanning tree of red faces for H. Let M' be the corresponding edges in R', and let L be the set of edges in R' not in M'. Each vertex w in V_1 belongs to exactly one red face (w, f, w', f') in M, since every other red face in M containing w also contains either f or f', and therefore the two red faces form a cycle and cannot both be in the spanning tree of red faces M. Therefore every vertex in R' is incident to exactly one edge in M', and so the two vertices u and v of degree 2 in R' are incident to exactly one edge in L, while the remaining vertices w of degree 3 in R' are incident to exactly two edges in L. That is, L consists of a path joining u and v in R', plus a collection of cycles in R', such that the path and the cycles are disjoint and cover all the vertices in R'. We show that L cannot contain a cycle in R', and so L is just a path joining u to v containing all vertices of R', that is, L is a Hamiltonian path from u to v in R'. Suppose L contains a cycle $e_1, e_2, \ldots, e_k, e_1$ in R'. Let f and f' be faces of R' inside and outside the cycle of e_i respectively. Since f and f' are vertices in the spanning tree of red faces M, there is a sequence of red faces $u_1, u_2, \ldots, u_l$ in M such that u_1 contains f, u_l contains f' and each pair u_{i-1}, u_i share a vertex f_i in V_2. In particular, if we denote $f_0 = f$ and $f_l = f'$, then for some pair f_i, f_{i+1} we must have the face f_i in R' inside and the face f_{i+1} outside the cycle of e_i. This implies that the red face u_i in M corresponds to one of the edges e_i in L and not in M', a contradiction. This completes the proof. ■

We obtain two Corollaries from this result.

Corollary 3 *Let G be a 3-connected cubic planar bipartite graph, and let H be the corresponding reduced graph. Suppose all vertices of H have degree 8. Then the question of whether H has a noncrossing Euler tour is NP-complete.*

PROOF: The reduced graphs H from Theorem 4 have all red faces as quadrilaterals, corresponding to octagons in G. If we collapse these red faces, we obtain a reduced graph H' with vertices of degree 8. By Proposition 2, H' has a noncrossing Euler tour if and only if H has a spanning tree of red faces, and this problem is NP-complete by Theorem 4. ■

Corollary 4 *Let G be a 3-connected cubic planar bipartite graph, and let H be the corresponding reduced graph. Suppose that the red faces in H are octagons and digons and the blue faces in H are triangles. Then the question of whether H has a spanning tree of arbitrary faces is NP-complete.*

PROOF: Let H be the reduced graph of Theorem 4, with red quadrilaterals and blue digons. If e and f are the two parallel edges of a blue digon, insert a vertex w in the middle of e and a vertex x in the middle of f, with w and x joined by two parallel edges. The blue digon splits thus into two blue triangles and a red digon, while the red quadrilaterals become red octagons, in the new reduced graph H'.

Suppose H has a spanning tree of red quadrilaterals M. Select the corresponding red octagons in H'. For a blue digon consisting of two edges e and f in H, if one of the two red quadrilaterals containing e or f is in M, then select the red digon joining the middle vertices w and x; if neither of the two red quadrilaterals containing e or f is in M, then select one of the two blue triangles containing w and x. The red and blue faces in H' thus selected, involving red octagons, red digons, and blue triangles, for a spanning tree of faces in H'.

Conversely, suppose H' has a spanning tree of faces M'. Let M be the set of red quadrilaterals in H such that the corresponding red octagon is in M'. Note that for each digon in H, only one of the corresponding two blue triangles and red digon in H' can be in M'. Thus M is a spanning tree of red faces.

Thus H' has a spanning tree of arbitrary faces if and only if H has a spanning tree of red faces, and NP-completeness follows from Theorem 4. ■

We finally show:

Theorem 5 *Suppose there exists a 3-connected cubic planar bipartite graph G_0 that is not Hamiltonian. Then the question of whether a 3-connected cubic planar bipartite graph G has a Hamiltonian cycle is NP-complete.*

PROOF: Takanori et al. [11]. showed that the question of whether a 2-connected cubic planar bipartite graph G has a Hamiltonian cycle is NP-complete. If such a G has two edges e and f that separate it into two components G' and G'', then their endpoints in either side are at odd distance (this can be inferred by examining the colors of the faces separated by e and f in a 3-coloring of the faces, and the two alternating colors surrounding one of these faces), so we may instead join the two endpoints of e and f in G' and G'' separately, and ask whether G' and G'' both contain a Hamiltonian cycle containing the added edge joining the endpoints of e and f. Repeating this decomposition process, we eventually reduce the question of whether G has a Hamiltonian cycle to the question of whether various G_i each contain a Hamiltonian cycle going through certain prespecified edges, with each G_i being 3-connected. Thus the question of whether a 3-connected cubic planar bipartite graph G has a Hamiltonian cycle going through certain prespecified edges is NP-complete.

Suppose G_0 is a minimal 3-connected cubic planar bipartite graph that is not Hamiltonian. We shall construct from G_0 a 3-connected cubic planar bipartite graph G_1 that has a Hamiltonian cycle, such that for some edge $e = (u, v)$ in G_1, every Hamiltonian cycle in G_1 goes through e. Furthermore, If f and g are the two edges other than e incident to u in G_1, then G_1 has a Hamiltonian cycle going through f and e and a Hamiltonian cycle going through g and e. Given a graph G with certain prespecified edges that a Hamiltonian cycle must go through, consider each such prespecified edge e' incident to a vertex u', and replace u' with the complement of u in G_1, so that the three edges incident to u' in G are replaced by the three edges incident to u in G, with e' corresponding to e in G_1. These replacements then force a Hamiltonian cycle for the new graph to correspond to a Hamiltonian cycle going through the prespecified edges e' in G. Therefore whether the resulting 3-connected cubic planar bipartite graph has a Hamiltonian cycle is NP-complete.

We construct G_1. We shall first do so to enforce that every Hamiltonian cycle in G_1 goes through e, and later guarantee that such a cycle can go through either f or g. Let R be a square in G_0. Suppose that if we remove two opposing sides of R from G_0 and replace the two paths of length 3 resulting from the remaining two sides of R by single edges, then we obtain a 3-connected graph G_1. By minimality of G_0, the graph G_1 has a Hamiltonian cycle. Furthermore, no Hamiltonian cycle for G_1 goes through either of the two edges of G_1 corresponding to the edges that were kept for the square R in G_0, since if either edge is in a Hamiltonian cycle for G_1, then we can extend this cycle to a Hamiltonian cycle visiting the four vertices of the square R in G_0, a contradiction. We have thus guaranteed that a Hamiltonian cycle in G_1 does not visit a particular edge e_0, and thus must visit an edge e adjacent to e_0, as desired.

Suppose instead that the two graphs G_1 obtained in the preceding construction by either of the two choices of two opposing edges of the square R to be removed are both only 2-connected. Then each choice of two opposing edges of R belongs to a cut of four edges in G_0 that separate G_0. Removing the two sets of four edges from the two cuts of four edges thus separates G_0 into four components C_1, C_2, C_3, C_4, with the removed edges of G_0 including an edge from C_1 to C_2, an edge from C_2 to C_3, an edge from C_3 to C_4, and an edge from C_4 to C_1, plus four edges from the four C_i going into the four vertices of the square R. That is, each C_i has three incoming edges that can be joined to a single vertex, since their three endpoints in C_i are at even distance (this can be inferred by examining the colors of the three faces surrounding C_i in a 3-coloring of the faces, and the two alternating colors surrounding one of these faces). By minimality of G_0 each such graph resulting from C_i with an additional vertex has a Hamiltonian cycle, yet it is not the case that each of the three choices of two edges going into each C_i yields a Hamiltonian cycle, since otherwise we would obtain a Hamiltonian cycle for G_0. Thus one of the three edges joining some C_i to the additional vertex must belong to every Hamiltonian cycle, thus giving a 3-connected cubic planar bipartite graph G_1 with an edge e that belongs to every Hamiltonian cycle of G_1.

It remains to ensure that a Hamiltonian cycle in G_1, which is forced to take $e = (u, v)$, can take either f or g out of u. Suppose instead that every Hamiltonian cycle is forced to take f as well. Consider the 3-connected cubic planar bipartite graph K of a cube with 8 vertices. Replace as before a vertex of K with the complement of u in G_1. This forces two particular edges e and f of the cube incident to the replaced vertex to be visited by a Hamiltonian cycle, yet the Hamiltonian cycle for the cube can still be chosen in two different ways so that either of the two edges f' and g' sharing the endpoint of e that was not replaced can be visited. This produces the required G_1' with edge e forced and the choice between f' and g' still available for a Hamiltonian cycle, completing the proof. ■

References

[1] R. Aldred, G. Brinkmann, and B.D. McKay, personal communication.

[2] D. Barnette, "Conjecture 5," Recent Progress in Combinatorics (Ed. W.T. Tutte), Academic Press, New York (1969) 343.

[3] G. Brinkmann, B.D. McKay, and U. von Nathusius, "Backtrack search and look-ahead for the construction of planar cubic graphs with restricted face sizes," http://cs.anu.edu.au/~bdm/papers/plantri_ad.pdf

[4] H.N. Gabow and M. Stellman, "Efficient algorithms for graphic intersection and parity," Proc. 12th Int. Conf. Automata, Languages, and Programming, Springer-Verlag LNCS 194, 1985, 210–220.

[5] M.R. Garey, D.S. Johnson, and R.E. Tarjan, "The planar Hamiltonian circuit problem is NP-complete," SIAM J. Comput. 5 (1976), 704–714.

[6] P.R. Goodey, "Hamiltonian circuits in polytopes with even sides," Israel Journal of Mathematics 22 (1975), 52–56.

[7] P.R. Goodey, "A class of Hamiltonian polytopes," J. Graph Theory 1 (1977), 181–185.

[8] D.A. Holton, B. Manvel, and B.D. McKay, "Hamiltonian cycles in cubic 3-connected bipartite planar graphs," Journal of Combinatorial Theory B 38 (1985) 279–297.

[9] L. Lovász, "Matroid matching and some applications," J. Combinatorial Theory Ser. B 28 (1980), 208–236.

[10] P.G. Tait, "Listing's topologie," Phil. Mag. 17 (1884), 30–46.

[11] Akiyama Takanori, Nishizeki Takao, and Saito Nobuji, "NP-completenss of the Hamiltonian cycle problem for bipartite graphs," Journal of Information Processing Abstract Vol. 03 No. 02–003.

[12] W.T. Tutte, "On Hamiltonian circuits," J. London Math. Soc. 21 (1946), 98–101.