Matchings and Cycle Decompositions in Hypercubes

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1 Introduction

The connection between perfect matchings and Hamiltonian cycles of the hypercube has been extensively studied. Fink [10] proved the conjecture of Kreweras [13] that for any perfect matching M of the d-dimensional hypercube, $d \geq 2$, there exists a second perfect matching M' such that the union $H = M \cup M'$ forms a Hamiltonian cycle. In fact Fink showed that this is still the case if M is a perfect matching of the complete graph induced by the vertices of the d-dimensional hypercube, $d \geq 2$, while M' is restricted to edges of the hypercube itself. In this paper we prove generalizations of this result.

After a long series of results, the authors of this paper [9] showed that if M_d is the number of perfect matchings of the d-dimensional hypercube and H_d is the number of Hamiltonian cycles of the d-dimensional hypercube, then $H_d = m_d^{2-o(1)}$.

2 Weak Decomposition into Cycles

A weak cycle in a graph is either a cycle or two parallel copies of a single edge, considered a cycle of length two. A weak r-cycle decomposition of a graph is a collection of disjoint paths and r weak cycles that span all the vertices of the graph.

Theorem 1 *Let G be the d-dimensional hypercube, $d \geq 1$, and let K be the clique on the vertices of G. Let M be a matching of K that is not perfect and such that M has s edges that are in G. Let $r \leq \lceil s/d \rceil$. Then G has a perfect matching M' such that $M \cup M'$ is a weak r-cycle decomposition of K.*

Proof. For the base case $d = 1$, M is empty and M' consists of the single edge of G. Suppose $d \geq 2$.

Some dimension of G has at most $\lfloor s/d \rfloor$ out of the s edges of M in G, say dimension 1. Thus out of these s edges, some s_1, s_2 are in the two $(d-1)$-dimensional hypercubes G_1, G_2 separated by dimension 1, with $s_1 + s_2 \geq \lceil (d-1)s/d \rceil$. Let M_1, M_2 be the matchings induced by M on the vertices of the cliques K_1, K_2 on the vertices of G_1, G_2 respectively. One of M_1, M_2, say M_1, is not a perfect matching. We may by induction find a weak r_1-cycle decomposition of K_1 given by $M_1 \cup M_1'$ for any $r_1 \leq \lceil s_1/(d-1) \rceil$. If a path of this decomposition goest for u to v in K_1, and both u and v have edges (u, u'), (v, v') in M that traverse at least dimenension 1, we add the corresponding edges (u', v') to M_2 to obtain M_2''. If M_2'' is not perfect, we may by induction find a weak r_2-cycle decompostion of K_2 given by $M_2'' \cup M_2'$ for any $r_2 \leq \lceil s_2/(d-1) \rceil$. This gives by induction any

$$r = r_1 + r_2 \leq \lceil s_1/(d-1) \rceil + \lceil s_2/(d-1) \rceil$$

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$$
\begin{aligned}
&\geq \lceil (s_1 + s_2)/(d-1) \rceil \\
&\geq \lceil \lceil (d-1)s/d \rceil /(d-1) \rceil \\
&\geq \lceil s/d \rceil .
\end{aligned}
$$

If M_2'' is perfect, we may remove an edge of M_2'' to obtain M_2''', and then find a weak (r_2-1)-cycle decomposition of G_2 given by $M_2''' \cup M_2'$, which after adding the removed edge gives a weak r_2-cycle decomposition of K_2 without paths. If $r_2 = 0$, we may increase in this case r_2 by 1 and decrease r_1 by 1, unless $r_1 = r_2 = 0$ in which case no cycles are needed and result follows by choosing instead dimension 1 to separate two vertices of G that are not matched in M. □

Corollary 1 *Let G be the d-dimensional hypercube, $d \geq 1$, and let K be the clique on the vertices of G. Let M be a perfect matching of K such that M has s edges that are in G. Let $r \leq \lceil s/d \rceil$. Then G has a perfect matching M' such that $M \cup M'$ is a weak r-cycle decomposition of K without paths.*

Proof. Remove any edge of M, apply the Theorem to obtain a decomposition having a single path joining the endpoints of the removed edge, and then add this edge to obtain the decomposition without paths. □

3 Strong Decomposition into Cycles

A strong r-cycle decomposition is a weak r-cycle decomposition without cycles of length 2 (parallel edges).

Theorem 2 *Let G be the d-dimensional hypercube, and let $r \geq 1$. Suppose that for every perfect matching M of the clique K on the vertices of G, there exists another perfect matching M' of G such that $M \cup M'$ is a weak r-cycle decomposition without paths. Then this is also the case for any $d' \geq d$, and for any $d'' \geq d2^{d-1}$ the decomposition can be chosen to be strong.*

Proof. Suppose the assumption holds for $d' \geq d$, and consider $d'+1$ dimensions. Let 1 be a dimension such that some edge of M traverses at least dimension 1. Then we may split G as G_1, G_2 across dimension 1, obtain a 0-cycle decomposition for G_1 as in the Theorem in the previous section, which again gives paths corresponding to additional edges giving a perfect matching M_2 for G_2. Now G_2 has d' dimensions, and by inductive hypothesis we obtain an r-cycle decomposition for G_2 that gives by extending the paths of G_1 joined across dimension 1 an r-cycle decomposition for G.

Suppose the assumption holds for d, and choose a large d''. If we choose dimension 1 to be traversed by at least $1/d''$ of the edges of M, the number of edges of M in one of the two subcubes across dimension 1 is reduced from $2^{d''-1}$ to at most $(d''-1)2^{d''-2}/d''$. Repeating this process again for dimension 2 traversed by at least $1/(d''-1)$ edges of M in the $(d''-1)$-dimensional subcube, and so on, we obtain in the $(d''-i)$-dimensional subcube at most $(d''-i)2^{d''-i-1}/d''$ edges of M. If $d = d''-i$, we will have no edges of M left in the d-dimensional cube if $d2^{d-1} < d''$. When we find the r cycles for d without edges of the original M, the decomposition will be strong for the original $d'' > d2^{d-1}$. □

For $r = 2$, the weak decomposition exists already at $d = 3$, and it is in fact strong unless M consists of alternating edges of a 6-cycle plus the edge joining the other 2 vertices at distance 3.

The decomposition for $r = 2$ is always strong at $d = 4$, as the difficult case arises only when for one cube with 2^3 vertices we choose alternating edges of the 6-cycle, for the other cube with 2^3 vertices we choose the opposite alteration of the 6-cycle, and join the corresponding unmatched vertices from the two cubes with 2^3 vertices. This is the only case that reduces to the case that is not strong for $d = 3$, and yet $r = 2$ cycles of length 8 can be found.

Corollary 2 *A weak 2-cycle decomposition without paths can always be found for any matching if $d \geq 3$, and A strong 2-cycle decomposition without paths can always be found for any matching if $d \geq 4$.*

4 Cycles and Trees

Theorem 3 *Let G be the Cartesian product of two cycles $C_{2^{d_1}}$, $C_{2^{d_2}}$, of lengths $2^{d_1}, 2^{d_2}$ respectively. Then G can be embedded as a subgraph of the hypercube H of dimension $d_1 + d_2$ by a one-to-one correspondence between vertices of G and H.*

Proof. $C_{2^{d_1}}$, $C_{2^{d_2}}$ are Hamiltonian cycles of the hypercubes H_{d_1}, H_{d_2} of dimensions d_1, d_2 respectively, and H is the Cartesian product of these two hypercubes, so taking Cartesian products of the two cycles and the two hypercubes correspondingly the result follows. □

Corollary 3 *Let $C_{\ell_1}, C_{\ell_2}, \ldots, C_{\ell_k}$ be k cycles C_{ℓ_i} of lengths ℓ_i respectively, so that the ℓ_i are even and add up to 2^d. Then the disjoint union of the C_{ℓ_i} can be embedded in the d-dimensional hypercube H_d by a one-to-one correspondence, and in such a way that some edge of C_{ℓ_i} and some edge of $C_{\ell_{i+1 \bmod p}}$ form opposite sides of a square, and the two edges so shared by C_{ℓ_i} with $C_{\ell_{i-1 \bmod p}}$ and $C_{\ell_{i+1 \bmod p}}$ are opposite edges of C_{ℓ_i}.*

Proof. Let $d_1 = d-1$ and $d_2 = 1$ in the theorem, so we have the partesian product of $C_{2^{d-1}}$ with an edge C_2 (a 2-cycle). The cycles can be embedded in this product by traversing along $C_{2^{d-1}}$ in parallel across C_2 two sets of paths $P_{\ell_1/2}, P_{\ell_2/2}, \ldots, P_{\ell_k/2}$ of half the lengths of the corresponding cycles. □

A binary tree is a tree with vertices of degrees $1, 2, 3$. A binary tree with 2^d vertices is balanced if it has 2^{d-1} vertices in each side of the bipartition. It has been conjectured that every such binary tree can be embedded into a d-dimensional hypercube by a one-to-one correspondence. We consider a family of special trees with an edge (u, v), paths of lengths a and b attached at u, and paths of length c and d attached at v.

Corollary 4 *Every balanced special tree can be embedded by a one-to-one correspondence into a d-dimensional hypercube.*

Proof. If $d = 1$ then we have just an edge, and the result follows. Otherwise for the tree to be balanced we must have either a, b, c, d all even, or a, b, c, d all odd, or one of a, b even and one odd (say a even) and one of c, d even and one odd (say c even). Then $a + c, b + d$ are even and say $a + c \geq 2$. We may attach the endpoints of the paths of length a, c to each other, attach the enpoints of the paths of length b, d to each other, and join the vertices in the paths of lengths a, c at distance 1 from u, v respectively to each other. Then we have a cycle of length $a + c$ and a cycle of length $b+d+2$ joined at a square, and the result follows from the previous corollary. □

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