Generalized Colourings (Matrix Partitions) of Cographs

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Abstract

Ordinary colourings of cographs are well understood; we focus on more general colourings, known as matrix partitions. We show that all matrix partition problems for cographs admit polynomial time algorithms and forbidden induced subgraph characterizations, even for the list version of the problems. Cographs are the largest natural class of graphs that have been shown to have this property. We bound the size of a biggest minimal M-obstruction cograph G, both in the presence of lists, and (with better bounds) without lists. Finally, we improve these bounds when either the matrix M, or the cograph G, is restricted.

1 Introduction

Cographs are a well understood class of graphs [4, 5, 14, 18]. A recursive definition is as follows. The one-vertex graph K_1 is a cograph; if G' and G'' are cographs, then so are the *disjoint union* $G' \cup G''$ and their *join* $G' + G''$ (obtained from $G' \cup G''$ by adding all edges joining vertices of G' to vertices of G''). It follows that the complement of a cograph is a cograph, and in fact the join of G' and G'' is the complement of the disjoint union of $\overline{G'}$ and $\overline{G''}$. It is not hard to show that G is a cograph if and only if it contains no induced path with four vertices [18]. Cographs can be recognized in linear time [5], and they can be represented, in the same time, by their *cotree* [5], which embodies the sequence of binary operations $\cup, +$, from the recursive definition, used in their construction. Many combinatorial optimization problems can be efficiently solved on the class of cographs, using the cotree representation [4, 5, 14]. This includes computing the chromatic number, and, more specifically, deciding if a cograph G is k-colourable. This suggests looking at more general colouring problems for the class of cographs. In fact, such investigations have already begun in [6, 19].

In [3, 7, 10, 11], a framework was developed, which encompasses many generalizations of colourings. Let M be a symmetric m by m matrix over $0, 1, *$. An *M-partition* of a graph G is a partition of the vertex set $V(G)$ into m parts $V_1, V_2, \ldots, V_m$ such that V_i is a clique (respectively independent set) whenever $M(i,i) = 1$ (respectively $M(i,i) = 0$), and there are all possible edges (respectively no edges) between parts V_i and V_j whenever $M(i,j) = 1$ (respectively $M(i,j) = 0$). Thus the diagonal entries prescribe when the parts are cliques or independent sets, and the off-diagonal entries prescribe when the parts are completely adjacent or nonadjacent (with $*$ meaning no restriction). A graph G that does not admit an M-partition is called an *M-obstruction*, and is also said to *obstruct M*. A *minimal M-obstruction* is a graph G which is an M-obstruction, but such that every proper induced subgraph of G admits an M-partition. If $\mathcal{M}$ is a set of matrices, we say that G is a minimal $\mathcal{M}$-obstruction if it is an M-obstruction for all $M \in \mathcal{M}$, but every proper induced subgraph of G admits an M-partition for some $M \in \mathcal{M}$.

Given a graph G, we sometimes associate lists with its vertices: a list $L(v)$ of a vertex v is a subset of $\{1, 2, \ldots, m\}$, and it prescribes the parts to which v can be placed. In other words, a *list M-partition* of G (with respect to the lists $L(v), v \in V(G)$) is an M-partition of G in which each vertex v belongs to a part V_i with $i \in L(v)$. Note that the trivial case when all lists are $L(v) = \{1, 2, \ldots, m\}$ corresponds to the situation when no lists are given. M-obstructions and minimal M-obstructions (as well as $\mathcal{M}$-obstructions and minimal $\mathcal{M}$-obstructions) for graphs G with lists L are defined in the obvious way.

In the *(list) M-partition problem*, we have a fixed matrix M, and are asked to decide whether or not a given graph G (with lists) does or does not admit a (list) M-partition (with respect to the given lists).

We shall mostly focus on matrices M which have no diagonal $*$'s. If M has a diagonal $*$, then every graph G admits an M-partition; however, if lists are involved we will allow diagonal $*$'s. A matrix without diagonal $*$'s may be written in a block form, by first listing the rows and columns with diagonal 0's, then those with diagonal 1's. The matrix falls into four blocks, a k by k diagonal matrix A with a zero diagonal, an ℓ by ℓ diagonal matrix B with a diagonal of 1's, and a k by ℓ off-diagonal matrix C and its transpose. We shall say that M is a *constant matrix*, if the off-diagonal entries of A are all the same, say equal to a, the off-diagonal entries of B are all the same, say b, and all entries of C are the same, say c. In this case, we also say that M is an *(a, b, c)-block matrix*. Note that we may assume that $a \neq 0$ and $b \neq 1$, or else we can decrease k or ℓ.

Let M be a fixed matrix; if we prove that all cographs that are minimal M-obstructions have at most K vertices, then we can characterize M-partitionability of cographs by a finite set of forbidden induced subgraphs.

The *complement* $\overline{M}$ of a matrix M has all 0's changed to 1's and vice versa. It is clear that G admits an M-partition if and only if $\overline{G}$ admits an $\overline{M}$-partition, and that this also applies in the obvious way to M-partitions with lists, and to $\mathcal{M}$-partitions.

If the matrix M is a $(*, *, *)$-block matrix, then an M-partition of G is precisely a partition of the vertices of G into k independent sets and ℓ cliques. Such partitions have been introduced in [1] (see also [10, 11, 17]), and further studied in [15, 16] for the class of chordal graphs (see also [12, 13]) and in [9] for the class of perfect graphs. More recently, they have been studied (without lists) for the class of cographs in [6, 19].

Suppose M is an m by m matrix; we shall refer to the integers $1, 2, \ldots, m$ as *parts*, since they index the set of parts in any M-partition of a graph. Given two sets of parts, $P, Q \subseteq \{1, 2, \ldots, m\}$, we define $M_{P,Q}$ to be the submatrix of M obtained by taking the rows in P and the columns in Q. We also let M_P denote $M_{P,P}$.

2 List Partition Problems

We first prove that for every matrix M the list M-partition problem for cographs can be solved in polynomial time, and characterized by finitely many forbidden induced subgraphs (with lists). By contrast, it is shown in [12, 13] that there exist matrices M for which the M-partition problem restricted to *chordal* graphs is NP-complete, even without lists.

Many of our arguments use the following observation. A disconnected graph $G = G_1 \cup G_2$ has an $\mathcal{M}$-partition if and only if G_1 has an M_P-partition and G_2 has an M_Q-partition, for some matrix $M \in \mathcal{M}$ and sets P, Q of parts such that $M_{P,Q}$ contains no 1. Of course the argument applies also with lists, if we view G_1, G_2 as inheriting the corresponding lists. We shall state this in the contrapositive form as follows.

Lemma 2.1 *Let $\mathcal{M}$ be fixed, and let $G = G_1 \cup G_2$ be a disconnected graph, with lists.*

Then G is an $\mathcal{M}$-obstruction if and only if for any matrix $M \in \mathcal{M}$ and any two sets P, Q of parts from M such that $M_{P,Q}$ does not contain a 1, the graph G_1 (with the corresponding lists) is an M_P-obstruction, or the graph G_2 (with the corresponding lists) is an M_Q-obstruction. $\square$

Suppose $\mathcal{M}$ is fixed, and $G = G_1 \cup G_2$ is disconnected.

Let $\mathcal{M}_1$ be a set of matrices M_P, where $M \in \mathcal{M}$ and P is a set of parts in M, such that G_1 is an M_P-obstruction, and let $\mathcal{M}_2$ be a set of matrices M_Q, where $M \in \mathcal{M}$ and Q is a set of parts in M, such that G_2 is an M_Q-obstruction. If, for any $M \in \mathcal{M}$, and any sets of parts P, Q of M such that $M_{P,Q}$ does not contain a 1, we have $M_P \in \mathcal{M}_1$ or $M_Q \in \mathcal{M}_2$, then the lemma ensures that for any subgraphs G'_1 of G_1 and G'_2 of G_2 which are $\mathcal{M}_1$-obstruction and $\mathcal{M}_2$-obstruction respectively, the subgraph $G' = G'_1 \cup G'_2$ of G is also an $\mathcal{M}$-obstruction. Thus the minimality of G also implies the minimality of G_1, G_2. Such sets $\mathcal{M}_1, \mathcal{M}_2$ can be always chosen - for instance as the sets of *all* matrices M_P such that G_1 is an M_P-obstruction, respectively all matrices M_Q such that G_2 is an M_Q-obstruction.

Corollary 2.2 *Let $\mathcal{M}$ be fixed, and let $G = G_1 \cup G_2$ be a disconnected graph, with lists. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be chosen as described above.*

Then G is an $\mathcal{M}$-obstruction if and only if G_1 is an $\mathcal{M}_1$-obstruction and G_2 is an $\mathcal{M}_2$-obstruction.

Moreover, if G is a minimal $\mathcal{M}$-obstruction, then G_1 is a minimal $\mathcal{M}_1$-obstruction, and G_2 is a minimal $\mathcal{M}_2$-obstruction. $\square$

Let $f(m)$ be the smallest integer such that for every m by m matrix M and every minimal M-obstruction cograph G with lists, G has at most $f(m)$ vertices. (In other words, $f(m)$ is the largest *size*, i.e., number of vertices, of a minimal M-obstruction cograph, over all m by m matrices M.)

Theorem 2.3 *For every integer m, we have*

$$f(m) \leq a^m m!$$

where $a = \frac{1}{ln(3/2)}$.

Proof. We apply Corollary 2.2 with $\mathcal{M}$ consisting of the single matrix M. Clearly a minimal $\mathcal{M}$-obstruction has size at most equal to the sum of the sizes of minimal M'-obstructions for all $M' \in \mathcal{M}$; thus we have

$$f(m) \leq 2 \sum_{i<m} \binom{m}{i} f(i).$$

By induction, letting $a = 1/ln(3/2)$, we have

$$f(m) \leq 2m!a^m \sum_{0<j\leq m} 1/(j!a^j) \leq 2m!a^m(e^{1/a} - 1) = a^m m!.$$

$\square$

Lemma 2.1 also yields an efficient algorithm to solve the list M-partition problem in the class of cographs. We consider the cotree of G, associating with each node t of the cotree (corresponding to a cograph G_t involved in the construction of G) a family of matrices $\mathcal{M}_t$. The family $\mathcal{M}_t$ consists of all matrices M_X, for $X \subseteq \{1, 2, \ldots, m\}$, such that G_t obstructs M_X. If t is a node of the cotree with children t', t'' corresponding to $G_t = G_{t'} \cup G_{t''}$, we know that G_t obstructs M_X if and only if for any $P \subseteq X, Q \subseteq X$ with $M_{P,Q}$ not containing 1, the graph $G_{t'}$ obstructs M_P or the graph $G_{t''}$ obstructs M_Q. Thus from the families $\mathcal{M}_{t'}, \mathcal{M}_{t''}$ we can compute the family $\mathcal{M}_t$. If $G_t = G_{t'} + G_{t''}$, we use complementation, as discussed earlier. Since the leaves of the cotree are single vertex cographs, each leaf t has $\mathcal{M}_t = \emptyset$. Then the given cograph G, is at the root r of the cotree, $G = G_r$, and we conclude G has a list M-partition if and only if $M \notin \mathcal{M}_r$.

Each set $\mathcal{M}_t$ has at most 2^m members, since there are at most 2^m subsets of $\{1, 2, \ldots, m\}$. Thus we obtain the following bound. (Note that $2^{O(m)}$ accounts also for the time to check if M is one of the matrices in $\mathcal{M}_r$.)

Corollary 2.4 *Every list M-partition problem for cographs can be solved in time $2^{O(m)}n$, linear in n.* $\square$

We could, of course, proceed similarly, to solve the cograph list $\mathcal{M}$-partition problem for a *family* $\mathcal{M}$ of matrices.

We note that in [6] there are efficient algorithms solving related partition problems for cographs, for special matrices M, but not necessarily of fixed size.

We now derive a lower bound on $f(m)$. The special m by m matrix M_m has m diagonal zeros, and all off-diagonal entries $*$. Thus a list M_m-partition of G is precisely a list m-colouring of G. It turns out that there are very large cograph minimal M_m-obstructions. Since we are dealing with list colourings, we shall use the corresponding terminology.

Theorem 2.5 *For every positive integer m, there exists a minimal M_m-obstruction cograph G, with lists, of size $(e-1-\epsilon(m))m!$, where $1 \geq \epsilon(m) = o(1)$.*

Proof. We shall construct a cograph G, with lists from the set $\{1,\dots,m\}$ of colours, that does not have a list colouring, but each of its proper induced subgraphs does. The construction will be done recursively. For each subset of colours, $K \subseteq \{1,2,\dots,m\}$, we shall construct a graph $G(K)$, with lists from $\{1,\dots,m\}$, such that

- $G(K)$ is list colourable with colours from a set $S \subseteq \{1,\dots,m\}$ if and only if $|S| \geq |K|$ and $S \neq K$, and,
- for each $v \in V(G)$, the subgraph $G(K)\backslash v$ is list colourable with colours from the set K.

Then $G = G(\{1,2,\dots,m\})$ will be a minimal M_m-obstruction, as desired.

The recursion starts with sets K consisting of a single element i. The graph $G(\{i\})$ is a single vertex with list $\{1,\dots,m\} \setminus i$. This graph clearly satisfies the above conditions. The graph $G(K)$ with $K \subseteq \{1,\dots,m\}$ and $|K| \geq 2$ is recursively defined as the disjoint union of all graphs $G(K \setminus j)$ for $j \in K$, together with an additional vertex v_K, with list $\{1,\dots,m\}$, that is adjacent to all other vertices. Note that each $G(K)$ is a cograph, by induction.

Let S be a set of colours such that $G(K)$ has a list colouring with colours from S, and let j_0 denote the colour of v_K in such a colouring. Then each graph $G(K \setminus j)$ has a list colouring using the colours from $S \setminus j_0$, and hence, by induction, $|S \setminus j_0| \geq |K \setminus j|$, and $S \neq K$. On the other hand, if we remove v_K, all components $G(K \setminus j)$ are colourable with colours from K by induction, and if we remove any other vertex $v \in G(K \setminus j_0)$, then, again by induction, we can colour $G(K \setminus j_0) \setminus v$ and all $G(K \setminus j)$, for $j \neq j_0$, with colours from $K \setminus j_0$, and colour v_K by j_0.

Thus $G = G(\{1, \ldots, m\})$ is a minimal M_m-obstruction (with lists). Let $g(k)$ denote the number of vertices of a graph $G(K)$ with $|K| = k$. Then $g(1) = 1$ and $g(k) = 1 + kg(k-1)$, and hence

$$g(m) = \sum_{i=0}^{m-1} \frac{m!}{(m-i)!} = m! \sum_{1 \leq i \leq m} 1/i! = m!(e - 1 - \epsilon(m)),$$

where $1 \geq \epsilon(m) = \sum_{i=m+1}^{\infty} \frac{1}{i!} = o(1)$. $\square$

Corollary 2.6 *For every integer m, we have*

$$(e - 1 - o(1))m! \leq f(m) \leq a^m m!$$

for $a = 1/ln(3/2)$. $\square$

3 Partition Problems Without Lists

For the remainder of the paper, we shall focus on the M-partition problem without lists. This implies that we now think of M in the block form, having k diagonal 0's and ℓ diagonal 1's, with $m = k + \ell$. Specifically, the parts $1, 2, \ldots, k$ will be independent sets, and the parts $k+1, k+2, \ldots, k+\ell = m$ will be cliques.

Given that we have no lists, we can improve the general bounds on the size of cograph minimal M-obstructions G. This is what we shall do in the present section. In the following two sections we shall obtain even better bounds when either the matrices M, or the cographs G, are restricted.

Lemma 3.1 *Let $\mathcal{M}$ be a collection of matrices, each of size at most m.*

If G is a minimal $\mathcal{M}$-obstruction cograph with maximum clique size r, then G has at most $g(m, r) \leq 2\binom{m+r}{r} + \binom{m+r-1}{r-1} - \binom{m+r-2}{r-2} - m - 1$ vertices.

The same conclusion applies if G has maximum independent set size r.

Proof. Suppose G has maximum clique size r. Since G is a cograph, its vertices can be partitioned into three graphs G_0, G_1, G_2 with no edges between G_0 and G_1, G_2, and with all edges between G_1 and G_2, where G_1 and G_2 are non-empty. We may assume that $G' = G_1 + G_2$ contains a clique of r vertices; in particular, there exists an integer $1 \le t \le r-1$ such that the maximum clique size in G_1 is $r-t$ and in G_2 is at most t. We now consider how many vertices are needed to ensure that G does not admit an M-partition for any matrix $M \in \mathcal{M}$. Note that no matrix $M \in \mathcal{M}$ can contain the submatrix M_r (defined above Theorem 2.5), since G is perfect, and hence r-colourable.

Let $g(m,r)$ denote the maximum number of vertices in a minimal $\mathcal{M}$-obstruction cograph G with maximum clique size r. We derive a recurrence on $g(m,r)$ by estimating separately G_0, G_1, and G_2. If G_0 is not empty, then G' has an M-partition for some $M \in \mathcal{M}$, and since M does not contain M_r, each clique of size r in G' is placed in some set P of $t \le r$ parts such that M_P contains a 1. This ensures that G_0 cannot use at least one part of M. Thus G_0 can be described as a minimal $\mathcal{M}'$-obstruction where all matrices in $\mathcal{M}'$ have size at most $m-1$, i.e., G_0 has at most $g(m-1,r)$ vertices. On the other hand, G_1 and G_2 have at most $g(m, r-t)$ respectively $g(m,t)$ vertices, as noted above. We obtain the recurrence

$$g(m,r) \le g(m-1,r) + g(m,r-t) + g(m,t),$$
$$g(0,r) = 1, g(m,1) \le m+1.$$

In order to bound $g(m,r)$ we consider the quantity

$$h(m,r) = 2\binom{m+r}{r} + \binom{m+r-1}{r-1} - \binom{m+r-2}{r-2}.$$

Using the well-known identity $\binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}$ we find that

$$h(m,r) - h(m,r-1) = h(m-1,r),$$

and thus

$$(h(m,r)-h(m,r-1))-(h(m,r-1)-h(m,r-2)) = h(m-1,r)-h(m-1,r-1)$$
$$= 2\binom{m+r-2}{r} + \binom{m+r-3}{r-1} - \binom{m+r-4}{r-2} \ge 0.$$

(Note that $\binom{m+r-2}{r} \ge \binom{m+r-4}{r-2}$.) Therefore $h(m, r-(t+1)) + h(m,t+1) \le h(m,r-t) + h(m,t)$ for $t+1 \le r-t$, and so $h(m,r-t) + h(m,t) \le$

$h(m,r-1)+h(m,1)$. Using the recursion for $g(m,r)$ we conclude inductively that $g(m,r) \leq h(m,r) - m - 1$, namely

$$\begin{aligned} g(m,r) &\leq h(m-1,r) - m - 2 + h(m,r-t) - m - 1 + h(m,t) - m - 1 \\ &\leq h(m-1,r) + h(m,r-1) + h(m,1) - 3m - 4 \\ &= h(m,r) + 2(m+1) + 1 - 3m - 4 = h(m,r) - m - 1. \end{aligned}$$

The case of maximum independent set size r follows by complementation.

□

Theorem 3.2 *Any minimal M-obstruction cograph G has at most $O(8^m/\sqrt{m})$ vertices.*

Proof. We shall consider a cotree for G, and associate with each node t of the cotree a set $\mathcal{M}_t$ of submatrices of M, obstructed by the graph G_t corresponding to the node t, and such that G_t obstructs $\mathcal{M}_t$ if and only if the two graphs $G_{t'}, G_{t''}$, corresponding to the two children t', t'' of t in the cotree, obstruct $\mathcal{M}_{t'}$ and $\mathcal{M}_{t''}$ respectively. This is analogous to the algorithm inherent in Corollary 2.2. The root t_0 of our cotree will have $\mathcal{M}_{t_0}$ consisting of the one (given) matrix M, and the corresponding (given) graph $G_{t_0} = G$. The total number of vertices of G is precisely the number of leaves in the cotree. If G_t has maximum clique size at most $\tilde{m}$, and if all matrices in $\mathcal{M}_t$ have size at most $\tilde{m}$, then the entire branch of the cotree rooted at t contains at most $2\binom{2\tilde{m}}{\tilde{m}} + \binom{2\tilde{m}-1}{\tilde{m}-1} - \binom{2\tilde{m}-2}{\tilde{m}-2} - \tilde{m} - 1$ leaves, by the above lemma. If $G_t = G_{t'} \cup G_{t''}$, and if both $G_{t'}$ and $G_{t''}$ contain a clique of size greater than k (the number of diagonal 0's in M), then we can choose $\mathcal{M}_{t'}$ and $\mathcal{M}_{t''}$ to consist of matrices with maximum size smaller than the maximum size of a matrix in $\mathcal{M}_t$. Indeed, any M'-partition of $G_{t'}$ (or of $G_{t''}$), with $M' \in \mathcal{M}_t$, uses a part j of M which is a clique ($j > k$), and which therefore cannot be used by $G_{t''}$ (respectively $G_{t'}$); thus it suffices to certify the non-partitionability of $G_{t'}$ and $G_{t''}$ for matrices of strictly smaller size. Similarly, if $G_t = G_{t'} + G_{t''}$, and both $G_{t'}$ and $G_{t''}$ contain an independent set of size greater than ℓ, it suffices to certify their non-partitionability for matrices of size strictly smaller than the maximum size of a matrix in $\mathcal{M}_t$.

We let $g(\tilde{m})$ denote the maximum size of a minimal $\mathcal{M}$-obstruction cograph G, over all sets $\mathcal{M}$ consisting of matrices of size at most $\tilde{m}$. Suppose $G = G' \cup G''$ (the case $G = G' + G''$ is similar), and the maximum clique sizes in G', G'' are c', c'' respectively, with $c' \geq c''$. We have observed above that if $c' \geq c'' > k$, then G has at most $2g(\tilde{m}-1)$ vertices. If $c'' \leq c' \leq \tilde{m}$, then both G' and G'' have size at most $2\binom{2\tilde{m}}{\tilde{m}} + \binom{2\tilde{m}-1}{\tilde{m}-1} - \binom{2\tilde{m}-2}{\tilde{m}-2} - \tilde{m} - 1$ by the lemma, whence G has size at most $2[2\binom{2\tilde{m}}{\tilde{m}} + \binom{2\tilde{m}-1}{\tilde{m}-1} - \binom{2\tilde{m}-2}{\tilde{m}-2} - \tilde{m} - 1]$. If $c' > \tilde{m}$ and $c'' \leq k$ we continue exploring the cotree, obtaining a sequence of graphs $G'_0, G'_1, \ldots, G'_s$, where $G'_i = G'_{i+1} \cup G''_{i+1}$ or $G'_i = G'_{i+1} + G''_{i+1}$. We always assume that if $G'_i = G'_{i+1} \cup G''_{i+1}$, then G'_{i+1} has a clique of size greater than $\tilde{m}$ and G''_{i+1} has maximum clique size at most k, and if $G'_i = G'_{i+1} + G''_{i+1}$, then G'_{i+1} has an independent set of size greater than $\tilde{m}$ and G''_{i+1} has maximum independent set size at most ℓ. We now argue that the sequence cannot be too long, namely, that $s \leq 2^{\tilde{m}}$. Indeed, we may assume that the sets $\mathcal{M}_t$ are *reduced*, in the sense that no $M_P, M_Q \in \mathcal{M}_t$ have $P \subseteq Q$ (as any graph obstructing M_P also obstructs M_Q). If we let N_i denote the set of all maximal sets P of parts (out of the m parts of M) such that $M_P \in \mathcal{M}_t$ corresponding to G'_i, then we see that $N_{i+1} \neq N_i$, otherwise G''_{i+1} is not needed. Thus one maximal set is dropped in each step from N_i to N_{i+1}. This implies that $s \leq 2^{\tilde{m}}$, and we obtain the general recurrence

$$\begin{aligned} g(\tilde{m}) &\leq 2^{\tilde{m}+1}[2\binom{2\tilde{m}}{\tilde{m}} + \binom{2\tilde{m}-1}{\tilde{m}-1} - \binom{2\tilde{m}-2}{\tilde{m}-2} - \tilde{m} - 2] + 2g(\tilde{m}-1) \\ &\leq O(2^{3\tilde{m}}/\sqrt{\tilde{m}}) + 2g(\tilde{m}-1), \end{aligned}$$

which solves to $g(m) \leq O(8^m/\sqrt{m})$. □

We now define $F(m)$ to be the size (number of vertices) of a largest minimal M-obstruction cograph G without lists, for any m by m matrix M. From the above theorem we have an upper bound on $F(m)$; the following lower bound will follow from Theorem 5.2.

Corollary 3.3 *We have*

$$m^2/4 \leq F(m) \leq O(8^m/\sqrt{m}).$$

□

4 Constant matrices

In this section we prove that for each *constant* matrix M with k diagonal 0's and ℓ diagonal 1's, all cograph minimal M-obstructions have size at most $(k+1)(\ell+1)$. These M-partitions for constant matrices M (i.e., for (a,b,c)-block matrices M) have been investigated in the classes of perfect and chordal graphs in [9, 12, 13], and, in the case of $(*,*,*)$-block matrices (corresponding precisely to partitions into k independent sets and ℓ cliques), in [6, 15, 16, 19]. Recall that we do not consider lists in this section.

We illustrate the technique in the special case of $(*,*,*)$-block matrices, proving the following result; special cases of this result have been proved, by a different technique, in [6], cf. also [19].

Theorem 4.1 *Let M be a $(*,*,*)$-block matrix. Then each minimal M-obstruction cograph is $(k+1)$-colourable, and partitionable into $\ell+1$ cliques.*

Proof. When $\ell = 0$, each minimal M-obstruction is a minimal cograph G that is not k-colourable. Since cographs are perfect, $G = K_{k+1}$, which is both $(k+1)$-colourable, and partitionable to $(0+1)$ cliques. The case $k = 0$ follows by complementation, and we can proceed by induction on $k + \ell$. Let the cograph G be a minimal M-obstruction; we may assume that G is disconnected, $G = G_1 \cup G_2$ (or we can consider $\overline{G}$ instead). We shall now use Corollary 2.2, with the set $\mathcal{M}$ consisting of the single matrix M, and with all lists equal to $\{1, 2, \ldots, m\}$ (i.e., without lists); we shall be taking into account the special form of M to choose particular families $\mathcal{M}_1, \mathcal{M}_2$.

Specifically, let j be the smallest integer such that G_1 has a partition into k independent sets and j cliques. (Note that $0 \leq j \leq \ell$, by the minimality of G.) Since G_1 has a partition into k independent sets and j cliques, G_2 does not have a partition into k independent sets and $\ell - j$ cliques (otherwise G is not an M-obstruction). Let M_1 be the $(*,*,*)$-block matrix with k diagonal 0's and $j-1$ diagonal 1's, and let M_2 be the $(*,*,*)$-block matrix with k diagonal 0's and $\ell - j$ diagonal 1's. We now let $\mathcal{M}_1$ consist of M_1 and all its submatrices, and let $\mathcal{M}_2$ consist of M_2 and all its submatrices. It is easy to check that these classes $\mathcal{M}_1, \mathcal{M}_2$ satisfy the conditions stated below Lemma 2.1. Indeed, if P, Q are such that $M_P \notin \mathcal{M}_1, M_Q \notin \mathcal{M}_2$, then M_P has at least j diagonal 1's (parts that are cliques), and M_Q has at least $\ell - j + 1$ diagonal 1's (parts that are cliques). This means that some part $i, i > k$, (part that is a clique) lies in both P and Q, whence $M_{P,Q}$ contains a 1.

We conclude, by Corollary 2.2, that G_1 is a minimal $\mathcal{M}_1$-obstruction, and G_2 is a minimal $\mathcal{M}_2$-obstruction, and hence a minimal M_1-obstruction and a minimal M_2-obstruction respectively (because of the special form of $\mathcal{M}_1, \mathcal{M}_2$).

Now, by the induction hypothesis, G_1 is $(k+1)$-colourable and partitionable into j cliques, while G_2 is $(k+1)$-colourable and partitionable into $\ell - j + 1$ cliques. It follows that G is both $(k+1)$-colourable and partitionable into $\ell + 1$ cliques. $\square$

Note that a clique can meet an independent set in at most one vertex. Thus we have an upper bound on the size of a minimal M-obstruction. In fact, we can conclude that a minimal M-obstruction cograph G can be described as follows. The vertices of G are $v_{i,j}, i = 0, 1, \ldots, k, j = 0, 1, \ldots, \ell$, with any two $v_{i,j}, v_{i',j}$ adjacent, and no two $v_{i,j}, v_{i,j'}$ adjacent. (There are additional constraints on when arbitrary $v_{i,j}, v_{i',j'}$ are adjacent, arising from the fact that G is a cograph. This aspect is examined in [6, 19].)

Corollary 4.2 *Let M be a $(*, *, *)$-block matrix. Then each cograph minimal M-obstruction has exactly $(k + 1)(\ell + 1)$ vertices.* □

We shall prove the general result in a form better able to support induction. Instead of obstructions to one single (a, b, c)-block matrix M with k diagonal 0's and ℓ diagonal 1's, we shall consider collections $\mathcal{M}$ consisting of (a, b, c)-block matrices $M_0, M_1, M_2, \ldots, M_r$, each having k_i diagonal 0's and ℓ_i diagonal 1's. We shall further assume that the collection $\mathcal{M}$ is *staircase-like*, meaning that $k_i \le k_j$ and $\ell_i \ge \ell_j$ for all $i < j$. If we have strict inequality everywhere, we call the collection *strictly staircase-like.* Clearly every collection of (a, b, c)-block matrices $\mathcal{N}$ contains a staircase-like subcollection $\mathcal{M}_1$ as well as a strictly staircase-like subcollection $\mathcal{M}_2$, such that a graph G is an $\mathcal{N}$-obstruction if and only if it is an $\mathcal{M}_1$-obstruction if and only if it is an $\mathcal{M}_2$-obstruction.

For notational convenience we shall allow matrices with $k_i = -1$ or $\ell_i = -1$. In this case we view each graph G as obstructing such a matrix. In particular, we shall set $k_{-1} = \ell_{r+1} = -1$.

Theorem 4.3 *Let a, b, c be fixed. Let $\mathcal{M} = \{M_i\}_{i=0}^r$ be a staircase-like collection of (a, b, c)-block matrices.*

Then the maximum size of a minimal $\mathcal{M}$-obstruction cograph is at most

$$f(\mathcal{M}) = \sum_{i=0}^{r}(k_i - k_{i-1})(\ell_i + 1) = \sum_{i=0}^{r}(\ell_i - \ell_{i+1})(k_i + 1).$$

Proof. Since the values of a, b, c are fixed, the matrices M_i are fully described by their parameters k_i, ℓ_i. To simplify the discussion, we shall write each M_i in the more descriptive form $M[k_i, \ell_i]$, and also write the bounding function $f(\mathcal{M})$ in the more descriptive form $f(\{(k_i, \ell_i)\}_{i=0}^r)$.

Let G be a minimal $\mathcal{M}$-obstruction. We may again suppose that G is disconnected, say $G = G_1 \cup G_2$, and shall derive an upper bound on G from upper bounds on G_1, G_2, using Corollary 2.2. Recall that we may assume that $a \neq 0$ and $b \neq 1$. We shall distinguish two main cases - when $c \neq 1$ and when $c = 1$.

CASE 1: $c \neq 1$.

We first consider the subcase when $a = *$. Thus $a = *, b \neq 1, c \neq 1$, and the matrices in $\mathcal{M}$ have no 1's, except those on the main diagonal. As in the proof of Theorem 4.1, the graph G obstructs $M[k_i, \ell_i]$ if and only if there exists some $0 \leq j_i \leq \ell_i + 1$ such that G_1 obstructs $M[k_i, j_i - 1]$ and G_2 obstructs $M[k_i, \ell_i - j_i]$ (and, moreover, if G is a minimal $M[k_i, \ell_i]$-obstruction, then G_1 is a minimal $M[k_i, j_i - 1]$-obstruction, and G_2 a minimal $M[k_i, \ell_i - j_i]$-obstruction). As $M[k_i, d]$ is a submatrix of $M[k_{i+1}, d]$ we can choose j_i so that $j_i \geq j_{i+1}$ and $\ell_i - j_i \geq \ell_{i+1} - j_{i+1}$. Using induction, and setting $j_{r+1} = 0$, we compute

$$\begin{aligned}
f(\{(k_i, \ell_i)\}_{i=0}^{r}) &= f(\{(k_i, j_i - 1)\}_{i=0}^{r}) + f(\{(k_i, \ell_i - j_i)\}_{i=0}^{r}) \\
&= \sum_{i=0}^{r} ((j_i - j_{i+1})(k_i + 1) + (\ell_i - j_i - \ell_{i+1} + j_{i+1})(k_i + 1)) \\
&= \sum_{i=0}^{r} (\ell_i - \ell_{i+1})(k_i + 1).
\end{aligned}$$

Now we consider the other subcase, when $a = 1$. Here $a = 1, b \neq 1, c \neq 1$, and there are off-diagonal ones between any parts j, j' that are independent sets ($j, j' \leq k$). Thus any two vertices that are placed in different independent sets must be adjacent. We can derive the following conditions from Corollary 2.2, or by the arguments given below.

The graph $G = G_1 \cup G_2$ has an $M[k_i, \ell_i]$-partition if and only if it has a partition where all parts i that are independent sets ($i \leq k$) are in one of G_1, G_2, or a partition in which there is only one part that is an independent set, and that set intersects both G_1 and G_2 (for this we must have $k_i \geq 1$). Equivalently, G obstructs $M[k_i, \ell_i]$ if and only if the following three conditions hold:

1. there exists a u_i with $0 \leq u_i \leq \ell_i + 1$ such that G_1 obstructs $M[0, u_i - 1]$ and G_2 obstructs $M[k_i, \ell_i - u_i]$,

2. symmetrically, there exists a v_i with $0 \leq v_i \leq \ell_i + 1$ such that G_2 obstructs $M[0, v_i - 1]$ and G_1 obstructs $M[k_i, \ell_i - v_i]$, and

3. if $k_i \geq 1$, there exists a w_i with $0 \leq w_i \leq \ell_i + 1$ such that G_1 obstructs $M[1, w_i - 1]$ and G_2 obstructs $M[1, \ell_i - w_i]$.

Note that we always can choose u_i and v_i such that

$$u_i + v_i \geq \ell_i + 1 \quad \text{for all } 0 \leq i \leq r. \tag{1}$$

If x denotes the largest value such that G_1 obstructs $M[0, x-1]$ we may actually assume that $u_i = \min\{x, \ell_i + 1\}$ and $v_i = \min\{y, \ell_i + 1\}$, where y denotes the largest value such that G_2 obstructs $M[0, y-1]$. In particular, this implies $u_i \geq u_{i+1}$ and $v_i \geq v_{i+1}$ for $0 \leq 1 \leq r-1$. Similarly, if i_0 is the smallest index such that $k_{i_0} \geq 1$ we may assume that $w_i = \min\{w_{i_0}, \ell_{i_0} + 1\}$.

Thus, in order to meet both conditions, it suffices that G_1 obstructs $M[0, u_0 - 1], M[1, w_{i_0} - 1]$ and $M[k_i, \ell_i - v_i]$ for $i \geq 0$, and G_2 obstructs $M[0, v_0 - 1], M[1, \ell_{i_0} - w_{i_0}]$ and $M[k_i, \ell_i - u_i]$. We may assume that the parameters k_i are strictly increasing, for if $k_i = k_{i+1}$ then, as $\ell_i > \ell_{i+1}$ any graph that obstructs $M[k_i, \ell_i]$ also must obstruct $M[k_{i+1}, \ell_{i+1}]$ and, furthermore $(k_{i+1} - k_i)(\ell_{i+1} + 1) = 0$. By Corollary 4.2, we may assume that $r \geq 1$ or $k_0 \geq 1$.

If $k_0 = 0$ and $k_1 = 1$, we may assume that $x = u_0$ and $y = v_0$ have been chosen such that $u_0 + v_0 = \ell_0 + 1$ (x and y not necessarily maximal). Also we may assume that $w_1 = \ell_1 - v_1$ as well as $\ell_1 - w_1 = \ell_1 - u_1$. Thus, also $u_0 - 1 = \ell_0 - v_0,\ w_1 = \ell_1 - v_1$ and $v_0 - 1 = \ell_0 - u_0$ and using induction we compute the size of G as the sum of the sizes of G_1 and G_2, at most

$$\begin{aligned}
&\textstyle\sum_{i=0}^{r}(k_i - k_{i-1})(\ell_i - v_i + 1) + \sum_{i=0}^{r}(k_i - k_{i-1})(\ell_i - u_i + 1) \\
&\quad = \textstyle\sum_{i=0}^{r}(k_i - k_{i-1})\left((\ell_i + 1) - (u_i + v_i - \ell_i - 1)\right) \\
&\quad\quad \leq \textstyle\sum_{i=0}^{r}(k_i - k_{i-1})\left(\ell_i + 1\right) = f(\mathcal{M}).
\end{aligned}$$

In order to complete this case it suffices to additionally consider the first three summands in the induction step. Assume first, that $k_0 \geq 2$. Then we have the first three summands in $f(G_1)$ are $u_0 + w_0 + (k_0 - 1)(\ell_0 + 1 - v_0)$ and for G_2 we have $v_0 + \ell_0 + 1 - w_1 + (k_0 - 1)(\ell_0 + 1 - u_0)$. Adding up these numbers yields

$$(k_0 + 1)(\ell_0 + 1) - (k_0 - 2)(u_0 + v_0 - \ell_0 - 1) \leq (k_0 + 1)(\ell_0 + 1).$$

If $k_0 = 1$ similar to the first case we may assume $w_0 = u_0 = \ell_0 + 1 - v_0$ and we compute

$$\begin{aligned}
& u_0 + w_0 + (k_1 - k_0)(2\ell_1 + 2 - v_1 - u_1) + v_0 + (\ell_0 + 1 - w_0) \\
= \ & (k_1 - k_0)(\ell_1 + 1) + 2(\ell_0 + 1) - (k_1 - k_0)(u_1 + v_1 - \ell_1 - 1) \\
\leq \ & (k_0 + 1)(\ell_0 + 1) + (k_1 - k_0)(\ell_1 + 1).
\end{aligned}$$

Finally, if $k_0 = 0$ and $k_1 \geq 2$ again we may assume $x + y = u_0 + v_0 = \ell_0$ and compute

$$\begin{aligned} & u_0 + w_1 + (k_1 - 1)(2\ell_1 + 2 - v_1 - u_1) + v_0 + \ell_1 + 1 - w_1 \\ = \ & (k_0 + 1)(\ell_0 + 1) + (k_1 - k_0)(\ell_1 + 1) - (k_1 - 1)(u_1 + v_1 - \ell_1 - 1) \\ \leq \ & (k_0 + 1)(\ell_0 + 1) + (k_1 - k_0)(\ell_1 + 1). \end{aligned}$$

Thus, in any case G has at most $f(\mathcal{M})$ vertices.

CASE 2: $c = 1$.

In this case $a \neq 0, b \neq 1, c = 1$, and a disconnected graph $G = G_1 \cup G_2$ has an $M[k, \ell]$-partition if and only if it has an $M[0, \ell]$-partition, or an $M[k, 0]$-partition. It follows from facts proved in [8], and is easy to see directly, that the only minimal $M[k, 0]$-obstruction is K_{k+1}, except in the case when $a = 1$ and $k \geq 2$, when the disjoint union of K_1 and K_2 is the only other minimal $M[k, 0]$-obstruction. Complements of these graphs are all the minimal $M[0, \ell]$-obstructions, the complement of $K_1 \cup K_2$, i.e., the path P_3 with three vertices, only if $\ell \geq 2$ and $b = 0$.

Suppose now G is an $\mathcal{M}$-obstruction. Let k, ℓ be largest integers such that G obstructs $M[k, 0], M[0, \ell]$; note that $k_r \leq k, \ell_0 \leq \ell$. We claim that G contains a disconnected induced subgraph H which obstructs $M[k_r, 0]$ and $M[0, \ell_0]$ and has size

$$k_r + \ell_0 + 1 = f(\{(0, \ell_0), (k_r, 0)\}) \leq f(\{(k_i, l_i)\}_{i=0}^{r}).$$

We may assume that both k_r and ℓ_0 are positive, as in case $k_r = 0$ or $\ell_0 = 0$ the claim holds trivially using the minimal $M[k_r, 0]$-obstructions and the minimal $M[0, \ell_0]$-obstructions.

If G contains K_{k_r+1} and $\overline{K_{\ell_0+1}}$ then, since in a cograph any maximum clique meets any maximum independent set (see, for instance Theorem 11.3.3 in [2]), the union of any two such sets can serve as H (with $k_r + \ell_0 + 1$ vertices).

Next, we consider the case that G contains K_{k_r+1} and P_3 and $\ell_0 \geq 2$. If these obstructions are in different components, then we let $H = K_{k_r+1} \cup P_3$, of size $k_r + 4 \leq (\ell_0 + 1) + k_r$, unless $\ell_0 = 2$. In the latter case we remove the midpoint v of P_3. Then $H \setminus v$ has the right size and contains $\overline{K_3}$. If K_{k_r+1} and P_3 are in the same component, then this component is not a clique. Hence, by connectivity, it contains a clique K of size $k_0 + 1 \geq 2$ and a vertex w which is adjacent to some vertex of K and non-adjacent to another. Now, $K + w$ contains a P_3 and has $k_r + 2 < (\ell_0 + 1) + k_r$ vertices.

If G contains $\overline{K_{\ell_0+1}}$ and $K_1 + K_2$ and $k_r \geq 2$, then we correspondingly find an independent set I with $\ell_0 + 1$ vertices, and a vertex w adjacent to some vertex in I and non-adjacent to another. Hence $I + w$ also contains $K_1 + K_2$.

Finally, if G contains P_3 as well as $K_1 + K_2$, then the P_3 plus a vertex from a different component yields an obstruction of size $4 < (\ell_0 + 1) + k_r$. □

Corollary 4.4 *If M is a constant matrix and G a minimal M-obstruction cograph, then G has at most $(k+1)(\ell+1)$ vertices.* □

If $c \neq 1$ and $b = *$, or if $c \neq 0$ and $a = *$, the bound from Theorem 4.3 is tight. We give a minimal obstruction of size $f(\mathcal{M})$ for the first case, the second follows by taking complements. Let G consist of the disjoint union of $\ell_r + 1$ cliques of size $k_r + 1$ and $\ell_i - \ell_{i+1}$ cliques of size $k_i + 1$ for $0 \leq i \leq r - 1$. We show that G cannot be partitioned into k_i independent sets and ℓ_i cliques. Assume it had such a partition. There are $\ell_i + 1$ cliques of size at least $k_i + 1$. At least one vertex of each of these cliques has to be mapped to a clique, a contradiction. In order to show that G is minimal let v be a vertex in a $k_i + 1$ clique. Then we have ℓ_i cliques of size $> k_i$ and the other cliques can be partitioned into k_i independent sets.

Theorem 4.3 also implies efficient algorithms for the M-partition problem, where M is an (a, b, c)-block matrix. Thus suppose that a, b, c are fixed; given a cograph G, we can find the strictly staircase-like collection dominating all the matrices M_i to which G is an obstruction, in time $O((k + \ell)n)$. Given a staircase-like collection of matrices $\mathcal{M}$, such that G contains an $\mathcal{M}$-obstruction, we can find an induced subgraph H of G, such that H has size at most $f(\mathcal{M})$ and H also contains an $\mathcal{M}$-obstruction, in time $O((k + \ell)n)$. (We always assume the cograph G is given by its cotree; note that the cotree can be found in linear time [5].) The algorithms find all minimal pairs (k, ℓ) such that a corresponding partition exists (along the boundary of the staircase) for each node in the cotree, testing each one in constant time as indicated by the cases in the proof, given the corresponding staircases for the two children in the cotree. Since the length of the boundary of the staircase is $O(k + \ell)$, and there are n nodes in the cotree, the time $O((k + \ell)n)$ follows.

We remark that the upper bound $(k+1)(\ell+1)$ does not hold in general even for the class of trees. For instance, in the case $k = 1$, $b = 0$, $c = *$, there is a tree with $(\ell/3)^2$ vertices that is a minimal M-obstruction [12, 13]. The more general bound $f(\mathcal{M})$ does not hold for trees even in the case $a = b = c = *$: take the stair-like collection $\mathcal{M}$ of two matrices M_0, M_1 with $k_0 = 0, k_1 = 1$, $\ell_0 = 7, \ell_1 = 4$ – we have $f(\mathcal{M}) = 13$, but there is a minimal

$\mathcal{M}$-obstruction with 14 vertices which is a tree, namely an edge $e = uv$ plus four attached paths of length 3, two attached at u and two attached at v. However, it is shown in [15] that the upper bound $(k+1)(\ell+1)$ does apply to collections consisting of one matrix, in the case of chordal graphs.

5 Unions of Cliques

In this section we study minimal obstructions that are unions of cliques. Unions of cliques are an interesting subclass of cographs - while cographs are precisely those graphs not containing an induced path on four vertices, unions of cliques are precisely those graphs not containing an induced path on three vertices.

Recall that we are no longer considering lists. We start the simplest case of a non-constant matrix.

Proposition 5.1 *Let M be an $m \times m$ matrix which has only 0's on the main diagonal, one off-diagonal 1 and $*$'s elsewhere. Then M has just two minimal obstructions that are cographs, namely K_{m+1} and $K_m \cup K_{m-1}$.*

Proof. An M-partition of a graph G is an m-colouring of G, in which two special colour classes are completely adjacent (each vertex of one is adjacent to each vertex of the other). Clearly both K_{m+1} and $K_m \cup K_{m-1}$ are minimal M-obstructions. Suppose G is an M-obstruction cograph not containing K_{m+1}. Then its maximum clique size must be m, as otherwise G, as a cograph, and hence a perfect graph, would be $m-1$ colourable, and so would admit an M-partition. Let A be a clique of size m in G.

Suppose $e = uv$ is any edge of A. The graph $G-u-v$ must have a clique B_e of size $m-1$, or else $G-u-v$ would be $m-2$- colourable, and u and v could be placed as the only vertices in the two classes that are completely adjacent, yielding an M-partition of G.

Suppose G is a minimal M-obstruction cograph. We now claim that the cliques B_e can be chosen so that no clique B_e can contain a vertex w adjacent to exactly one vertex of the edge $e = uv$, say w adjacent to v. (In other words, each vertex $w \in B_e$ is adjacent to either both or to neither of u, v.) Otherwise, let G_e be a smallest induced subgraph of G containing A and B_e without an M-colouring placing u and v as the only vertices in the special classes that are completely adjacent: indeed, considering the cotree of G_e we find that the $\cup$-node where the directed paths from u resp. w to the root meet must be a descendent of the $+$-node, where both meet the path from v to the root. Let U, W be the graphs defined by the children of that $\cup$-node such that $u \in U$ and $w \in W$ and $v \in S$ the graph defined by the child of the +node. The minimality of G_e implies that $G_e \setminus W$ can be placed, and the maximality of the clique A in graph G implies that the largest clique in W is no larger than the clique $U \cap A$. Given the placement for $G_e \setminus W$, we may then place W in the parts where the clique $U \cap A$ is placed, since these parts are joined by $*$, and W can be colored with $|U \cap A|$ colors, thus placing all of G_e, a contradiction.

We may choose e in A joining two sets S and S' closest to the root of the cotree of G. If A and B_e are in different components of G then $A \cup B_e = K_m \cup K_{m-1}$ is an obstruction, while if A and B_e are in the same component of G then each vertex w in B_e is adjacent to at least one endpoint of e, and thus to both, giving the obstruction $B_e \cup \{u, v\} = K_{m+1}$. □

For general matrices M (with k diagonal 0's and ℓ diagonal 1's) we derive the following bounds on possible M-obstructions that are unions of cliques. Recall that we view M as a block matrix with a diagonal matrix A (having zero diagonal) and B (having a diagonal of 1s), and an off-diagonal matrix C and its transpose.

We shall consider the function

$$f(k,\ell) = \begin{cases} (k+1)(\ell+1) & \text{if} \quad k \le \ell+2 \\ k(\ell+2)-1 & \text{if} \quad \ell+2 \le k \le 2\ell+4 \\ \lfloor (k+2\ell+4)^2/8 - 1 \rfloor & \text{if} \quad k \ge 2\ell+4. \end{cases}$$

We note that $f(k,\ell) = \max((k+1)(\ell+1), \Theta(k^2))$, i.e., there exists a function $h(k) = \Theta(k^2)$ such that $f(k,\ell) = \max((k+1)(\ell+1), h(k))$.

Theorem 5.2 *For each k and ℓ there exists a matrix M with k 0s and ℓ 1s on the diagonal, which admits a minimal M-obstruction G with $f(k,\ell)$ vertices that is a union of cliques.*

Proof. The case $k \le \ell+2$ follows from Corollary 4.2, thus assume $k \ge \ell+2$. Let $2 \le 2r \le k$ and start M with r blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the diagonal. This is followed by a constant matrix of size $k - 2r$ with 0's on the diagonal and 1's off-diagonal. All other off-diagonal entries are $*$'s. Let G be a disjoint union of $\ell + r$ cliques of size $k - 2r + 2$ and one of size $k - 2r + 1$. This is an obstruction since removing ℓ of the cliques, we are left with r cliques of size $k-2r+2$ that we can partition at best into an edge and a clique of size $k-2r$. As the cliques are pairwise non-adjacent each of these edges has to use a different block, leaving one vertex of the clique of size $2k-2r+1$ pending. This already shows how to partition $G \setminus v$ if v is in the smaller clique. If v belongs to a clique of size $k-2r+2$ then we can map one vertex of each of the cliques of size $k-r+1$ to the same element of one of the small blocks and, otherwise, proceed as above. Now, choosing $r = \max\{1, \lceil (k-2(l+1))/4 \rceil\}$ yields the desired bounds. □

Theorem 5.3 *Let f be defined as above. If M is any matrix with k 0's and ℓ 1's on the diagonal, then each minimal M-obstruction that is a disjoint union of cliques has at most $f(k,\ell)+(k+1)\ell$ vertices.*

If the block C of M contains no 1, then each minimal M-obstruction that is a disjoint union of cliques has at most $f(k,\ell)$ vertices.

Proof. Suppose a minimal M-obstruction G is the disjoint union of cliques $K_1,\dots,K_t$, with $|K_i| \geq |K_j|$ for $i \leq j$. Removing $v \in K_t$ we have a partition assigning each K_i to a set of parts S_i for $i < t$. The submatrix M_i corresponding to S_i must have at least one 1, as otherwise each part of S_i would be an independent set, i.e., it would contain at most one vertex from K_i; since $|K_t| \leq |K_i|$, we could additionally assign K_t to parts of S_i and not have an obstruction. If M_i has a 1 on the diagonal, we may assume that $S_i = \{s_i\}$ consists of this entry alone. The sets S_i, thus, are partitioned into a collection T of S_i's all having $S_i = \{s_i\}$ and $M_i = (1)$ and a collection R of S_j's where M_j has 0's on the diagonal and some 1 off-diagonal $M(r_j, c_j) = 1$. Note, that part S_j must have at least one vertex, that is in class r_j as well as one that is in c_j. We may further assume that for $S_i \in T$ and $S_j \in R$ we always have $|K_i| \geq |K_j|$, since s_i can absorb any clique. Hence, the clique $K_t \backslash v$ is assigned to parts in R and no pair of cliques from $K_1,\dots,K_{t-1}, K_t \backslash v$ may share the parts s_i, r_j, c_j. Since G is minimal the K_i in T are of size at most $k+1$, as this already enforces them to use a 1 on the diagonal of M. Let $r = |R|$. A set $S_j \in R$ must not use r_k, c_k for $k \neq j$, since K_j is non-adjacent to K_k. Hence, such an S_j has size at most $k-2r+2$ and K_t has size at most $k-2r+1$, for $K_t - v$ avoids all pairs c_j, r_j (note, that $j < t$).

If part C of M has no 1's, then also the $S_i = \{s_i\} \in T$ correspond to cliques of size at most $k-2r+2$. For if, say K_i has size $k_i > k-r+2$ and $v \in K_i$, then $G \setminus v$ has an M-partition where we may assume that $K_i \setminus v$ is one of the ℓ cliques, contradicting G being an M-obstruction. Therefore, in this case, $|V(G)| \leq (\ell+r+1)(k-2r+2)-1$ which is maximized at $f(k,\ell)$. If $r=0$ we have ℓ cliques in T of size $k+1$ and $|K_t| \leq k+1$ adding up to $(\ell+1)(k+1) = f(k,\ell)$. This proves the upper bound for this special case.

Continuing with the general case, the $S_i = \{s_i\} \in T$ correspond to cliques of size at most $k+1$, giving at most $(k+1)\ell$ additional vertices, so $|V(G)| \leq f(k,\ell) + (k+1)\ell$. $\square$

Theorem 5.4 *There exists a matrix M with the k by k block A with 0 diagonal having no off-diagonal 0s, the ℓ by ℓ block B with 1 diagonal having all off-diagonal entries $*$, and the k by ℓ block C having all entries $*$, such that M admits a minimal M-obstruction with $f'(k,\ell) = \Theta(k\ell + k^{1.5})$ vertices, that is a disjoint union of cliques.*

To be more precise, letting

$$r = \max(1, \lceil -1/2 - \ell/3 + \sqrt{(1/2 + \ell/3)^2 + 2(k+1)/3} \rceil),$$

so that $r = \Theta(1)$ if $k \leq \ell$, $r = \Theta(k/\ell)$ if $\ell \leq k \leq \ell^2$, and $r = \Theta(\sqrt{k})$ if $k \geq \ell^2$, and $t = k + 1 - (r^2 + r)/2$, so that $t = \Theta(k)$, we have

$$f'(k,\ell) = (t+r)\ell + tr + (r^2+r)/2.$$

Proof. We may interpret any matrix $A = (k)_{ij}$ with no off-diagonal zeros as a kind of adjacency matrix $A(H)$ of a simple graph H on the k vertices $\{1, 2, \ldots, k\}$: two vertices i, j are adjacent if $k_{ij} = 1$ and nonadjacent if $k_{ij} = *$. Vice versa, with any simple graph H we can associate this way a unique matrix $A(H)$ of the described type.

Let t, r be positive integers, and H be the disjoint union of t isolated vertices and $r - 1$ cliques of sizes $2, 3, \ldots r$ respectively. The corresponding matrix $A = A(H)$ is a $k \times k$-matrix where $k = t - 1 + (r^2 + r)/2$.

Now let G be the graph that is the disjoint union of r cliques of sizes $t+r, t+r-1, \ldots, t+1$ respectively, and an additional ℓ cliques of size $t+r$. Thus $|V(G)| = q = (t+r)\ell + tr + (r^2+r)/2$. First, we show, by induction on r, that G is an obstruction for the matrix M with the A part as described above. If G had an M-partition, then each of the ℓ parts corresponding to a 1 diagonal can be used for a clique of G, and we may put in such parts the largest cliques possible, that is, the ℓ additional cliques of size $t+r$. The remaining r cliques must go to A, so we reduce the problem to A-partition after removing the ℓ additional cliques of size $t+r$ from G. The clique K_G of size $t+1$ in G had to use at least one vertex of a non-trivial clique K_H of H. Since G is the disjoint union of cliques, the other cliques of G may use one and only one vertex of H if and only if K_G uses only one vertex of K_H. Let $\tilde{G}$ arise from G by deleting K_G and one vertex of each of the other non-trivial cliques of G. Then G has an M-partition only if $\tilde{G}$ has an $M(H \setminus K_H)$ partition, which is not the case by inductive assumption (G does not have an $M(\tilde{H})$ partition for any graph $\tilde{H}$ consisting of t isolated vertices and $r-1$ non-trivial cliques; the base case $r = 1$ has G consisting of a clique of size $t+1$ but $\tilde{H}$ has no non-trivial cliques).

We still have to show that the obstruction G is minimal. Assume v is a vertex in the clique of size $t+r-i$, then $G \setminus v$ has $r-i$ cliques of size at most $t+r-i-1$. These can be mapped to into the t isolated vertices of H and to one vertex of each of the $r-i-1$ cliques of size at most $r-i$ of H. From each of the remaining cliques K_j of size $t+r-j$, $0 \leq j \leq i-1$ of G we map t vertices each to the isolated vertices of H and the remaining $r-j$ vertices of K_j to the clique of size $r-j$ in H.

It remains to choose r to maximize

$$q = q_r = (k + 1 - r^2/2 + r/2)\ell + (k + 3/2 - r^2/2)r.$$

We note that

$$q_{r+1} - q_r = -3/2(r^2 + 2(1/2 + \ell/3)r - 2(k+1)/3),$$

so the maximum occurs at

$$r = \max(1, \lceil -1/2 - \ell/3 + \sqrt{(1/2 + \ell/3)^2 + 2(k+1)/3} \rceil).$$

□

Theorem 5.5 *Suppose the block submatrix A with 0 diagonal has no 0 off diagonal. Let f' be defined as above, satisfying $f'(k,\ell) = \Theta(k\ell + k^{1.5})$, and let*

$$g'(k,\ell) = f'(k,\ell) + (k+1)\ell.$$

If M has k 0's and ℓ 1's on the diagonal, then any minimal M-obstruction that is a disjoint union of cliques has at most $g'(k,\ell)$ vertices.

If in addition the block C contains no 1, then any obstruction that is a disjoint union of cliques has at most $f'(k,\ell)$ vertices.

Proof. We proceed as in the proof of Theorem 5.3, and assume a minimal M-obstruction G is the disjoint union of cliques $K_1, \ldots, K_t$ with $|K_i| \geq |K_j|$ for $i \leq j$. Removing $v \in K_t$ we have a partition assigning each K_i to parts from S_i for $i < t$. The sets S_i are partitioned into a collection T of S_i's having $S_i = \{s_i\}$ and $M_i = (1)$ and a collection R of S_j's where M_j has 0's on the diagonal and 1,* off-diagonal. Let U_j be the set of indices that are used exclusively by $S_j \in R$ and D be the set of indices that are used by at least two $S_i \in R$. We may order the sets $U_1, \ldots, U_{r-1}$ nonincreasingly. Then $|U_i| \geq r+1-i$, since otherwise we may U be a set of size $r-i$ consisting of one element from each of $U_i, \ldots, U_{r-1}$, and assign the cliques $K_i, \ldots, K_{r-1}$ and K_t to $U \cup D$, contrary to the fact that G is an obstruction. If we let t be the number of parts in A that do not correspond to $r+1-i$ chosen elements out of U_i, then $k \geq t+2+3+\ldots+r = t-1+(r^2+r)/2$ so $t \leq k+1-(r^2+r)/2$.

If part C of M has no 1's, then also the $S_i = \{s_i\} \in T$ correspond to cliques of size at most $t+r$. For if, say K_i has size $k_i > t+r$ and $v \in K_i$, then $G \setminus v$ has an M-partition where we may assume that $K_i \setminus v$ is one of the ℓ cliques, contradicting G being an obstruction. Therefore $|V(G)| \leq (t+r)\ell + tr + (r^2+r)/2$ which is maximized at $f'(k,\ell)$.

Continuing with the general case, we estimate the largest clique by $k+1$, so $|V(G)| \leq (k+1)\ell + tr + (r^2+r)/2 \leq (k+1)\ell + f'(k,\ell) = g'(k,\ell)$. $\square$

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