On a Conjecture of Erdös, Faber, and Lovasz

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Abstract

A conjecture of Erdös, Faber, and Lovasz states that if a graph G is the union of n edge-disjoint cliques of size n, then G has vertex chromatic number n. An equivalent version requires the cliques to be of size *at most* n, and claims that in that case the vertex chromatic number of G is *at most* n. We prove this equivalent version in the case where the graph G is also a clique, and then study the dual graph H (where the role of vertices and cliques is reversed) and the corresponding bipartite graph J (where one side corresponds to G and the other side to H). We prove the conjecture when J is planar, and when H is not too dense and the cliques in H have at least three vertices. We also prove some cases of H having cliques that are triangles and edges.

1 Introduction

We state the conjecture of Erdös, Faber, and Lovasz, as follows.

Conjecture 1 *If a graph $G = (U, E)$ is the union of at most n edge-disjoint cliques K^i, $1 \leq i \leq n$, where each K^i has at most n vertices, then G has vertex chromatic number at most n.*

In the conjecture, we may assume the following five properties.

(1) Each vertex in G belongs to at least 2 and at most $n-1$ cliques K^i. If a vertex u belongs to only one K^i, we may remove u, and if the vertices of the resulting graph can be colored with n colors then the removed vertices can be colored as well, since each K^i has at most n vertices. If a vertex u belongs to all n cliques, then these cliques share no other vertices and can be colored.

(2) Each clique K^i in G has size at least 2 and at most $n-1$. Clearly the cliques must have at least two vertices, and by (1) each vertex of K^i belongs to at least one other clique, so there can be at most $n-1$ such vertices.

(3) Any two K^i, K^j share a vertex. Otherwise add a new vertex v_{ij} shared by these two cliques, and the sizes of the cliques will never exceed $n-1$. We shall not always assume this property.

(4) Each clique K^i in G has size at least three. A clique K^i of size two can be removed, and then the conjecture would say that the resulting graph with $n-1$ cliques of size at most $n-1$ can be colored with $n-1$ colors. If we add K^i back, a new color can be used if necessary for one of the two vertices of K^i.

(5) At least one vertex in G belongs to at least three cliques K^i. This last claim we shall prove later in the dual graph H.

Consider now the dual graph H whose vertices correspond to cliques K^i in G. If a vertex u in G belongs to $r_u \geq 2$ cliques K^i, then u corresponds to a clique Q_u of size r_u in H. The equivalent conjecture is then as follows.

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Conjecture 2 *If a graph $H = (V, F)$ with $|V| = n$ is the union of edge-disjoint cliques Q^j, then the cliques Q^j can be colored with at most n colors so that any two cliques Q^j of the same color share no vertices.*

The five corresponding properties about H that we may assume are then as follows.

(1) Each clique Q^j in H has size at least 2 and at most $n-1$. This is equivalent to property (1) for G.

(2) Each vertex in H belongs to between at least 2 and at most $n-1$ cliques Q^j. This is equivalent to property (2) for G.

(3) The graph H itself is a clique. This is the same as property (3) for G. We shall not always assume this property.

(4) Each vertex in H belongs to at least three cliques Q^j. This is the same as property (4) for G.

(5) At least one clique Q^j in H has at least three vertices. This is the same as property (5) for G. If all Q^j have two vertices, then the conjecture follows by partitioning the edges of the clique H into at most n matchings by the Walecki construction [1, 3, 7]. More precisely, $n-1$ perfect matchings decompose H if n is even, and n matchings each matching all but one vertex are needed if n is odd.

We finally consider a third equivalent formulation of the conjecture. To this purpose we define a bipartite graph $J = (U, V, L)$, where U is the vertex set of G (corresponding to cliques in H), V is the vertex set of H (corresponding to cliques in G), and L joins vertices such that the corresponding vertex in G for U belongs to the corresponding clique in G for V. Since no two cliques in G share two vertices, J must be *square free*, that is, no two vertices of U in J have two common neighbors in V. The graph J^2, with edges joining vertices at distance two in J induces G on U and induces H on V.

The equivalent third formulation of the conjecture in terms of J is then as follows.

Conjecture 3 *If a bipartite graph $J = (U, V, L)$ with $|V| = n$ is square-free, and the vertices in U and V have degrees at least two, then J^2 can be colored with at most n colors.*

We may again assume the following five properties for J equivalent to those of G and H.

(1) The vertices in U have degrees at least 2 and at most $n-1$.

(2) The vertices in V have degrees at least 2 and at most $n-1$.

(3) Every pair of vertices in V has a common neighbor in U. Again, we shall not always assume this property.

(4) Each vertex in V has degree at least three.

(5) At least one vertex in U has degree at least three.

We state some known results in terms of H. If all Q^j have exactly two vertices, that is, the Q^j are edges, then by Vizing's theorem [10], the Q^j can be colored with d or $d+1$ colors, where d is the maximum degree in H. If all Q^j have exactly k vertices, with k fixed, and the maximum degree is d, then as $d \to \infty$, the Q^j can be colored with $(1+o(1))d/(k-1)$ colors, by a result of Pippenger and Spencer [8]. If the Q^j have arbitrary sizes, and the maximum degree is d, then as $d \to \infty$, the Q^j can be colored with $(1+o(1))d$ colors, see Kahn [5, 6]. In particular, we have that $n + o(n)$ colors suffice. Finally, the proof of the Erdös-Hanani conjecture, known as the Rödl nibble, implies that there exists H formed with cliques Q^j of any fixed size k, so that the number of edges of H is $(1-o(1))\binom{n}{2}$, see [9].

We prove the conjecture when G is a clique, when J is planar, and when H is not too dense and the cliques Q^j in H have at least three vertices. We also prove some cases of H where the cliques Q^j are triangles and edges, including generalizations of the case of the Walecki construction mentioned above.

2 The Case where G is a Clique

In this section we prove Conjecture 1 when G is a clique. This is equivalent to Conjecture 2 when H is such that every pair of Q^j share a vertex.

Theorem 1 *If H is such that every pair of Q^j share a vertex, then the number m of Q^j satisfies $m \leq n$, so Conjecture 2 holds in this case.*

Proof. Let r be the size of smallest clique $Q = Q^j$. Name the vertices of Q $v_1, v_2, \ldots, v_r$, and say v_i belongs to s_i cliques Q^j other than Q. Order the v_i so that $s_1 \geq s_2 \geq \cdots \geq s_r$.

If $s_2 = 0$, then all the cliques Q^j meet Q at v_1, and each clique has at least one vertex other than v_1, so $m < n$.

Otherwise $0 < s_2 \leq s_1$. The total number of Q^j is $m = s_1 + s_2 + \cdots + s_r + 1$. Since each of the s_2 cliques sharing v_2 with Q must meet each of the s_1 cliques sharing v_1 with Q at a different vertex, and Q has r vertices, we have $n \geq s_1 s_2 + r$ vertices.

Note that $s_2 + 1 \geq r$. If $0 < s_2 < s_1$, then

$$m \leq s_1 + (r-1)s_2 + 1$$

$$\leq s_1 s_2 + s_1 + 1 - (s_1 - s_2)s_2$$

$$\leq s_1 s_2 + s_1 + 1 - (s_1 - 1)$$

$$\leq s_1 s_2 + 2$$

$$\leq s_1 s_2 + r \leq n.$$

If $s_2 = s_1$, suppose $s_2 + 1 > r$. Then

$$m \leq r s_1 + 1$$

$$\leq s_1 s_2 + 1$$

$$\leq s_1 s_2 + r \leq n.$$

Finally if $s_2 = s_1$ and $r = s_2 + 1$, then

$$m \leq r s_1 + 1$$

$$\leq s_1(s_2 + 1) + 1$$

$$\leq s_1 s_2 + r = n.$$

□

By duality, we then have the following.

Theorem 2 *If G is a clique, union of at least two edge-disjoint cliques K^i, then the number m of vertices in G satisfies $m \leq n$, where n is the number of cliques K^i in G. Therefore Conjecture 1 holds if G is a clique.*

3 The Case where J is Planar

In this section we consider the case where J is planar.

Theorem 3 *If $J = (U, V, L)$ is planar, then Conjecture 3 holds.*

Proof. Notice that all faces of a planar embedding of J have an even number of sides at least 6. We show that we can assume all faces have exactly 6 sides without increasing n, the number of vertices in V, only increasing the number of vertices in U. Let a face be $v_1u_1v_2u_2 \cdots v_tu_t$. If this face has at least 8 sides, then $t \geq 4$, and we can add either a path $v_1u'v_3$ or a path $v_2u''v_4$ inside this face, while staying square-free. This makes the number of sides of the face 2 less and adds a face with 6 sides.

As in the introduction, we may also assume all vertices $v \in V$ have degree at least 3, and at least one vertex $u \in U$ has degree at least 3. Consider a face $v_1u_1v_2u_2v_3u_3$, where u_2 and the v_i have degree at least 3, and consider identifying u_1 with u_3 and v_2 with v_3, and corresponding edges, thus shrinking one 6-cycle. This operation reduces n by 1, so we may proceed by induction to color the u_i with $n-1$ colors, and after undoing the identifying give a new color to u_1, while keeping the color at u_3.

The problem with the identifying operation is that we may no longer have a square-free graph. This can only happen if J contains a 6-cycle C that is not a face. In that case, choose such a C having fewest vertices inside, and consider the subgraph J' of J induced by the vertices inside C or on C. If J' still has a vertex in U of degree at least 3, then we again consider a face containing such a u_2, and the result still follows by induction because there is no 6-cycle that is not a face inside C. Regardless of whether u_2 of degree at least 3 is in the boundary of J' or not, we can choose the face F bounded with u_2 so that at most one edge of the face F is in the boundary of J'. Therefore the shrinking of F will not turn the boundary of J' into a 4-cycle.

In the final case, the subgraph J' only has vertices of U of degree 2. Notice that this corresponds to a graph H' that is triangulated, so by Euler's formula there must be a vertex v of V in J' of degree 3,4, or 5. If there is only one vertex of V in J outside C, then only one clique Q^j of the corresponding H is of size strictly greater than 2, so as in the Introduction, by the Walecki construction, we may in the even n case use a new color for this large Q^j, since Walecki for n even uses only $n-1$ colors; and in the odd n case, choose Q^j of give size at least 3 so that for the almost perfect matching of color 1, if one endpoint of a matched edge of color 1 is included in Q^j then so is the other endpoint, and then Q^j can be given color 1.

In the remaining case, J' only has vertices of U of degree 2, and there are at least two vertices of V in J outside C. If v is of degree 3 in V inside C, then some path P of length 2 from v goes to a vertex of V of degree at most $n-2$, so removing this path P leaves us at the end of the path with degrees 2 and at most $n-3$, so there is always a color available for the middle vertex of the path P. We may then remove the other two paths of length two from v, and remove v, thus reducing n by one. The induction then follows after we turn the faces into 6-cycles as before.

Finally, if v has degree 4 of 5, then the path P of length 2 can be chosen to join to a vertex v' inside C as well, and v' will not be joined by paths of length 2 to at least one vertex on C and to the at least two vertices of V outside C, so v' has degree at most $n-4$. After removing P, v will have degree at most 4 and v' will have degree at most $n-5$, so again a color is available for the middle vertex of P. We similarly remove all paths of length 2 starting at v of degree 4 or 5, and remove v, thus reducing n by one. The induction then follows after we turn the faces into 6-cycles as before. □

If we assume instead that J has bounded genus γ, the induction above fails when the 6-cycle that is not a face cannot be shrunk to a point. However, it seems this would affect only base cases with n small enough, so some shifted bound $n + f(\gamma)$ should be provable.

4 The Case where H is not too dense and $|Q^j| \geq 3$

We first show the conjecture when H is not too dense, and the cliques Q^j have size at least 3.

Theorem 4 *Let k be the size of the smallest clique Q^j in H, or equivalently, the minimum degree of a vertex in U for J. If $k \geq \sqrt{n}+1$, then Conjectures 2 and 3 hold. If $k \geq 3$ and H has at most $\frac{1}{2}(1-\frac{1}{k})^2 n^2$ edges, then Conjecture 2 holds. Note that this bound is a constant away from the maximum possible number of edges, namely $\binom{n}{2} = \frac{n(n-1)}{2}$ for H.*

Proof. If $k \geq \sqrt{n}+1$ is the minimum size of a clique Q^j in H, then each Q^j has at least

$$\binom{k}{2} = \frac{k(k-1)}{2} \geq \frac{(\sqrt{n}+1)\sqrt{n}}{2} > \frac{n}{2}$$

edges, a fraction of at least $\frac{1}{n-1}$ of the possible $\binom{n}{2}$ edges of H. Therefore the number of Q^j is at most $n-1$, so clearly n colors suffice to color the Q^j. The result for J follows by equivalence of the two conjectures.

If $k \geq 3$, and H is not too dense, we shall show that some Q^j in H is adjacent to at most $n-1$ cliques $Q^{j'}$, so we can remove this Q^j, proceed to color the resulting H' by induction, and then color the remaining Q^j with a color out of n colors that did not occur in the at most $n-1$ neighboring $Q^{j'}$.

Suppose to the contrary that each Q^j has at least n neighboring $Q^{j'}$, and choose first a $Q = Q^j$ that has size k. Since all $Q^{j'}$ containing a vertex v of Q have at least k vertices, they each give degree at least $k-1$ to v, so there can be at most $\frac{n-1}{k-1}$ such $Q^{j'}$, or at most $\frac{n-1}{k-1} - 1$ such $Q^{j'}$ if we do not count Q containing v. On the other hand, since at least n cliques $Q^{j'}$ meet Q, we have that at least $\frac{n}{k}$ cliques $Q^{j'}$ meet Q at some v of Q, since Q has k vertices, or at least $\frac{n}{k}+1$ such $Q^{j'}$ meet at v is we count Q as well.

We therefore have some number

$$\frac{n}{k} + 1 \leq x \leq \frac{n-1}{k-1}$$

of cliques Q^j meeting at v. But then for each such Q^j, at least $n-x$ cliques $Q^{j'}$ must meet each Q^j at vertices other than v, which gives $n-x$ edges of J coming out of the vertices other than v in V for each of these x cliques Q^j, plus at least k edges of J in Q^j. Thus the number of edges of J is at least

$$x(n-x+k) \geq x(n-x) + k \geq \frac{n}{k}(n - \frac{n}{k}) + k$$

by a minimization on x of $x(n-x)$. The result for H follows by observing that if J has $k' \geq k$ edges corresponding to a clique Q^j in H, then H has $\binom{k'}{2} \geq k' \frac{k-1}{2}$ edges for Q^j. This shows that H has number of edges at least $\frac{k-1}{2}$ times the number of edges of J, which is too dense for J. $\square$

The following result proves in particular the conjecture for $n \leq 10$ if all Q^j in H have size at least 3.

Theorem 5 *Let k be the size of the smallest Q^j in H. If H has $n \leq (k-1)(k+2)$ vertices, then Conjecture 2 holds.*

Proof. Write $n = (k-1)x + y$, where $0 \leq x \leq k+1$ and $1 \leq y \leq k-1$. Let $Q = Q^j$ be a clique of size k for H. Then each vertex v of Q belongs to at most x cliques $Q^{j'}$, since each such $Q^{j'}$ has at least $k-1$ neighbors for v. Thus, not counting Q, we have at most $x-1$ neighbor cliques for each of the k vertices of Q. The number of cliques sharing vertices with Q, other than Q, is thus at most

$$k(x-1) = (k-1)x + x - k \leq (k-1)x + y = n$$

with equality only if $x = k+1$ and $y = 1$, namely only $n = k^2$. Thus unless $n = k^2$, the clique Q has at most $n-1$ neighboring cliques $Q^{j'}$, and we can remove Q, proceed by induction, and then assign to Q the remaining color out of n, different from the colors used for the $n-1$ neighboring cliques $Q^{j'}$.

If $n = k^2$, and the inequality above is strict, then the induction follows as well. But equality holds if each vertex v of Q belongs to exactly $k+1$ cliques of size k, giving full degree $(k-1)(k+1) = n-1$ to each vertex v of Q. But then Q could also be taken to be any of the neighboring cliques at v, covering all vertices of H, so H must be a clique with all Q^j of size k.

So suppose H is a clique of size $n = k^2$ whose edges are partitioned into $k(k+1)$ cliques of size k. Notice that each clique $Q = Q^j$ has k^2 neighboring cliques $Q^{j'}$, thus at least $k-1$ non-neighbor cliques $Q^{j''}$. We can thus choose to give to Q and one such $Q^{j'}$ the same color. We shall select k such pairs with same color, reducing the number of colors for the $k(k+1)$ cliques to $k(k+1) - k = k^2 = n$ colors as required.

We denote the k pairs to be chosen as $(Q^1, R^1), \ldots, (Q^k, R^k)$. The initial (Q^1, R^1) was already chosen. So assume we have chosen up to (Q^{j-1}, R^{j-1}), with all chosen Q^i with $2 \leq i \leq j-1$ sharing a common vertex v with Q^1, and with each such Q^i sharing each a different vertex with R^1. Then choose a vertex v' from R^1 different from the vertices shared by the Q^i with $2 \leq i \leq j-1$, and let Q^j be the clique containing the edge (v, v'). We then let R^j be one of the $k-1$ cliques that do not share a vertex with Q^j, and different from the $j-2 \leq k-2$ cliques R^i with $2 \leq i \leq j-1$ already chosen. This completes the choice of k pairs of cliques (Q^j, R^j) to be assigned the same color, completing the proof. □

Finally, we prove a bound on the number of edges of J in a uniform case.

Theorem 6 *Let $J = (U, V, L)$ with $|U| = m$ and $|V| = n$ be a square-free bipartite graph with vertices of degree at least 2 as in Conjecture 3. Assume all vertices of U have same degree d_1 and all vertices of V have same degree d_2. Then $|L| < \min(n\sqrt{2m}, m\sqrt{2n})$. Furthermore, there exist such graphs J with $n = p^2$, $m = ps$, and $|L| = p^2 s = m\sqrt{n}$ for p prime and any $1 \leq s \leq p+1$.*

Proof. Any vertex v in V has $d_2(d_1 - 1) \leq n-1$ vertices at distance 2, so $d_1 d_2 < 2n$. This gives $(\frac{|L|}{m})(\frac{|L|}{n}) < 2n$, and therefore $|L| \leq n\sqrt{2m}$. By symmetry, we also have $|L| \leq m\sqrt{2n}$.

For the construction with p prime, we consider the graph H associated with J such that H is a graph with $n = p^2$ vertices decomposed into s partitions of the vertices into p cliques of size p, for a total of $m = ps$ cliques. We label the vertices of H as (i, j) with $0 \leq i, j < p$ and i, j considered as integers modulo p. The partition number t for $0 \leq t < p$ will have cliques Q^i, $0 \leq i < p$ having vertices $(j, i + tj)$, $0 \leq j < p$. The partition number p, if $s = p+1$, will have cliques Q^i, $0 \leq i < p$ having vertices (i, j), $0 \leq j < p$. □

5 Tight Colorings of Partially Complete Triple Systems

A *triple system* is a graph H with n vertices whose edges are decomposed into edge-disjoint triangles. The triple system is *complete* if H is a complete graph on n vertices. A *coloring* of a triple system is an assignment of colors to the triangles in the triple system so that if two triangles share a vertex then the two triangles have different colors.

A complete triple system needs at least $k = \frac{n-1}{2}$ colors to be colored, since each vertex participates in k triangles. It has been conjectured that n colors suffice to color such a complete triple system (a special case of a conjecture of Erdös, Faber and Lovasz), and it has been shown that $\frac{n}{2} + o(n)$ colors suffice.

We consider the case where some t special vertices in G are chosen, it is required that each of these t vertices participate in k triangles, and that every triangle include at least one of the t vertices. We call such triple systems partially complete, and we ask for which values of t does it suffice to have only k colors. We show the following.

Theorem 7 *If $n \geq 1 + 10(t-1)$, then k colors are necessary and sufficient.*

Proof. For each $1 \leq i \leq t$, we define S_i to be the set of k dijoint edges e which together with the ith special vertex (this vertex is not in e) form a triangle in the triple system. The t sets S_i each have size k. We then define a graph L whose vertices are the union of the sets S_i. Two vertices in different S_i, say corresponding to e_i, e_j in S_i, S_j are joined by an edge if the two triangles (i, e_i) and (j, e_j) can be given the same color. If $i \in S_j$ (and thus $j \in S_i$) as well, then the only neighbor if e_i in S_j is e_j (and the only neighbor of e_j in S_i is e_i). Otherwise, two e_i, e_j are adjacent if and only if they do not share an element.

The aim is to decompose L into k cliques of size t, one per color, thus each clique containing exacly one vertex from each S_i. Suppose we have done this by induction up to S_{t-1}. Note that at most one element e_t of S_t can be joined by a single edge to each S_i, $1 \leq i \leq t-1$, for a total of $s \leq t-1$ such elements e_t. Note also that each remaining e_t forbids in addition at most two elements e_i of each S_i that share an endpoints with e_t, therefore there are at most three forbidden such e_i for each such e_t, for a total of at most $2(t-1)+s$ over all $1 \leq i \leq t-1$, giving degree from e_t to the cliques of size $t-1$ over all $1 \leq i \leq t-1$ at least $k - 2(t-1) - s$. A similar argument shows degree at least $k - 2(t-1) - s$ from the cliques of size $t-1$ to S_t as well.

Suppose there is no way to match S_t to the cliques of size k. Then we have a bipartite graph with $k - s$ vertices on each side and degrees at least $k - 2(t-1) - s$ such that there is a set T in the left whose neighbors U in the right have $|U| < |T|$ by Hall's Theorem. If $|T| \leq k - 2(t-1) - s$ then $|U| \geq |T|$ because each vertex in T has degree at least $k - 2(t-1) - s$. If $|T| > 2(t-1)$ then each vertex not in U would not be adjacent to the s uniquely matched vertices and the other more than $2(t-1)$ vertices in T, which is not possible, so U would be all of the right side, a contradiction. It thus suffices to have $2(t-1) \leq k - 3(t-1)$, or $k \geq 5(t-1)$, or $n \geq 1 + 10(t-1)$. □

A special case with a stronger result is the following.

Corollary 1 *If $t = 3u$ and u triangles cover of the triple system cover all of the t special vertices, then $n \geq 3 + 8(t-1)$ is enough to guarantee k colors suffice.*

Proof. Order the S_i by selecting the three elements of the u special in some order first, then going through all u triangles Now not only at most $t-1$ elements of S_t have a unique neighbor among the preceeding S_i, in fact only one such element e_t will have a unique neighbor, namely the one that participates in the special triangle out of u.

Therefore the first case in the theorem becomes $|T| \leq k-1-2(t-1)$, the second case is $|T| > 2(t-1)$ as before, and therefore $k \geq 1+4(t-1)$ suffices, or $n \geq 3+8(t-1)$. $\square$

Triple systems generalize to r-systems, where G is decomposed into edge-dijoint cliques of size r. The partially complete case is where $n = 1+(r-1)k$, some t special vertices participate in k such cliques of size r, and every such clique of size r contains at least one of the t vertices. We consider the special case $t = ru$, and u special cliques of size r cover the t special vertices.

Theorem 8 *In this $t = ru$ case, it is enough that $n \geq r+2(r-1)^2(t-1)$ for k colors to suffice.*

Proof. Now the S_i fo $1 \leq i \leq t$ are sets of size k consisting of disjoint $r-1$ cliques K_i, so that (i, K_i) is a k-clique in the system. If we do not consider the special clique on the t vertices, we are left with sets S'_i of size $k-1$. Each element of S_t forbids $r-1$ elements of each other S_i, for a total of at most $(r-1)(t-1)$ forbidden elements. Therefore the degree for S_t, and in the other direction as well, is at least $k-1-(r-1)(t-1)$. As before, if there is no matching, we must have $|T| > |U|$, where U is the neighbors of T. If $|T| \leq k-1-(r-1)(t-1)$ then this is not possible because of degrees, and if $|T| > (r-1)(t-1)$ this is not possible since each vertex in the side of U would have to have a neighbor in T.

It thus suffices that $(r-1)(t-1) \leq k-1-(r-1)(t-1)$, or $k \geq 1+2(r-1)(t-1)$, or $n \geq r+2(r-1)^2(t-1)$. $\square$

6 Walecki Constructions

Walecki showed that a complete graph with n vertices can be decomposed into $n-1$ perfect matchings if n is even, and into n matchings that leave only one vertex unmatched if n is odd, thus verifying for this case Conjecture 2 for H.

We consider first the case where we are given two numbers $k \geq 2, \ell \geq 3$, and the graph G with $n = k\ell$ is a clique decomposed into k disjoint cliques of size ℓ, with the remaining edges of the clique G forming themselves individual cliques. Here we show the following.

Theorem 9 *In this case with $n = k\ell$, it is necessary and sufficient to use $n-\ell+1$ colors if n is even, and $n-\ell+2$ colors if n is odd.*

Proof. For all n, an obvious lower bound on the number of colors is $n-\ell+1$, the number of cliques that each vertex belongs to, namely one clique of size ℓ plus $(n-1)-(\ell-1)$ cliques that are individual edges. If n is odd, and we consider a color not used for any of the cliques of size ℓ, clearly this color will avoid at least one vertex that is not matched by this color, increasing the lower bound to $n-\ell+2$.

If k is even, first consider a clique G' of size k obtained from G by shrinking the k cliques of size ℓ to single vertices, and decompose this clique into $k-1$ perfect matchings. After de-shrinking G' back to G, each edge of a matching in G' corresponds to the edges of ℓ matchings of size ℓ, giving a decomposition into $(k-1)\ell$ perfect matchings in G plus one color for the k disjoint cliques of size ℓ, for a total of $(k-1)\ell+1 = n-\ell+1$ colors.

If $k = 2k'+1$ is odd, then again we consider the clique G' of size k obtained from G by shrinking the k cliques of size ℓ to single vertices, and decompose G' into k' sets that are unions of disjoint cycles that cover all the vertices (e.g. Hamiltonian cycles). This can be done by labelling the vertices of G' as $1, 2, \ldots, k$ modulo k and connecting the vertices that are i apart in this labelling for $i = 1, 2, \ldots, k'$.

If ℓ is even, then for each such union of cycles, we consider the decomposition of a clique of size ℓ into $\ell - 1$ perfect matchings, and each edge of the oriented union of cycles will correspond to a choice of each one of these $2(\ell - 1)$ possible matchings from one clique of size ℓ in G to the next in both directions, for a total of $k'2(\ell - 1)$ perfect matchings. It remains to match the jth vertex of a clique of size ℓ to the jth vertex of other cliques of size ℓ, which can be done with k matchings that avoid one clique which is also selected with the same color. The total number of colors is thus $(k - 1)(\ell - 1) + k = n - \ell + 1$.

Finally, if k, ℓ are both odd, then a clique of size ℓ is decomposed into 2ℓ matchings plus orientation for the oriented union of k' cycles. We need in addition k matchings in G', which can have one edge included in the $k'2\ell k$ matchings for G a total of ℓ times for the first $2k'\ell$ of these matchings, plus one additional matching. Adding one color for the cliques of size ℓ gives $k'2\ell+2 = n-\ell+2$ colors. □

The Walecki construction for n even describes the clique of size n as a regular polygon in the plane with $n-1$ vertices plus the center, and each perfect matching consists of parallel edges in the plane for the polygon, with the remaining vertex matched by a perpendicular edge to the center. For n odd, we omit the center.

Theorem 10 *If n is even, consider G being a clique of size n with the edges that are not cliques foming cliques that are triangles joined at the central vertex. If the number of triangles is even, then $n - 1$ colors are necessary and sufficient, while if the number of triangles is odd, then n colors are sufficent (but only known as necessary for one triangle). If n is odd, then n colors suffice for any number of triangles joined at one vertex plus the remaining edges to form a clique G of size n.*

Proof. For n even, clearly one vertex will be in no triangle, showing the lower bound of $n - 1$. If there is only one triangle, then the color for this triangle will be avoided by one of the remaining odd number of vertices, showing the lower bound of n.

For n even and an even number of tringles, these triangles can be chosen to join the center to almost diametrically opposite sides of the polygon for each pair of triangles. The color of the polygon side edge is chosen for the triangle, which is normally the color used to join the center to the opposite vertex to this edge in the polygon, but this color is here omitted since it is replaced by the color used by the almost opposite triangle. Thus in some sense colors at the center get flipped among almost opposite triangles, giving the coloring with a tight number of $n - 1$ colors.

For n even and an odd number of triangles, the remaining triangle is given a new color, for a total of n colors. Instead, we could color the remaining traingle as the polygon edge suggests, and use the new color for the opposite edge. The opposite edge can be omitted if we reduce the number of vertices to $n-1$ to make n odd, and then $n-1$ colors will suffice, or n colors for n odd. □

Theorem 11 *If n is odd and one vertex is joined to all the other vertices by triangles with all other edges left the same, then $n - 2$ colors are necessary and sufficient for $n \geq 7$, and 4 colors are necessary and sufficient for $n = 5$.*

Proof. The upper and lower bounds are obvious for $n = 5$ and two triangles, so we consider $n \geq 7$ odd. The lower bound is obvious by considering the number $n - 2$ of cliques for the vertices that are not the vertex joining all the triangles.

For the $n-2$ upper bound, consider the Walecki construction for $n-1$ even which gives $(n-1)-1 = n-2$ colors. Find an additional perfect matching on the $n-1$ vertices which uses one edge of each color, which can be done for $n-1 \geq 6$ even. The first idea, if the vertices are $1, 2, \ldots, 2k-1 = n-2$ around the polygon, and 0 at the center is to match 0 to 1, i to $k+i-1$ for $i = 2, \ldots, k$. This produces no parallel edges, and works if k is odd since in this case no edge is perpendicular to the edge joining 0 and 1. If $k = 2k'$, still join 0 to 1, join i to $2k'+i$ for $i = 2, \ldots, 2k'-2$, join $2k'-1$ to $2k'$, and join $2k'+1$ to $4k'-1$.

Once the perfect matching has been found, with different colors, use this matching joined with a new vertex to form triangles with the same different colors. $\square$

Corollary 2 *If n is even and one vertex is joined to all but one of the other vertices by triangles with all other edges left the same, then $n-1$ colors are necessary and sufficient for $n \geq 6$, and 4 colors are necessary and sufficient for $n = 4$.*

Proof. This follows from the previous result by removing one vertex other than the vertex joined to all the triangles from the odd case. $\square$

We next consider the case where G is a complete graph and the cliques are edges except for triangles containing at least one of two vertices x, y. For n even, each of x, y is joined by triangles to a complete matching of G minus x, y. For n odd, there is a triangle xyz, and each of x, y is joined by triangles to a complete matching of G minus x, y, z. In both cases, the two matchings joined to x, y form a union of even length cycles.

Theorem 12 *Suppose the even length cycles in the union of the two matchings for x, y have lengths congruent to 2 modulo 4. Then if n is odd, $n-1$ colors suffice and $n-2$ colors are needed, and if n is even, $n-2$ colors suffice and $n-3$ colors are needed. In general, if n is odd, n colors suffice and $n-2$ colors are needed, and if n is even, $n-1$ colors suffice and $n-3$ colors are needed. Suppose next that the even length cycles in the union of the two matchings for x, y have lengths congruent to 4 modulo 8. Then if n is odd, $n-1$ colors suffice and $n-2$ colors are needed. Suppose finally that there is a single even length cycle, of length $n-3$ for n odd. Then $n-2$ colors are necessary and sufficient.*

Proof. Suppose n is odd. Then $n-2$ colors are needed since z participates in $n-2$ cliques of which only one is a triangle. For the $n-1$ construction, consider the Walecki construction for G minus x, y having $n-2$ vertices, with $n-2$ odd, so the construction has no center. The vertices other than z must belong to cycles of length 2 modulo 4 $(6, 10, \ldots)$. These cycles will be paths on the perifery of the $(n-2)$-gon with the last edge coming back. We must show that the two corresponding matchings formed by these cycles have edges with all different colors. Clearly the edges on the perifery of the polygon all have different colors, but the edge closing the cycle repeats a color on the path of the same cycle. However, the repeated color appears in two different matchings, since the length of the cycle is congruent to 2 modulo 4. Therefore $n-2$ colors suffice, plus one color for the tringle xyz, giving the total of $n-1$ colors.

If the even lengths are just assumed to be different from 4, then we can pair up the cycles that have lengths multiples of 4, and exchange the two rightmost vertices of the left cycle with the two leftmost vertices of the right cycle. This makes the cycles act like cycles of length 2 modulo 4. There may be a cycle left of length multiple of 4 that will require one more color, costing n instead of $n-1$. Finally, if there are cycles of length 4 among the cycles of lengths multiples, then put a cycle of length 4 to the left of a cycle of length multiple of 4 (possibly 4 as well), flip the two rigthmost vertices of the left cycle and exchange these two vertices with the single leftmost vertex of the right cycle. Again, there may be a cycle of length multiple of 4 that will require one more color, costing n instead of $n-1$. The key is that all edges in this construction have an even number of vertices on the perifery of the polygon as well.

Suppose next n is even. Then $n-3$ colors are needed since each vertex other than x, y participates in $n-3$ cliques of which two are triangles. For the $n-2$ construction, consider again the Walecki construction for G minus x, y having $n-2$ vertices, with $n-2$ even, so this time the construction has a center. All vertices must belong to cycles of length 2 modulo 4. The cycles are defined as before, except for one cycle that goes through the center vertex instead of a single edge closing the cycle. Thus we must be careful not to repeat colors for the two edges joining the center. This can be done for one of these two edges by choosing the repeated color that could occur in the opposite side on the perifery to be on the other matching, but we cannot then guarantee that the other edge joining the center will not repeat on the perifery, so we need a new color besides the $n-3$ colors on the Walecki construction, for a total of $n-2$ colors. One perifery color will not be used in any of the even length cycles, and can be used to join x, y, so $n-2$ colors suffice. For the general case, we proceed to do the switch as in the odd case. We may use two new colors for the two edges joining the center instead of just one, but the extra color used for the case of a left over cycle of length multiple of 4 could be one of these two, costing $n-1$ instead of $n-2$.

Suppose that the even cycles have lengths congruent to 4 modulo 8, and n is odd. The lower bound $n-2$ is as before, and the upper bound $n-1$ is obtained as before by considering the Walecki construction without center for G minus x, y having $n-2 = 2k+1$ vertices. We label the vertices along the perifery as $0, 1, \ldots, 2k$, and let one of the two matchings be $(1, k+1), (2, k+2), (k, 2k)$, with 0 being the vertex z. Notice that the edges of this matching are not parallel, so they all have different colors. Suppose one even length cycle has length 4ℓ congruent to 4 modulo 8. For this cycle we use the vertices $1, \ldots, 2\ell, k+1, \ldots, k+2\ell$ and include in the other matching the edges $(1,2), (3,4), \ldots, (2\ell-1, 2\ell), (k+2, k+3), (k+4, k+5), \ldots, (k+2\ell-2, k+2\ell-1), (k+1, k+2\ell)$. Notice that the edge $(k+1, k+2\ell)$ is parallel to $(k+\ell, k+\ell+1)$ in the perifery, with ℓ odd, which was not included. Therefore all these essentially perifery edges all get different colors. For the next cycle, of length $4\ell'$, shift the construction by adding 2ℓ, and so on and so forth. In the end, as before, we use $n-2$ colors plus one color for xyz, giving the upper bound $n-1$.

Suppose finally that there is a single even length cycle, of length $n-3$ for n odd. Then $n-2$ colors are necessary as before, and sufficient as follows. Consider again the Walecki construction for $n-2 = 2k+1$ odd, with the center omitted and $n-2$ colors. The cycle of length $2k$ omitting 0 out of $0, 1, \ldots, 2k$ is obtained by selecting all the edges $(i, i+2)$ of different colors, where addition is modulo $2k+1$, then omitting the edge $(2k, 1)$ that has the color missing at $z = 0$ which will be used for the triangle xyz, replacing the edge $(k-1, k+1)$ with the edge $(1, 2k-1)$ of the same color, the edge $(k, k+2)$ with the edge $(2, 2k)$ of the same color, the edge $(0, 2)$ with the edge $(k+2, k+1)$ of the same color, and the edge $(2k-1, 0)$ with the edge $(k, k-1)$ of the same color. $\square$

Theorem 13 *In the Walecki construction, suppose n is odd (thus no center of the polygon), and label the vertices along the perifery $1, 2, \ldots, n$. Then we may consider only having triangles $i-1, i, i+1$ (modulo n), and color them with the missing color i, using n colors. If n is even, then we add the center, and this construction with $n-1$ colors works as well, provided though that each chosen triangle on the perifery has an almost diametrically opposite chosen triangle on the perifery. Additional special triangles that are vertex-disjoint can be added if we use n colors.*

Proof. The case of n odd follows directly from the missing color i at i already appearing at the edge $(i-1, i+1)$.

For n even, with almost diametrically opposite triangles $(i-1, i, i+1)$ and $(i'-1, i', i'+1)$, we may join the center with i using color i' and join the center with i' using color i, flipping colors. □

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