Homomorphic Image d-Cubes of $d+1$-Cubes are Retracts

Tomás Feder* Carlos Subi†

June 26, 2009

Theorem 1 *Suppose a d-cube H_d is a homomorphic image of a $d+1$-cube H_{d+1} (with loops included). Then H_{d+1} contains a d-subcube H'_d such that the homomorphism restricted to H'_d is an isomorphism to H_d.*

Proof: If $d = 0$ then any vertex H'_d maps to the vertex H_d by an isomorphism. If $d = 1$ then any edge H'_d that maps to the edge that is not a loop in H_d has H'_d mapping to H_d by an isomorphism. So assume $d \geq 2$.

If some vertex v in H_d is the image of just one vertex w in H_{d+1}, then the $d+1$ edges incident to w must map to at least the d edges incident to v. Choose d edges incident to w that map to the d such distinct edges incident to v other than the loop. We may assume w is all 0s, and that these d edges leave fixed d distinct vertices adjacent to w that have exactly one 1. Then the vertex adjacent to two such vertices with exactly one 1, having exactly two 1s, is also fixed, so as the fixed points of an endomorphism form a 2-satisfiability instance, and the 2-satisfiability instance has no clauses, we have that H'_d involving w and all these d dimensions has all vertices fixed, so if we set $H_d = H'_d$ then the homomorphism is a retraction mapping H'_d to H_d by an isomorphism, as desired.

Otherwise every vertex in H_d has at least two vertices in H_{d+1} that map to it, thus exactly two. Suppose v, v' are antipodal in H_d, thus at distance d, and have preimages in H_{d+1} given by w_1, w_2 and w'_1, w'_d, where w_i and w'_i are at distance d or $d+1$ from each other. Then w_1 and w_2 are at distance 1 or 2 from each other.

Say w_1 and w_2 are at distance 2 from each other. Ignoring every dimension other than the first three (such dimensions are set to 0), we may assume $w_1 = 000$ and $w_2 = 110$. Consider the vertices 001 and 111. The vertex 011 is adjacent to $001, 111, 010$ so two of these three must be mapped to the same vertex. Similary the vertex 101 is adjacent to $001, 111, 100$ so two of these tree must be mapped to the same vertex. So either 001 is identified with 111, or say 001 with 010 and 111 with 100. Since dimension 3 was chosen arbitrarily, we can assume that the second choice happens for at most one choice of dimension 3, so for all but at most one choice of dimension 3, we have that 001 is identified with 111. But then 000 has at least d neighbors that map to different neighbors, without choosing both 010 and 100, so

*268 Waverley St., Palo Alto, CA 94301. E-mail: `tomas@theory.stanford.edu`.

†E-mail: `carlos subi@hotmail.com`.

as in the proof above we obtain a 2-satisfiability instance without clauses, giving an H'_d that maps to H_d by an isomorphism.

Finally if w_1, w_2 are at distance 1 from each other, then changing any other dimension from both w_1 and w_2 we obtain again two adjacent vertices that map to the same vertex, so as before we can choose d edges coming out of w_1 that map to different edges that are not the loop, and we get a 2-satisfiability instance without clauses as in the proof above, giving an H'_d that maps to H_d by an isomorphism, completing the proof. ■

Theorem 2 *A $d+1$-cube can retract to a d-cube in exactly $d(d+1)/2+2d$ non-isomorphic manners for $d \geq 2$.*

PROOF: Let $f(d)$ be the number of such retractions, up to isomorphism. Let H_0, H_1 be the two d-cubes, such that H_0 stays fixed and H_1 maps to H_0. Let $f(d)$ be the number of retractions. If H_1 is mapped by an isomorphism to H_0, one can show that there are exactly three mappings, namely fix the cube, flip an edge, or rotate a square (this last one for $d \geq 2$). Otherwise H_1 decomposes into H_{10} and H_{11}, with H_{11} mapping by projection to H_{10}. If H_{10} never comes back to H_{11}, there are $f(d-1)$ possibilities for H_{10}. Otherwise, we proceed further to project H_{111} to H_{110} and so on, for $2 \leq i \leq d$ steps, and the resulting subcube goes back to H_{11}, in $d-1$ possible manners. We thus have $f(1) = 3$, $f(d) = f(d-1) + 3 + d - 1$, giving $f(d) = d(d+1)/2 + 2d$. ■