DISTANCE-TWO COLORINGS OF CUBIC PLANAR GRAPHS

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Abstract. A distance-two r-coloring of a graph G is an assignment of r colors to the vertices of G so that any two vertices at distance at most two have different colors. The distance-two four-coloring problem for cubic planar graphs is known to be NP-complete. We prove that the problem remains NP-complete for tri-connected bipartite cubic planar graphs, which we call Barnette graphs. For Goodey graphs, i.e., Barnette graphs all of whose faces have size four or six, the distance-two four-coloring problem becomes polynomial. In fact, we are able to fully describe all Goodey graphs that admit such a coloring, and the coloring is unique (up to a permutation of colors).

More generally, we call tri-connected cubic plane graphs whose face sizes are $3, 4, 5$, or 6, type-two Barnette graphs. By contrast with the above Barnette graphs, the distance-two four-coloring problem is polynomial for type-two Barnette graphs. In addition to the above Goodey graphs (with face sizes 4 or 6), only graphs with face sizes 3 or 6 admit such a coloring.

For plane quartic graphs, the analogue of type-two Barnette graphs are graphs with face sizes 3 or 4. For this class, the distance-two four-coloring problem is also polynomial; in fact, we can again fully describe all colorable instances – there are exactly two such graphs, and they again have a unique coloring (up to a permutation of colors).

1. Introduction

Tait conjectured in 1884 [20] that all cubic polyhedral graphs, i.e., all tri-connected cubic planar graphs, have a Hamiltonian cycle; this was disproved by Tutte in 1946 [22], and the study of Hamiltonian cubic planar graphs has been a very active area of research ever since, see for instance [1, 10, 16, 18]. Barnette formulated two conjectures that have been at the centre of much of the effort: (1) that *bipartite* tri-connected cubic planar graphs are Hamiltonian (the case of Tait's conjecture where all face sizes are even) [4], and (2) that tri-connected cubic planar graphs with all face sizes $3, 4, 5$ or 6 are Hamiltonian, cf. [3, 19]. Goodey [11, 12] proved the second conjecture when all faces have sizes 4 or 6. When all faces have sizes 5 or 6, this was a longstanding open problem, especially since these graphs (tri-connected cubic planar graphs with all face sizes 5 or 6) are the popular *fullerene graphs* [8]. The second conjecture has now been affirmatively resolved in full [17]. For the first conjecture, two of the present authors have shown in [9] that if the conjecture is false, then the Hamiltonicity problem for tri-connected cubic planar graphs is NP-complete. In view of these results and conjectures, in this paper we call bipartite tri-connected cubic planar graphs *Barnette graphs*, or *type-one Barnette graphs*; we call cubic plane graphs with all face sizes $3, 4, 5$ or 6 *type-two Barnette graphs*; and finally we call tri-connected cubic plane graphs with all face sizes 4 or 6 *Goodey graphs*. Note that it would be more logical, and historically accurate, to assume tri-connectivity also for type-two Barnette graphs and for Goodey graphs; under those definitions Goodey graphs are Barnette graphs of both type-one and type-two. However, we do not need the assumption of tri-connectivity to prove our positive results for these graphs, and hence we do not assume it.

Cubic planar graphs have been also of interest from the point of view of colorings [6, 14]. In particular, they are interesting for distance-two colourings. Let G be a graph with degrees at most d. A *distance-two r-coloring* of G is an assignment of colors from $[r] = \{1, 2, \ldots, r\}$ to the vertices of G such that if a vertex v has degree $d' \leq d$ then the $d' + 1$ colors of v and of all the neighbors of v are all distinct. (Thus a distance-two coloring of G is a classical coloring of G^2.) It was conjectured by Wegner [23] that a planar graph with maximum degree d has a distance-two r-colouring where $r = 7$ for $d = 3$, $r = d + 5$ for $d = 4, 5, 6, 7$, and $r = \lfloor 3d/2 \rfloor + 1$ for all larger d. The case $d = 3$ has recently been settled in the positive by Hartke, Jahanbekam and Thomas [13], cf. also [21].

For cubic planar graphs in general it was conjectured in [13] that that if a cubic planar graph is tri-connected, or has no faces of size five, then it has a distance-two six-coloring. We propose a weaker version of the second case of the conjecture, namely, we conjecture that *a bipartite cubic planar graph can be distance-two six-colored.* We prove this in one special case, which of course also confirms the conjecture of Hartke, Jahanbekam and Thomas in that case.

Note that at least four colors are needed in any distance-two coloring of a cubic graph since a vertex and its three neighbours must all receive distinct colors. Moreover, Heggerness and Telle [15] have shown that the problem of distance-two four-coloring cubic planar graphs is NP-complete.

Much attention has also been focused on the relation of distance-two colorings and the girth [5, 14]. For example, Borodin and Ivanova [5] have shown that planar subcubic graphs can be distance-two colored with four colors provided there are the girth is at least 22.

We mostly focus the distance-two four-coloring problem for type-one and type-two Barnette graphs. The former turns out to be NP-complete, while the latter is polynomial; in fact the type-two Barnette graphs that are distance-two four-colorable will be fully described.

(Note that we use the term plane graph when the actual embedding is used, e.g., by discussing the faces; when the embedding is unique, as in tri-connected graphs, we stick with writing "planar".)

2. Relation to edge-colorings

Distance-two colorings have a natural connection to edge-colorings.

Theorem 1. *Let G be a graph with degrees at most d that admits a distance-two $(d+1)$-coloring, with d odd. Then G admit an edge-coloring with d colors.*

Proof. The even complete graph K_{d+1} can be edge-colored with d colors by the Walecki construction [2]. We fix one such coloring c, and then consider a distance-two $(d + 1)$-coloring of G. If an edge uv of G has colors ab at its enpoints, we color uv in G with the color $c(ab)$. It is easy to see that this yields an edge-coloring of G with d colors. □

We call the resulting edge-coloring of G the *derived edge-coloring* of the original distance-two coloring.

In this paper, we mostly focus on the case $d = 3$ (the *subcubic* case). Thus we use the edge-coloring of K_4 by colors red, blue, green. This corresponds to the unique partition of K_4 into perfect matchings. Note that for every vertex v of K_4 and every edge-color i, there is a unique other vertex u of K_4 adjacent to v in edge-color i. Thus if we have the derived edge-coloring we can efficiently recover the original distance-two coloring. In the subcubic case, in turns out to be sufficient to have just one color class of the edge-coloring of G.

Theorem 2. *Let G be a subcubic graph, and let R be a set of red edges in G. The question of whether there exists a distance-two four-coloring of G for which the derived edge-coloring has R as one of the three color classes can be solved by a polynomial time algorithm. If the answer is positive, the algorithm will identify such a distance-two coloring.*

Proof. We may assume in K_4 red joins colors $13, 24$, blue joins colors $12, 34$ and green joins colors $14, 23$. Note that we may also assume that R is a matching that covers at least all vertices of degree three, otherwise we answer in the negative. We may further assume that some vertex v gets an even color (2 or 4). The parity of the color of a vertex u determines the parity of the color of its neighbors, namely the parity is the same if they are adjacent by an edge in R, and they are of different parity otherwise. We may thus extend from v the assignment of parities to all the vertices, unless an inconsistency is reached, in which case no coloring exists. Otherwise, at this point all vertices have only two possible colors, namely $1, 3$ for odd and $2, 4$ for even.

Define an auxiliary graph G' with vertices $V(G') = V(G)$, and edges xy in $E(G')$ if xy is a red edge in $E(G)$ or if there is a path xzy without red edges in $E(G)$. Note that these edges xy join vertices of the same parity, and x, y must have different colors. If G' has an odd cycle, then no solution exists. Otherwise G' is bipartite, and we may choose $1, 3$ in different sides of a bipartition of G' for odd vertices, and $2, 4$ in different sides for even vertices.

Each vertex u will have at most one neighbor x of the same parity in G, namely the one joined to it by the red edge, and ux is an edge of G'. This guarantees different colors for u, x. The at most two other neighbors y, z of u have different parity from u, x, and the path yuz in G ensures the edge yz is in G'. This guarantees different colors for y, z. Thus the colors for u, x, y, z are all different at each vertex u, and we have a distance-two coloring of G. □

3. Distance-two colorings of bipartite cubic plane graphs

Let G be a bipartite cubic plane graph. The faces of G can be three-colored, say, red, blue and green. This face coloring induces a three-edge-coloring of G as follows. Any edge joins exactly one pair of faces of the same color; we color the edge by this color. It is easy to see that this is an edge-coloring, i.e., that incident edges have distict colors. We call an edge-coloring that arises this way from some face-coloring of G a *special three-edge-coloring* of G. We first ask when is a special three-edge-coloring of G the derived edge-coloring of a distance-two four-coloring of G.

Theorem 3. *A special three-edge-coloring of G is the derived edge-coloring of some distance-two four-coloring of G if and only if the size of each face is a multiple of* 4.

Proof. The edges around a face f alternate in colors, and the vertices of f can be colored consistently with this alternation if and only if the size of f is a multiple of 4. This proves the "only if" part. For the "if" part, suppose all faces have size multiple of 4. If there is an inconsistency, it will appear along a cycle C in G. If there is only one face inside C, there is no inconsistency. Otherwise we can join some two vertices of C by a path P inside C, and the two sides of P inside C give two regions that are inside two cycles C', C''. The consistency of C then follows from the consistency of each of C', C'' by induction on the number of faces inside the cycle. □

Corollary 1. *Let G be a cubic plane graph in which the size of each face is a multiple of four. Then G can be distance-two four-colored.*

We now prove a special case of the conjecture stated in the introduction, that all bipartite cubic plane graphs can be distance-two six-colored. Recall that the faces of any bipartite cubic plane graph can be three-colored.

Theorem 4. *Suppose the faces of a bipartite cubic plane graph G are three-colored red, blue and green, so that the red faces are of arbitrary even size, while the size of each blue and green face is a multiple of 4. Then G can be distance-two six-colored.*

Proof. Let G' be the multigraph obtained from G by shrinking each of the red faces. Clearly G' is planar, and since the sizes of blue and green faces in G' are half of what they were in G, they will be even, so G' is also bipartite. Let us label the two sides of the bipartition as A and B respectively. Now consider the special three-edge coloring of G associated with the fave coloring of G. Each red edge in this special edge-coloring joins a vertex of A with a vertex of B; we orient all red edges from A to B. Now traversing each red edge in G in the indicated orientation either has a blue face on the left and green face on the right, or a green face on the left and blue face on the right. In the former case we call the edge *class one* in the latter case we call it *class two*. The red edges around each red face in G are alternatingly in class 1 and class 2. Each vertex of G is incident with exactly one red edge. We assign colors $1, 2, 3$ to vertices incident to class one red edges and colors $4, 5, 6$ to vertices incident to class two red edges. It remains to decide how to choose from the three colors available for each vertex. A vertex adjacent to red edges in class i has only three vertices within distance two in the same class, namely the vertex across the red edge, and the two vertices at distance two along the red face in either direction. Therefore distance-two coloring for class i corresponds to three-coloring a cubic graph. Since neither class can yield a K_4, such a three-coloring exists by Brooks' theorem [7]. This yields a distance-two six-coloring of G. □

4. Distance-two four-coloring of Barnette graphs is NP-complete

We begin with our main intractability result. First we derive a weaker version of our claim.

Theorem 5. *The distance-two four-coloring problem for bipartite planar subcubic graphs is NP-complete.*

Proof. Consider the graph H in Figure 1.

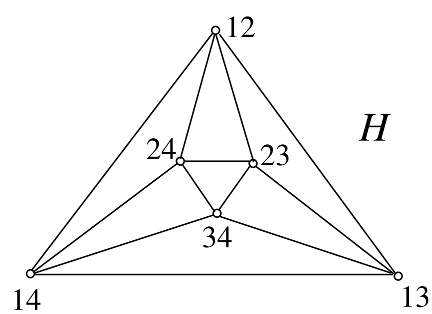

Figure 1. The graph H for the proof of Theorem 5

We will reduce the problem of H-coloring planar graphs to the distance-two four-coloring problem for bipartite planar subcubic graphs. In the *H-coloring problem* we are given a planar graph G and and the question is whether we can color the vertices of G with colors that are vertices of H so that adjacent vertices of G obtain adjacent colors. This can be done if and only if G is three-colorable, since the graph H both contains a triangle and is three-colorable itself. (Thus any three-coloring of G is an H-coloring of G, and any H-coloring of G composed with a three-coloring of H is a three-coloring of G.) It is known that the three-coloring problem for planar graphs is NP-complete, hence so is the H-coloring problem.

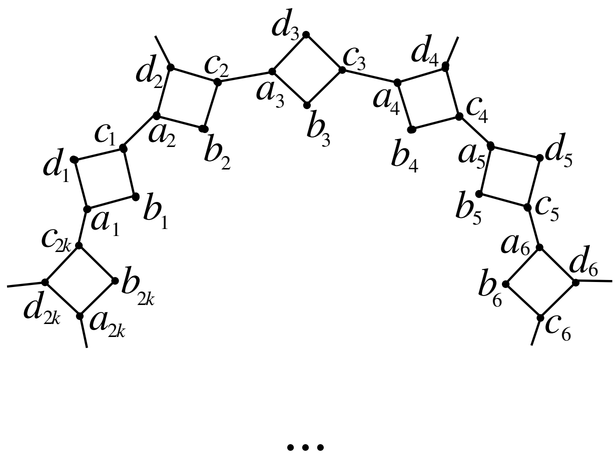

FIGURE 2. The ring gadget

Thus suppose G is an instance of the H-coloring problem. We form a new graph G' obtained from G by replacing each vertex v of G by a *ring* gadget depicted in Figure 2. If v has degree k, the ring gadget has $2k$ squares. A *link* in the ring is a square $a_ib_ic_id_ia_i$ followed by the edge c_ia_{i+1}. A link is *even* if i is even, and *odd* otherwise. Every even link in the ring will be used for a connection to the rest of the graph G', thus vertex v has k available links. For each edge vw of G we add a new vertex f_{vw} that is adjacent to a vertex d_s in one available link of the ring for v and a vertex d'_t in one available link of the ring for w. (We use primed letters for the corresponding vertices in the ring of w to distinguish them from those in the ring of v.) The actual choice of (the even) subscripts s, t does not matter, as long as each available link is only used once. The resulting graph is clearly subcubic and planar. It is also bipartite, since we can bipartition all its vertices into one independent set A consisting of all the vertices $a_i, c_i, b_{i+1}, di + 1$ with odd i in all the rings, and another independent set B consisting of the vertices $a_i, c_i, b_{i+1}, di + 1$ with even i in all the rings. Moreover, we place all vertices f_{vw} into the set A. Note that in any distance-two four-coloring of the ring, each link must have four different colors for vertices a_i, b_i, c_i, d_i, and the same color for a_i and a_{i+1}. Thus all a_i have the same color and all c_i have the same color. The pair of colors in b_i, d_i is also the same for all i; we will call it the *characteristic pair of the ring for v*. For any pair ij of colors from $1, 2, 3, 4$, there is a distance-two coloring of the ring that has the characteristic pair ij.

We claim that G is H-colorable if and only if G' is distance-two four-colorable.

In an H-coloring of G, the vertices of G are actually assigned unordered pairs from $\{1, 2, 3, 4\}$, since the vertices of H are labeled by pairs. (Note that two vertices of H are adjacent if and only if the pairs they are labeled with intersect in exactly one element.) Thus suppose that we have an H-coloring ϕ of G. If $\phi(v) = ij$ (i.e., the vertex v of G is assigned the vertex of H labeled by the pair ij), then we colour the ring of v so that its characteristic pair is ij. This still leaves a choice of which of the colors i, j is in which b_s, d_s, in each of the links a_s, b_s, c_s, d_s. Since ϕ is an H coloring, adjacent vertices vw are assigned pairs that intersect if exactly one element. This makes it possible to color each b_s, d_s so that all colors at distance at most two are distinct. For instance if vertices v and w are adjacent in G and colored by $12, 13$ by ϕ, and if f_{vw} is adjacent to the vertices d_s in the ring for v and d_t in the ring for w, then both b_s in the ring for v and b_t in the ring for w are colored 1, as is f_{vw}, while d_s in the ring for v and d_t in the ring for w are colored 2 and 3 respectively. It is easy to see that this is a distance-two four-coloring of G'.

Conversely, in any distance-two four-coloring of G', the color of a vertex f_{vw} determines the same color in the b's of its adjacent links of the rings for v and w, whence the characteristic pairs of these two rings intersect in exactly one element. Thus we may define a mapping ϕ of $V(G)$ to $V(H)$ by assigning to each vertex $v \in V(G)$ the characteristic pair of the ring for v. Then ϕ is an H-coloring of G, since adjacent vertices of G are assigned pairs that are adjacent in H. □

The NP-completeness result can be strengthened to tri-connected cubic graphs. We use the previous construction, adjusted so the resulting graph is both cubic and tri-connected.

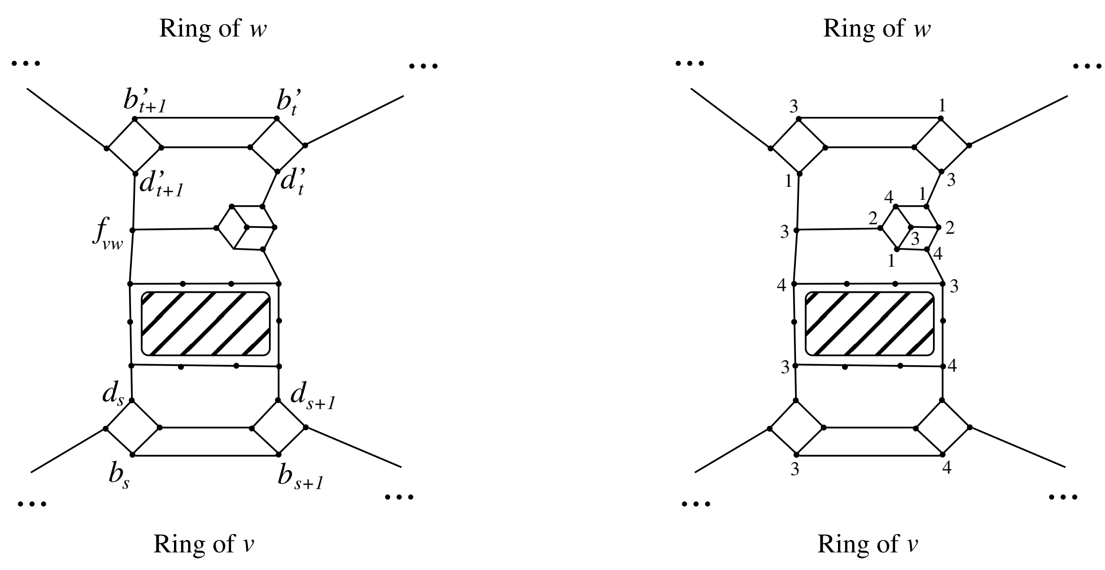

FIGURE 3. The edge-gadget for f_{vw} (left) and its unique distance-two coloring (right)

Theorem 6. *The distance-two four-coloring problem for tri-connected bipartite cubic planar graphs is NP-complete.*

Proof. The construction of the graph G' is modified as suggested in in the left half of Figure 3. Recall that in the construction of G', for each edge vw of G a separate vertex f_{vw} was made adjacent to d_s in the ring of v and d'_t in the ring of w. Recall that both s and t are even, and the vertices d_{s+1}, d'_{t+1} (with both subscripts odd) remained available for connection. We now make a new edge-gadget around the vertex f_{vw}, making it directly adjacent to d'_{t+1}, and connected to d_s by a path, as depicted in Figure 3. In both rings, the two "b" type vertices in the two consecutive links are joined together by an additional edge; specifically, we add the edges $b_s b_{s+1}$ and $b'_t b'_{t+1}$. (Note that this forces the corresponding "d" type vertices d_s and d_{s+1} to be colored differently in any distance-two four-coloring, and similarly for d'_t and d'_{t+1}). Moreover, further vertices and edges are added, as depicted in Figure 4. The shaded ten-sided region is identified with the ten-sided exterior face of the graph depicted in Figure 4, which has a unique distance-two four-coloring, shown there. (The heavy edges correspond to the ten-sided shaded figure.) Note that the construction is not symmetric, as it depends on which ring is viewed as the "bottom" ring for the vertex f_{vw}. (The depicted figure has the ring of v on the bottom, but the conclusions are the same if it were the ring of w.) We can choose either way, independently for each edge vw of G. It can be seen that the resulting graph, which we denote by G'', is bipartite, planar, and cubic. We may assume that G is bi-connected (the three-coloring problem for biconnected planar graphs is still NP-complete), and therefore G'' is also tri-connected (as no two faces share more than one edge). Using the unique distance-two four-colouring of the graph in Figure 4, it also follows that in any distance-two four-coloring of G'' the vertices d_s and d'_{t+1} have different colors, while both vertices b_s and b'_{t+1} have the same color (the color of f_{vw}), in any distance-two four-coloring of G''. To facilitate checking this, we show in the right half of Figure 3 a partial distance-two four-coloring, by circles, squares, up triangles, and down triangles; this coloring is forced by arbitrarily coloring f_{vw} and its three neighbours by four distinct colors. Since the colors of the pair b_s, d_s and the pair b'_{t+1}, d'_{t+1} have exactly one color in common, the previous NP-completeness proof applies, i.e., G is H-colorable if and only if G'' is distance-two four-colorable. □

We remark that (with some additional effort) we can prove that the problem is still NP-complete for the class of tri-connected bipartite cubic planar graphs with no faces of sizes larger than 44.

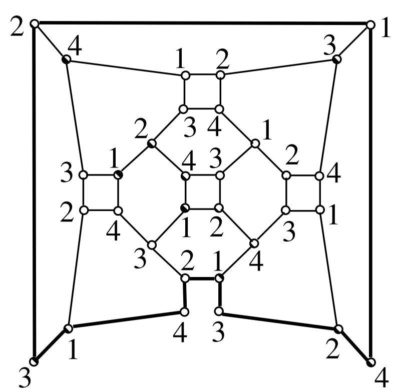

Figure 4. The graph for the shaded region, with its unique distance-two four-coloring

5. Distance-two four-coloring of Goodey graphs is polynomial

Recall that Goodey graphs are type-two Barnette graph with all faces of size four and six [11, 12]. In other words, a *Goodey graph* is a cubic plane graph with all faces having size 4 or 6. By

Euler's formula, a Goodey graph has exactly six square faces, while the number of hexagonal faces is arbitrary.

A *cyclic prism* is the graph consisting of two disjoint even cycles $a_1a_2\cdots a_{2k}a_1$ and $b_1b_2\cdots b_{2k}b_1, k \geq 2$, with the additional edges a_ib_i, $1 \leq i \leq 2k$. It is easy to see that cyclic prisms have either no distance-two four-coloring (if k is odd), or a unique distance-two four-coloring (if $k \geq 2$ is even). Only the cyclic prisms with $k = 2, 3$ are Goodey graphs, and thus from Goodey cyclic graphs only the cube (the case of $k = 2$) has a distance-two coloring, which is moreover unique.

In fact, all Goodey graphs that admit distance-two four-coloring can be constructed from the cube as follows. The Goodey graph C_0 is the cube, i.e., the cyclic prism with $k = 2$. The Goodey graph C_1 is depicted in Figure 6. It is obtained from the cube by separating the six square faces and joining them together by a pattern of hexagons, with three hexagons meeting at a vertex tying together the three faces that used to meet in one vertex. The higher numbered Goodey graphs are obtained by making the connecting pattern of hexagons higher and higher. The next Goodey graph C_2 has two hexagons between any two of the six squares, with a central hexagon in the centre of any three of the squares, the following Goodey graph C_3 has three hexagons between any two of the squares and three hexagons in the middle of any three of the squares, and so on. Thus in general we replace every vertex of the cube by a triangular pattern of hexagons whose borders are replacing the edges of the cube. We illustrate the vertex replacement graphs in Figure 5, without giving a formal description. The entire Goodey graph C_1 is depicted in Figure 6. (Note that the graph in Figure 4 was obtained from the graph in Figure 6 by the deletion of two edges.)

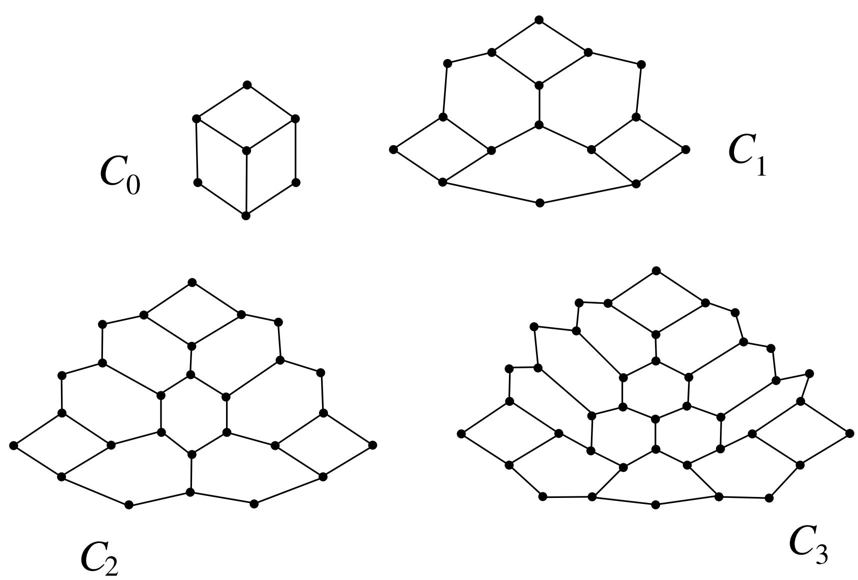

FIGURE 5. The vertex replacements for Goodey graphs C_0, C_1, C_2, and C_3

Theorem 7. *The Goodey graphs $C_k, k \geq 0$, have a unique distance-two four-coloring, up to permutation of colors.*

Proof. We described C_k as eight triangular regions R, each consisting of $\binom{k}{2}$ hexagons, one region R for each vertex of the cube. Each R has three squares at the corners, which we

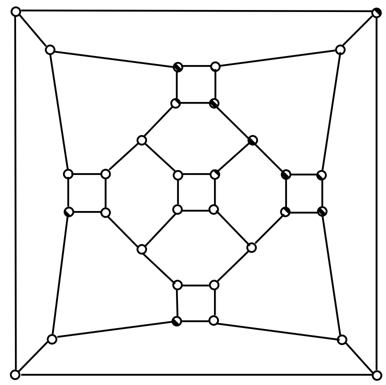

FIGURE 6. The Goody graph C_1

describe as two squares joined by a chain of k hexagons horizontally at the bottom, and a third square on top. (See Figure 5.)

We partition the vertices into $k+2$ horizontal paths P_i, $0 \leq i \leq k+1$, with each P_i having endpoints of degree 2 and internal vertices of degree 3. The path P_0 has length $2k+2$, and the remaining paths P_i, $i \geq 1$ have length $2k+6-2i$. In particular the last P_{k+1} has length 4, and is the only P_i that is actually a cycle, pictured as the square at the top. See Figure 7.

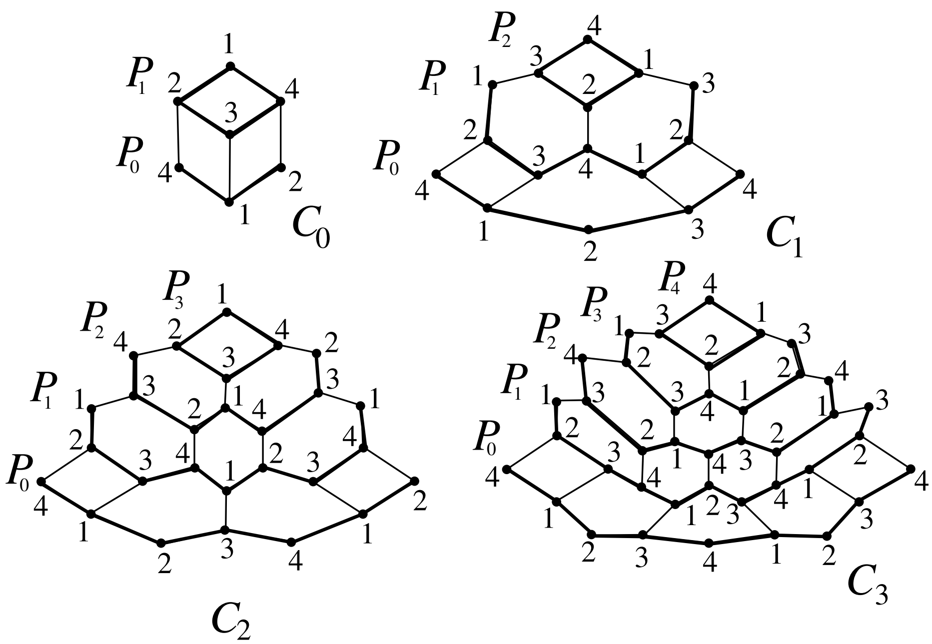

FIGURE 7. The paths P_i and the resulting distance-two colorings

We denote $P_i = v_i^0 v_i^1 \cdots v_i^{\ell_i}$. The edges between P_0 and P_1 are $v_0^0 v_1^1$, $v_0^j v_1^{j+1}$ for $1 \leq j \leq \ell_0 - 1$, j odd, and $v_0^{\ell_0} v_1^{\ell_1 - 1}$. We can choose the permutation of colors for the square $v_0^0 v_0^1 v_1^2 v_1^1$ to be 4132, forcing for neighbors v_1^0, v_1^3, v_0^2 the colors $1, 4, 2$, and completing the adjacent square, or hexagon with the assignment to v_1^4, v_0^3 of colors $1, 3$. This forced process extends similarly through the chain of hexagons until the last square.

We have derived the beginning of P_0 as 4123 and the beginning of P_1 as 12341. After the forced extension, P_0 will be an initial segment of $(4123)^*$ and P_1 will be an initial segment of $(1234)^*$.

For $i \geq 1$, the edges between P_i and P_{i+1} are $v_i^0 v_{i+1}^1$, $v_i^j v_{i+1}^{j-1}$ for $3 \leq j \leq \ell_i - 3$, j odd, and $v_i^{\ell_i} v_{i+1}^{\ell_{i+1}-1}$.

A similar process derives the beginning of P_i for i odd as 1234 and the beginning of P_i for i even as 4321. After the forced extension, P_i for i odd will be an initial segment of $(1234)^*$ and P_i for i even will be an initial segment of $(4321)^*$.

This gives a unique coloring for the triangular region after coloring one square S, which is uniquely extended to the four triangular regions surrounding S, and then uniquely extended to the four triangular regions surrounding S' opposite to S. □

Theorem 8. *The Goodey graphs $C_k, k \geq 0$, are the only bipartite cubic planar graphs having a distance-two four-coloring.*

Proof. Consider a Goodey graph G with a fixed distance-two four-coloring. Recall that Goodey graphs have exactly six squares. Each of the squares is joined by four chains of hexagons to four squares. We consider the dual six-vertex graph G' whose vertices are squares in G, with ab an edge in G' if and only if there is a chain of hexagons joining squares a and b. It can be readily verified that such a chain cannot cross itself or another chain in G. Indeed, the colors in the fixed distance-two four-coloring are uniquely forced along such chains and they don't match if the chains should cross. It follows that the graph G' is planar. A similar argument shows that a chain cannot return to the same square, and two chains from the square a cannot end at the same square b. Thus G' has no faces of size one or two, and by Euler's formula it has 12 edges and 8 faces; therefore all faces of G' must be triangles, and G' is the octahedron.

Let T be a triangular face in G', let s be a side of T with the smallest number d of hexagons in G. Then it can again be checked using the coloring that the other two sides of T will also have d hexagons in G. Then T corresponds to a triangular region R as in Theorem 7, and the octahedron G' yields $G = C_k$ for $k = d$. □

Corollary 2. *The distance-two four-coloring problem for Goodey graphs is solvable in polynomial time.*

Recognizing whether an input Goodey graph is some C_k can be achieved in polynomial time; in the same time bound G can actually be distance-two four-colored.

We also derive the following corollary of Theorem 4 and the above discussion of distance-two coloring of cyclic prisms.

6. Distance-two four-coloring of type-two Barnette graphs is polynomial

We now return to general type-two Barnette graphs, i.e., cubic plane graphs with face sizes $3, 4, 5$, or 6. As a first step, we analyze when a general cubic plane graph admits a distance-two four-coloring which has three colors on the vertices of every face of G.

Theorem 9. *A cubic plane graph G has a distance-two four-coloring with three colors per face if and only if*

(1) *all faces in G have size which is a multiple of* 3*,*
(2) *G is bi-connected, and*

(3) *if two faces share more than one edge, the relative positions of the shared edges must be in congruent modulo* 3 *in the two faces.*

The last condition means the following: if faces F_1, F_2 meet in edges e, e' and there are n_1 edges between e and e' in (some traversal of) F_1, and n_2 edges between e and e' in (some traversal of) F_2, then $n_1 \equiv n_2(\mod 3)$.

Proof. Suppose G has a distance-two four-coloring with three colors in each face. The unique way to distance-two color a cycle with colors $1, 2, 3$ is by repeating them in some order $(123)^*$ along one of the two traversals of the cycle. Therefore the length is a multiple of 3 so (1) holds. Moreover, there can be no bridge in G as that would imply a face that self-intersects and is traversed in opposite directions along any traversal of that face, disagreeing with the order $(123)^*$ in one of them; thus (2) also holds. Finally, (3) holds because the common edges must have the same colors in both faces.

Suppose the conditions hold, and consider the dual G^D of G. (Note that each face of G^D is a triangle.) We find a distance-two coloring of G as follows. Let F be a face in G; according to conditions (1-2), its vertices can be distance-two colored with three colors. That takes care of the vertex F in G^D. Using condition (3), we can extend the coloring of G to any face F' adjacent to F in G^D. Note that we can use the fourth colour, 4, on the two vertices adjacent in F' to the two vertices of a common edge. In this way, we can propagate the distance-two coloring of G along the adjacencies in G^D. If this produces a distance-two coloring of all vertices of G, we are done. Thus it remains to show there is no incosistency in the propagation. If there is an inconsistency, it will appear along a cycle C in G^D. If there is only one face inside of C, then C is a triangle corresponding to a vertex of G, and there is no inconsistency. Otherwise we can join some two vertices of C by a path P inside C, and the two sides of P inside C give two regions that are inside two cycles C', C''. The consistency of C then follows from the consistency of each of C', C'' by induction on the number of faces inside the cycle. □

It turns out that conditions (1 - 3) are automatically satisfied for cubic plane graphs with faces of sizes 3 or 6.

Corollary 3. *Type-two Barnette graphs with faces of sizes* 3 *or* 6 *are distance-two four-colorable.*

Proof. Such a graph must be bi-connected, i.e., cannot have a bridge, since no triangle or hexagon can self-intersect. Moreover, only two hexagons can have two common edges, and it is easy to check that they must indeed be in relative positions congruent modulo 3 on the two faces. (Since all vertices must have degree three.) Thus the result follows from Theorem 9. □

Theorem 10. *Let G be type-two Barnette graph. Then G is distance-two four-colorable if and only if it is one of the graphs $C_k, k \geq 0$, or all faces of G have sizes* 3 *or* 6.

Proof. If there are faces of size both 3 and 4 (and possibly size 6), then there must be (by Euler's formula) two triangles and three squares, and as in the proof of Theorem 8, the squares must be joined by chains of hexagons, which is not possible with just three squares.

If there is a face of size 5, then there is no distance-two four-coloring since all five vertices of that face would need different colors. □

7. Distance-two coloring of quartic graphs

A *quartic graph* is a regular graph with all vertices of degree four. Thus any distance-two coloring of a quartic graph requires at least five colors. A *four-graph* is a plane quartic graph whose faces have sizes 3 or 4. The argument to view these as analogues of type-two Barnette graphs is as follows. For cubic plane Euler's formula limits the numbers of faces that are triangles, squares, and pentagons, but does not limit the number of hexagon faces. Similarly, for plane quartic graphs, Euler's formula implies that such a graph must have 8 triangle faces, but places no limits on the number of hexagon faces.

We say that two faces are *adjacent* if they share an edge.

Lemma 1. *If a four-graph can be distance-two five-colored, then every square face must be adjacent to a triangle face. Thus G can have at most 24 square faces.*

Proof. We view the numbers $1, 2, 3, 4$ modulo 4, and number 5 is separate. Let $u_1u_2u_3u_4$ be a square face that has no adjacent triangle face. (This is depicted in Figure 8 as the square in the middle.) Color u_i by i. Let the adjacent square faces be $u_iu_{i+1}w_{i+1}v_i$. One of v_i, w_i must be colored 5 and the other one $i+2$. Then either all v_i or all w_i are colored 5, say all w_i are colored 5, and all v_i are colored $i+1$. Then $v_iu_iw_i$ cannot be a triangle face, or w_i, w_{i+1} would be both colored 5 at distance two. Therefore $t_iv_iu_iw_i$ must be a square face. (In the figure, this is indicated by the corner vertices being marked by smaller circles; these must exist to avoid a triangle face.) This means that the original square is surrounded by eight square faces for $u_1u_2u_3u_4$, and t_i must have color $i+3$, since u_i, v_{i+3}, v_i, w_i have colors $i, i+1, i+2, 5$.

But then there cannot be a triangle face $x_iv_iw_{i+1}$, since x_i is within distance two of u_i, u_{i+1}, v_i, t_i, w_{i+1} of colors $i, i+1, i+2, i+3, 5$, so each of the adjacent square faces $u_iu_{i+1}w_{i+1}v_i$ for $u_1u_2u_3u_4$ has adjacent square faces as well. This process of moving to adjacent square faces eventually reaches all faces as square faces, contrary to the fact that there are 8 triangle faces. □

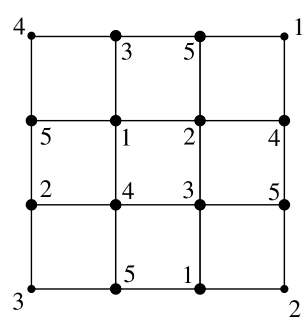

Figure 8. One square without adjacent triangles implies all faces must be squares

Since the number of vertices of a graph with 32 faces of size three and four is finite, we have the following conlusion.

Corollary 4. *The distance-two five-coloring problem for four-graphs is polynomial.*

However, it turns out that we can say more: we can fully describe all four-graphs that are distance-two five-colorable. Consider the four-graphs G_0, G_1 given in Figure 9. The graph G_0

has 8 triangle faces and 4 square faces, the graph G_1 has 8 triangle faces and 24 square faces. Note that G_0 is obtained from the cube by inserting two vertices of degree four in two opposite square faces. Similarly, G_1 is obtained from the cube by replacing each vertex with a triangle and inserting into each face of the cube a suitably connected degree four vertex. (In both figures, these inserted vertices are indicated by smaller size circles.)

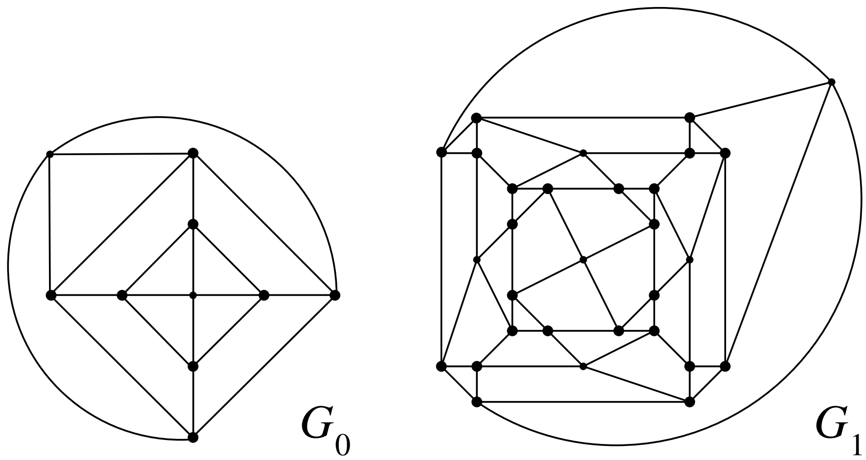

FIGURE 9. The only four-graphs that admit a distance-two five-coloring

Theorem 11. *The only four-graphs G that can be distance-two five-colored are G_0, G_1. These two graphs can be so colored uniquely up to permutation of colors.*

Proof. We show that if G can be so colored, then either G is G_0 or every triangle in G must be surrounded by six square faces, in which case G is G_1.

Suppose G has two adjacent triangles $u_5u_1u_2$ and $T = u_5u_2u_3$. The vertices adjacent to T must be given two colors other than those of u_5, u_2, u_3. If T has two adjacent squares, then it has five adjacent vertices, which must be given the two colors in alternation, a contradiction. Similarly if T is adjacent to three triangles then the three vertices adjacent would need three new colors, a contradiction.

We may thus assume a triangle $u_5u_3u_4$. If there is a square $u_1u_5u_4t$, this square plus the two adjacent triangles would need six colors, a contradiction, so $u_5u_4u_1$ is a triangle, completing u_5 adjacent to the four-cycle $u_1u_2u_3u_4$. Color u_i with color i. Then the additional vertex v_i adjacent to u_i for $1 \le i \le 4$ must be given color $i+2$ (modulo 4), so these v_i form a 4-cycle, and any additional vertex adjacent to a v_i must get color 5, so there is a single additional v_5 with color 5. This gives a uniquely colored G_0, up to permutation of colors.

In the remaining case, each triangle $T' = u_1u_2u_3$ has adjacent squares $u_iu_{i+1}w_{i+1}v_i$, with addition modulo 3. The vertices v_i, w_{i+1} must be given the two colors different from those of T', and in alternation around T', so there cannot be a triangle $u_iv_iw_i$ else w_i, w_{i+1} with the same color would be at distance two. So there are squares $u_iv_it_iw_i$, and T' is surrounded by six squares.

By Lemma 1, we must have a triangle adjacent to the square $u_iv_it_iw_i$, either $v_it_ix_i$ or $w_it_iy_i$, but not both since six colors would be needed. Let such a triangle be T_i, and we link T' to the three T_i. These triangles viewed as vertices linked form a cubic graph without triangles G', since a triangle face would be three triangles joined in G, which would need to have only three squares inside by Lemma 1. The graph G' has 8 vertices for the 8 triangles, so this graph is the cube C. Replacing each vertex corresponding to a triangle by the corresponding triangle gives a graph D.

Suppose the triangles adjacent to T' are $v_i t_i x_i$ for $1 \leq i \leq 3$. Then going around a face of C we notice only one vertex inside this face by Lemma 1, giving the construction of G_1. If we assign to the vertex inside this face the color 5, we notice that the surrounding triangles in D must use three colors at most 4, and each must omit a different color of 4. This implies that all vertices in centers of square faces must be 5, and only opposite triangles for C use the same 3 out of 4 colors. This proves existence and uniqueness up to permutation of colors of the distance-two 5-coloring of G_1.

Suppose instead the adjacent triangles are $T_1 = w_1 t_1 y_1$, $T_2 = v_2 t_2 x_2$, and $T_3 = v_3 t_3 x_3$. If there is no triangle $v_1 w_2 x$, then the three squares Q_1, Q_2, Q_3 between T_1 and T_2 are respectively adjacent to squares Q'_1, Q'_2, Q'_3, and Q'_2 must be adjacent to a triangle by Lemma 1. There must be triangles at both ends of the Q'_i and these are adjacent to T_1 and T_2, a contradiction.

Finally, suppose again the adjacent triangles are $T_1 = w_1 t_1 y_1$, $T_2 = v_2 t_2 x_2$, and $T_3 = v_3 t_3 x_3$, but there is a triangle $v_1 w_2 x$. This triangle faces T', and T_1 faces T_3. Triangles facing each other give two diagonals in the square faces of C, which implies two opposite faces without such diagonals in C, while the four sets of two diagonals form a matching of the 8 vertices of C. If the center of a face without diagonals gets assigned 5, then the adjacent triangles will be assigned a subset of $1 \leq i \leq 4$. Then joining the sets of two diagonals assigns a 5 to a vertex of each remaining triangle, which is not possible to the center of the remaining face without diagonals. $\square$

We close this section by recalling that Wegner's conjecture [23] claims that any planar graph with maximum degree four can be nine-colored. The graph in Figure 10 has nine vertices and diameter four, hence actually requires nine colors. Thus the bound in Wegner's conjecture, if the conjecture is true, cannot be lowered even for four-graphs. Similarly, the Wegner bound for planar graphs with maximum degree three is 7 [13], and the bound is again achieved by a type-two Barnette graph, specifically the graph obtained from K_4 by subdividing three incident edges, resulting in one triangle face and three pentagon faces.

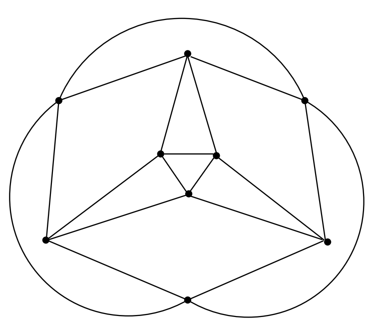

FIGURE 10. A four-graph requiring nine colors in any distance-two coloring

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