

DISTANCE-TWO COLORINGS OF BARNETTE GRAPHS

TOMÁS FEDER, PAVOL HELL, AND CARLOS SUBI

ABSTRACT. Barnette identified two interesting classes of cubic polyhedral graphs for which he conjectured the existence of a Hamiltonian cycle. We examine these classes from the point of view of distance-two colorings. A distance-two r -coloring of a graph G is an assignment of r colors to the vertices of G so that any two vertices at distance at most two have different colors. A cubic graph obviously needs at least four colors, and the distance-two four-coloring problem for cubic planar graphs is known to be NP-complete. We prove the problem remains NP-complete for tri-connected bipartite cubic planar graphs, which we call type-one Barnette graphs, since they are the first class identified by Barnette. By contrast, we show the problem is polynomial for cubic plane graphs with face sizes 3, 4, 5, or 6, which we call type-two Barnette graphs, because of their relation to Barnette's second conjecture. By contrast, we show that the problem is polynomial for cubic plane graphs with face sizes 3, 4, 5, or 6, which we call type-two Barnette graphs, because of their relation to Barnette's second conjecture. In fact, the colorable instances that are bipartite can be fully described, and those that are non-bipartite can be characterized by their face sizes. We have similar results for quartic plane graphs: the analogue of type-two Barnette graphs are graphs with face sizes 3 or 4. For this class, the corresponding distance-two five-coloring problem is also polynomial; in fact, we can again fully describe all colorable instances – there are exactly two such graphs.

It has recently been proved that every planar subcubic graph can be distance-two colored with at most seven colors, and conjectured that six colors suffice if the planar subcubic graph can be drawn without faces of size five. We consider a weaker version of the conjecture, stating that six colors suffice for a bipartite cubic planar graph, i.e., a cubic plane graph with all faces even. We prove this conjecture in the case when all faces have sizes divisible by four, and in another more general case, when the faces are colored in three colors, of which two colors consist of faces with sizes divisible by four.

1. INTRODUCTION

Tait conjectured in 1884 [22] that all cubic polyhedral graphs, i.e., all tri-connected cubic planar graphs, have a Hamiltonian cycle; this was disproved by Tutte in 1946 [24], and the study of Hamiltonian cubic planar graphs has been a very active area of research ever since, see for instance [1, 11, 17, 19]. Barnette formulated two conjectures that have been at the centre of much of the effort: (1) that *bipartite* tri-connected cubic planar graphs are Hamiltonian (the case of Tait's conjecture where all face sizes are even) [4], and (2) that tri-connected cubic planar graphs with all face sizes 3, 4, 5 or 6 are Hamiltonian, cf. [3, 20]. Goodey [12, 13] proved that the conjectures hold on the intersection of the two classes, i.e., that tri-connected cubic planar graphs with all face sizes 4 or 6 are Hamiltonian. When all faces have sizes 5 or 6, this was a longstanding open problem, especially since these graphs (tri-connected cubic planar graphs with all face sizes 5 or 6) are the popular *fullerene graphs* [8]. The second conjecture has now been affirmatively resolved in full [18]. For the first conjecture, two of the present

authors have shown in [10] that if the conjecture is false, then the Hamiltonicity problem for tri-connected cubic planar graphs is NP-complete. In view of these results and conjectures, in this paper we call bipartite tri-connected cubic planar graphs *type-one Barnette graphs*; we call cubic plane graphs with all face sizes 3, 4, 5 or 6 *type-two Barnette graphs*; and finally we call cubic plane graphs with all face sizes 4 or 6 *Goodey graphs*. Note that it would be more logical, and historically accurate, to assume tri-connectivity also for type-two Barnette graphs and for Goodey graphs. However, we prove our positive results without needing tri-connectivity, and hence we do not assume it.

Cubic planar graphs have been also of interest from the point of view of colorings [6, 15]. In particular, they are interesting for distance-two colourings. Let G be a graph with degrees at most d . A *distance-two r -coloring* of G is an assignment of colors from $[r] = \{1, 2, \dots, r\}$ to the vertices of G such that if a vertex v has degree $d' \leq d$ then the $d' + 1$ colors of v and of all the neighbors of v are all distinct. (Thus a distance-two coloring of G is a classical coloring of G^2 .) Clearly a graph with maximum degree d needs at least $d + 1$ colors in any distance-two coloring, since a vertex of degree d and its d neighbours must all receive distinct colors. It was conjectured by Wegner [25] that a planar graph with maximum degree d has a distance-two r -colouring where $r = 7$ for $d = 3$, $r = d + 5$ for $d = 4, 5, 6, 7$, and $r = \lfloor 3d/2 \rfloor + 1$ for all larger d . The case $d = 3$ has been settled in the positive by Hartke, Jahanbekam and Thomas [14], cf. also [23]. Specifically, they proved that a subcubic planar graph can always be distance-two colored with at most seven colors. (This is optimal, as K_4 with three incident edges subdivided is a subcubic planar graph with seven vertices and diameter two, and hence requires seven colors.) It was also conjectured in [14] that six colors suffice if the planar graph can be drawn without pentagonal faces. We propose a weaker version of the conjecture as follows.

Conjecture 1.1. *A bipartite cubic planar graph can always be distance-two with at most six colors.*

In other words, we conjecture that six colors suffice if the cubic planar graph can be drawn without odd faces. We prove the conjecture in two special cases (Corollary 2.4 and, more generally, Theorem 2.5), which of course also confirms the conjecture of Hartke, Jahanbekam and Thomas for those cases.

Heggerness and Telle [16] have shown that the problem of distance-two four-coloring cubic planar graphs is NP-complete. On the other hand, Borodin and Ivanova [5] have shown that subcubic planar graphs of girth at least 22 can be distance-two four-colored. In fact, there has been much attention focused on the relation of distance-two colorings and the girth, especially in the planar context [5, 15].

Our results focus on distance-two four-colorings of cubic planar graphs, with particular attention on Barnette graphs, of both types. It turns out that the distance-two four-coloring problem for type-one Barnette graphs is still NP-complete, while for type-two Barnette graphs it is not only polynomial, but the positive instances can be explicitly described. They include one infinite family of Goodey graphs (cubic plane graphs with all faces of size 4 or 6), and all type-two Barnette graphs which have all faces of size 3 or 6. Interestingly, there is an analogous result for quartic (four-regular) graphs: all quartic planar graphs with faces of only

sizes 3 or 4 that have a distance-two five coloring can be explicitly described; there are only two such graphs.

Note that we use the term “plane” graph when the actual embedding is used, e.g., when discussing the faces; on the other hand, when the embedding is unique, as in tri-connected graphs, we stick with writing “planar”.

A preliminary version of this paper was presented as a conference communication [9]; here we include all proofs.

2. RELATIONS TO EDGE-COLORINGS AND FACE-COLORINGS

Distance-two colorings have a natural connection to edge-colorings.

Theorem 2.1. *Let G be a graph with degrees at most d that admits a distance-two $(d + 1)$ -coloring, with d odd. Then G admit an edge-coloring with d colors.*

Proof. The even complete graph K_{d+1} can be edge-colored with d colors by the Walecki construction [2]. We fix one such coloring c , and then consider a distance-two $(d + 1)$ -coloring of G . If an edge uv of G has colors ab at its endpoints, we color uv in G with the color $c(ab)$. It is easy to see that this yields an edge-coloring of G with d colors. \square

We call the resulting edge-coloring of G the *derived edge-coloring* of the original distance-two coloring.

In this paper, we mostly focus on the case $d = 3$ (the *subcubic* case). Thus we use the edge-coloring of K_4 by colors red, blue, green. This corresponds to the unique partition of K_4 into perfect matchings. Note that for every vertex v of K_4 and every edge-color i , there is a unique other vertex u of K_4 adjacent to v in edge-color i . Thus if we have the derived edge-coloring we can efficiently recover the original distance-two coloring.

In the subcubic case, it turns out to be sufficient to have just one color class of the edge-coloring of G .

Theorem 2.2. *Let G be a subcubic graph, and let R be a set of edges in G . The question of whether there exists a distance-two four-coloring of G for which the derived edge-coloring has R as the set of red edges, can be solved by a polynomial time algorithm. If the answer is positive, the algorithm will identify such a distance-two coloring.*

Proof. We may assume in K_4 red joins colors 13, 24, blue joins colors 12, 34 and green joins colors 14, 23. Note that we may also assume that R is a matching that covers at least all vertices of degree three, otherwise we answer in the negative. We may further assume that some vertex v gets an even color (2 or 4). The parity of the color of a vertex u determines the parity of the color of its neighbors, namely the parity is the same if they are adjacent by an edge in R , and they are of different parity otherwise. We may thus extend from v the assignment of parities to all the vertices, unless an inconsistency is reached, in which case no coloring exists. Otherwise, at this point all vertices have only two possible colors, namely 1, 3 for odd and 2, 4 for even.

Define an auxiliary graph G' with vertices $V(G') = V(G)$, and edges xy in $E(G')$ if xy is a red edge in $E(G)$ or if there is a path xzy without red edges in $E(G)$. Note that these edges xy join vertices of the same parity, and x, y must have different colors. If G' has an odd cycle,

then no solution exists. Otherwise G' is bipartite, and we may choose 1, 3 in different sides of a bipartition of G' for odd vertices, and 2, 4 in different sides for even vertices.

Each vertex u will have at most one neighbor x of the same parity in G , namely the one joined to it by the red edge, and ux is an edge of G' . This guarantees different colors for u, x . The at most two other neighbors y, z of u have different parity from u, x , and the path yuz in G ensures the edge yz is in G' . This guarantees different colors for y, z . Thus the colors for u, x, y, z are all different at each vertex u , and we have a distance-two coloring of G . \square

There is also a relation to face-colorings. It is a folklore fact that the faces of any bipartite cubic plane graph G can be three-colored [21]. This three-face-coloring induces a three-edge-coloring of G by coloring each edge by the color not used on its two incident faces. (It is easy to see that this is in fact an edge-coloring, i.e., that incident edges have distinct colors.) We call an edge-coloring that arises this way from some face-coloring of G a *special three-edge-coloring* of G . We first ask when is a special three-edge-coloring of G the derived edge-coloring of a distance-two four-coloring of G .

Theorem 2.3. *A special three-edge-coloring of G is the derived edge-coloring of some distance-two four-coloring of G if and only if the size of each face is a multiple of four.*

Proof. The edges around a face f alternate in colors, and the vertices of f can be colored consistently with this alternation if and only if the size of f is a multiple of four. This proves the “only if” part. For the “if” part, suppose all faces of G have size multiple of four. Start with a fixed three-colouring of the faces of G and the corresponding special three-edge-coloring of G . (Say the edge colours are 1, 2, 3.) Now we can try to recover a distance-two four-colouring of G for which this is the derived edge-colouring as suggested above Theorem 2.2. If there is an inconsistency, it will appear along a cycle C in G . If there is only one face inside C , there is no inconsistency. Otherwise we can join some two vertices of C by a path P inside C , and the two sides of P inside C give two regions that are inside two cycles C', C'' . The consistency of C then follows from the consistency of each of C', C'' by induction on the number of faces inside the cycle. \square

Corollary 2.4. *Let G be a cubic plane graph in which the size of each face is a multiple of four. Then G can be distance-two colored with four colors.*

This proves a special case of Conjecture 1.1. We now prove a more general case as follows. Recall that the faces of any bipartite cubic plane graph can be three-colored.

Theorem 2.5. *Suppose the faces of a bipartite cubic plane graph G are three-colored red, blue and green, so that the red faces are of arbitrary even size, while the size of each blue and green face is a multiple of four. Then G can be distance-two colored with four colors.*

Proof. Let G' be the multigraph obtained from G by shrinking each of the red faces. Clearly G' is planar, and since the sizes of blue and green faces in G' are half of what they were in G , they will be even, so G' is also bipartite. Let us label the two sides of the bipartition as A and B respectively. Now consider the special three-edge coloring of G associated with the face coloring of G . Each red edge in this special edge-coloring joins a vertex of A with a vertex of B ; we orient all red edges from A to B . Now traversing each red edge in G in the indicated

orientation either has a blue face on the left and green face on the right, or a green face on the left and blue face on the right. In the former case we call the edge *class one* in the latter case we call it *class two*. Each vertex of G is incident with exactly one red edge; the vertex inherits the class of its red edge. The vertices around each red face in G are alternately in class 1 and class 2. We assign colors 1, 2, 3 to vertices of class one and colors 4, 5, 6 to vertices of class two. It remains to decide how to choose from the three colors available for each vertex. A vertex adjacent to red edges in class i has only three vertices within distance two in the same class, namely the vertex across the red edge, and the two vertices at distance two along the red face in either direction. Therefore distance-two coloring for class i corresponds to three-coloring a cubic graph. Since neither class can yield a K_4 , such a three-coloring exists by Brooks' theorem [7]. This yields a distance-two six-coloring of G . \square

3. DISTANCE-TWO FOUR-COLORING OF TYPE-ONE BARNETTE GRAPHS IS NP-COMPLETE

We now state our main intractability result.

Theorem 3.1. *The distance-two four-coloring problem for tri-connected bipartite cubic planar graphs is NP-complete.*

We will begin by deriving a weaker version of our claim.

Theorem 3.2. *The distance-two four-coloring problem for bipartite planar subcubic graphs is NP-complete.*

Proof. Consider the graph H in Figure 1.

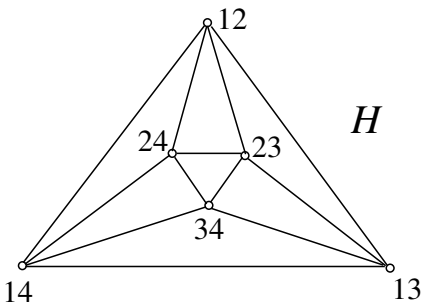


FIGURE 1. The graph H for the proof of Theorem 3.2

We will reduce the problem of H -coloring planar graphs to the distance-two four-coloring problem for bipartite planar subcubic graphs. In the H -coloring problem we are given a planar graph G and the question is whether we can color the vertices of G with colors that are vertices of H so that adjacent vertices of G obtain adjacent colors. This can be done if and only if G is three-colorable, since the graph H both contains a triangle and is three-colorable itself. (Thus any three-coloring of G is an H -coloring of G , and any H -coloring of G composed with a three-coloring of H is a three-coloring of G .) It is known that the three-coloring problem for planar graphs is NP-complete, hence so is the H -coloring problem.

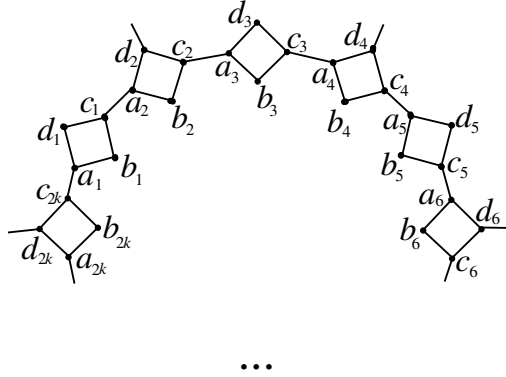


FIGURE 2. The ring gadget

Thus suppose G is an instance of the H -coloring problem. We form a new graph G' obtained from G by replacing each vertex v of G by a *ring gadget* depicted in Figure 2. If v has degree k , the ring gadget has $2k$ squares. A *link* in the ring is a square $a_i b_i c_i d_i a_i$ followed by the edge $c_i a_{i+1}$. A link is *even* if i is even, and *odd* otherwise. Every even link in the ring will be used for a connection to the rest of the graph G' , thus vertex v has k available links. For each edge vw of G we add a new vertex f_{vw} that is adjacent to a vertex d_s in one available link of the ring for v and a vertex d'_t in one available link of the ring for w . (We use primed letters for the corresponding vertices in the ring of w to distinguish them from those in the ring of v .) The actual choice of (the even) subscripts s, t does not matter, as long as each available link is only used once. The resulting graph is clearly subcubic and planar. It is also bipartite, since we can bipartition all its vertices into one independent set A consisting of all the vertices $a_i, c_i, b_{i+1}, d_{i+1}$ with odd i in all the rings, and another independent set B consisting of the vertices $a_i, c_i, b_{i+1}, d_{i+1}$ with even i in all the rings. Moreover, we place all vertices f_{vw} into the set A . Note that in any distance-two four-coloring of the ring, each link must have four different colors for vertices a_i, b_i, c_i, d_i , and the same color for a_i and a_{i+1} . Thus all a_i have the same color and all c_i have the same color. The pair of colors in b_i, d_i is also the same for all i ; we will call it the *characteristic pair of the ring for v* . For any pair ij of colors from $1, 2, 3, 4$, there is a distance-two coloring of the ring that has the characteristic pair ij .

We prove that G is H -colorable if and only if G' is distance-two four-colorable. In an H -coloring of G , the vertices of G are actually assigned unordered pairs from $\{1, 2, 3, 4\}$, since the vertices of H are labeled by pairs. (Note that two vertices of H are adjacent if and only if the pairs they are labeled with intersect in exactly one element.) Thus suppose that we have an H -coloring ϕ of G . If $\phi(v) = ij$ (i.e., the vertex v of G is assigned the vertex of H labeled by the pair ij), then we colour the ring of v so that its characteristic pair is ij . This still leaves a choice of which of the colors i, j is in which b_s, d_s , in each of the links a_s, b_s, c_s, d_s . Since ϕ is an H coloring, adjacent vertices vw are assigned pairs that intersect in exactly one element. This makes it possible to color each b_s, d_s so that all colors at distance at most two are distinct. For instance if vertices v and w are adjacent in G and colored by $12, 13$ by ϕ , and if f_{vw} is adjacent to the vertices d_s in the ring for v and d'_t in the ring for w , then both b_s in

the ring for v and b_t in the ring for w are colored 1, as is f_{vw} , while d_s in the ring for v and d_t in the ring for w are colored 2 and 3 respectively. It is easy to see that this is a distance-two four-coloring of G' .

Conversely, in any distance-two four-coloring of G' , the color of a vertex f_{vw} determines the same color in the b 's of its adjacent links of the rings for v and w , whence the characteristic pairs of these two rings intersect in exactly one element. Thus we may define a mapping ϕ of $V(G)$ to $V(H)$ by assigning to each vertex $v \in V(G)$ the characteristic pair of the ring for v . Then ϕ is an H -coloring of G , since adjacent vertices of G are assigned pairs that are adjacent in H . \square

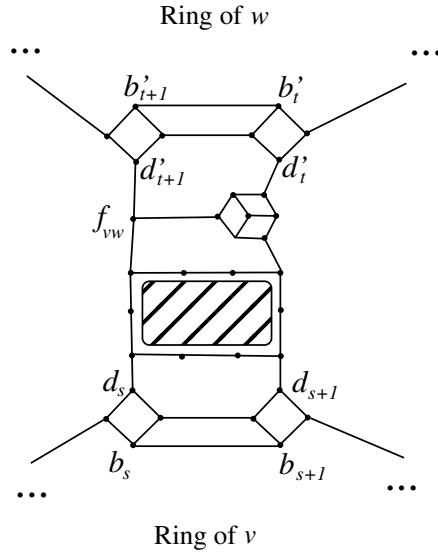


FIGURE 3. The modified edge-gadget for f_{vw}

To prove the full Theorem 3.1, the construction of the graph G' is modified as suggested in Figure 3. Recall that in the construction of G' , for each edge vw of G a separate vertex f_{vw} was made adjacent to d_s in the ring of v and d'_t in the ring of w . Recall that both s and t are even, and the vertices d_{s+1}, d'_{t+1} (with both subscripts odd) remained available for connection. We now make a new edge-gadget around the vertex f_{vw} , making it directly adjacent to d'_{t+1} , and connected to d_s by a path, as depicted in Figure 3. In both rings, the two “ b ” type vertices in the two consecutive links are joined together by an additional edge; specifically, we add the edges $b_s b_{s+1}$ and $b'_t b'_{t+1}$. (Note that this forces the corresponding “ d ” type vertices d_s and d_{s+1} to be colored differently in any distance-two four-coloring, and similarly for d'_t and d'_{t+1}). Moreover, further vertices and edges are added, as depicted in Figure 3. The shaded ten-sided region is identified with the ten-sided exterior face of the graph depicted in Figure 5, which has a unique distance-two four-coloring, shown there. (The heavy edges correspond to the ten-sided shaded figure.) (Note that the graph in Figure 5 was obtained from the graph in Figure 7 by the deletion of two edges.) Note that the construction is not symmetric, as it depends on which ring is viewed as the “bottom” ring for the vertex f_{vw} . (The depicted

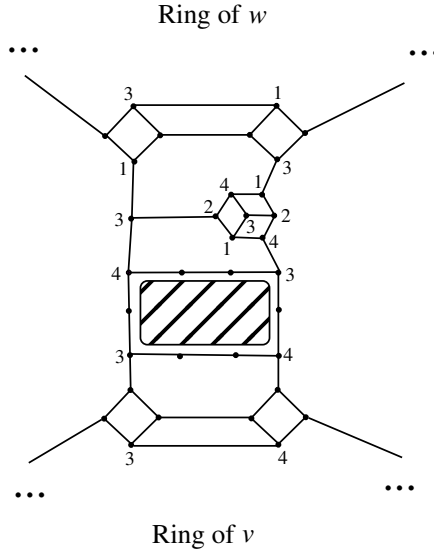


FIGURE 4. With its unique distance-two coloring

figure has the ring of v on the bottom, but the conclusions are the same if it were the ring of w .) We can choose either way, independently for each edge vw of G . It can be seen that the resulting graph, which we denote by G'' , is bipartite, planar, and cubic. We may assume that G is bi-connected (the three-coloring problem for biconnected planar graphs is still NP-complete), and therefore G'' is also tri-connected (as no two faces share more than one edge). Using the unique distance-two four-colouring of the graph in Figure 5, it also follows that in any distance-two four-coloring of G'' the vertices d_s and d'_{t+1} have different colors, while both vertices b_s and b'_{t+1} have the same color (the color of f_{vw}), in any distance-two four-coloring of G'' . To facilitate checking this, we show in Figure 4 a partial distance-two four-coloring, by circles, squares, up triangles, and down triangles; this coloring is forced by arbitrarily coloring f_{vw} and its three neighbours by four distinct colors. Since the colors of the pair b_s, d_s and the pair b'_{t+1}, d'_{t+1} have exactly one color in common, the previous NP-completeness proof applies, i.e., G is H -colorable if and only if G'' is distance-two four-colorable.

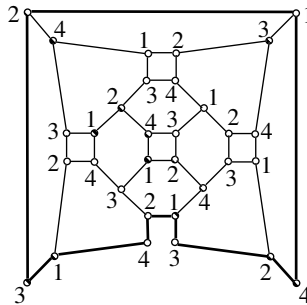


FIGURE 5. The graph for the shaded region, with its unique distance-two four-coloring

We remark that (with some additional effort) we can prove that the problem is still NP-complete for the class of tri-connected bipartite cubic planar graphs with no faces of sizes larger than 44.

4. DISTANCE-TWO FOUR-COLORING OF GOODEY GRAPHS

Recall that Goodey graphs are type-two Barnette graph with all faces of size 4 and 6 [12, 13]. In other words, a *Goodey graph* is a cubic plane graph with all faces having size 4 or 6. By Euler’s formula, a Goodey graph has exactly six square faces, while the number of hexagonal faces is arbitrary.

A *cyclic prism* is the graph consisting of two disjoint even cycles $a_1a_2 \cdots a_{2k}a_1$ and $b_1b_2 \cdots b_{2k}b_1$, $k \geq 2$, with the additional edges $a_i b_i$, $1 \leq i \leq 2k$. It is easy to see that cyclic prisms have either no distance-two four-coloring (if k is odd), or a unique distance-two four-coloring (if $k \geq 2$ is even). Only the cyclic prisms with $k = 2, 3$ are Goodey graphs, and thus from Goodey cyclic graphs only the cube (the case of $k = 2$) has a distance-two coloring, which is moreover unique.

In fact, all Goodey graphs that admit distance-two four-coloring can be constructed from the cube as follows. The Goodey graph C_0 is the cube, i.e., the cyclic prism with $k = 2$. The Goodey graph C_1 is depicted in Figure 7. It is obtained from the cube by separating the six square faces and joining them together by a pattern of hexagons, with three hexagons meeting at a vertex tying together the three faces that used to meet in one vertex. The higher numbered Goodey graphs are obtained by making the connecting pattern of hexagons higher and higher. The next Goodey graph C_2 has two hexagons between any two of the six squares, with a central hexagon in the centre of any three of the squares, the following Goodey graph C_3 has three hexagons between any two of the squares and three hexagons in the middle of any three of the squares, and so on. Thus in general we replace every vertex of the cube by a triangular pattern of hexagons whose borders are replacing the edges of the cube. We illustrate the vertex replacement graphs in Figure 6, without giving a formal description. The entire Goodey graph C_1 is depicted in Figure 7.

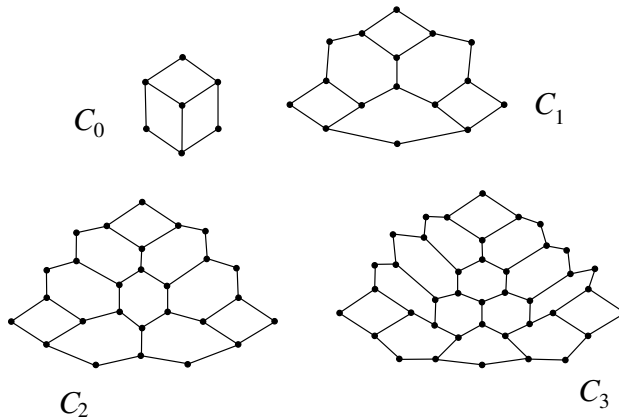


FIGURE 6. The vertex replacements for Goodey graphs C_0, C_1, C_2 , and C_3

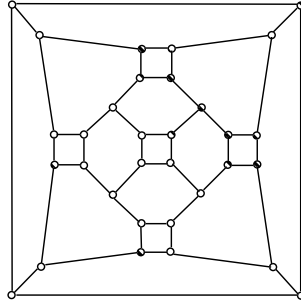


FIGURE 7. The Goody graph C_1

We have the following results.

Theorem 4.1. *The Goodey graphs $C_k, k \geq 0$, have a unique distance-two four-coloring, up to permutation of colors.*

Proof. We described C_k as eight triangular regions R , each consisting of $\binom{k}{2}$ hexagons, one region R for each vertex of the cube. Each R has three squares at the corners, which we describe as two squares joined by a chain of k hexagons horizontally at the bottom, and a third square on top. (See Figure 6.)

We partition the vertices into $k + 2$ horizontal paths $P_i, 0 \leq i \leq k + 1$, with each P_i having endpoints of degree 2 and internal vertices of degree 3. The path P_0 has length $2k + 2$, and the remaining paths $P_i, i \geq 1$ have length $2k + 6 - 2i$. In particular the last P_{k+1} has length 4, and is the only P_i that is actually a cycle, pictured as the square at the top. See Figure 8.

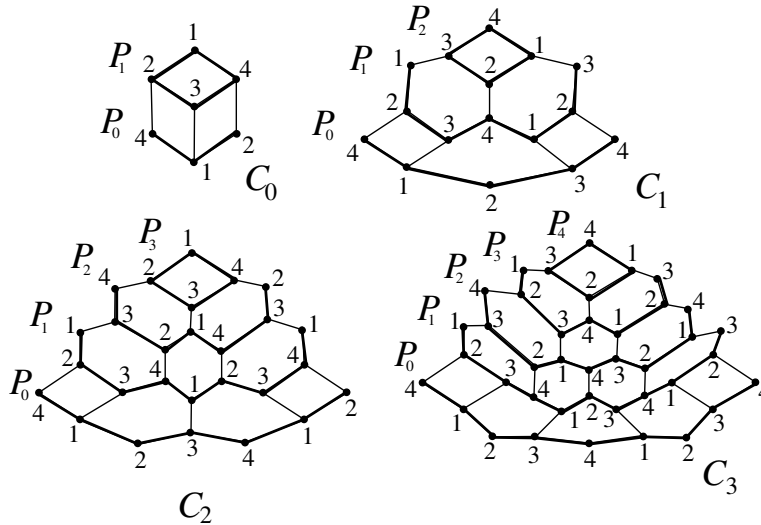


FIGURE 8. The paths P_i and the resulting distance-two colorings

We denote $P_i = v_i^0 v_i^1 \cdots v_i^{\ell_i}$. The edges between P_0 and P_1 are $v_0^0 v_1^1, v_0^j v_1^{j+1}$ for $1 \leq j \leq \ell_0 - 1, j$ odd, and $v_0^{\ell_0} v_1^{\ell_1 - 1}$. We can choose the permutation of colors for the square $v_0^0 v_0^1 v_1^2 v_1^1$ to be

4132, forcing for neighbors v_1^0, v_1^3, v_0^2 the colors 1, 4, 2, and completing the adjacent square, or hexagon with the assignment to v_1^4, v_0^3 of colors 1, 3. This forced process extends similarly through the chain of hexagons until the last square.

We have derived the beginning of P_0 as 4123 and the beginning of P_1 as 12341. After the forced extension, P_0 will be an initial segment of $(4123)^*$ and P_1 will be an initial segment of $(1234)^*$.

For $i \geq 1$, the edges between P_i and P_{i+1} are $v_i^0 v_{i+1}^1, v_i^j v_{i+1}^{j-1}$ for $3 \leq j \leq \ell_i - 3$, j odd, and $v_i^{\ell_i} v_{i+1}^{\ell_{i+1}-1}$.

A similar process derives the beginning of P_i for i odd as 1234 and the beginning of P_i for i even as 4321. After the forced extension, P_i for i odd will be an initial segment of $(1234)^*$ and P_i for i even will be an initial segment of $(4321)^*$.

This gives a unique coloring for the triangular region after coloring one square S , which is uniquely extended to the four triangular regions surrounding S , and then uniquely extended to the four triangular regions surrounding S' opposite to S . \square

Theorem 4.2. *The Goodey graphs $C_k, k \geq 0$, are the only bipartite cubic planar graphs having a distance-two four-coloring.*

Proof. Consider a Goodey graph G with a fixed distance-two four-coloring. Recall that Goodey graphs have exactly six squares. Each of the squares is joined by four chains of hexagons to four squares. We consider the dual six-vertex graph G' whose vertices are squares in G , with ab an edge in G' if and only if there is a chain of hexagons joining squares a and b . It can be readily verified that such a chain cannot cross itself or another chain in G . Indeed, the colors in the fixed distance-two four-coloring are uniquely forced along such chains and they don't match if the chains should cross. It follows that the graph G' is planar. A similar argument shows that a chain cannot return to the same square, and two chains from the square a cannot end at the same square b . Thus G' has no faces of size one or two, and by Euler's formula it has 12 edges and 8 faces; therefore all faces of G' must be triangles, and G' is the octahedron.

Let T be a triangular face in G' , let s be a side of T with the smallest number d of hexagons in G . Then it can again be checked using the coloring that the other two sides of T will also have d hexagons in G . Then T corresponds to a triangular region R as in Theorem 4.1, and the octahedron G' yields $G = C_k$ for $k = d$. \square

We can therefore conclude the following.

Corollary 4.3. *The distance-two four-coloring problem for Goodey graphs is solvable in polynomial time.*

Recognizing whether an input Goodey graph is some C_k can be achieved in polynomial time; in the same time bound G can actually be distance-two four-colored.

5. DISTANCE-TWO FOUR-COLORING OF TYPE-TWO BARNETTE GRAPHS IS POLYNOMIAL

We now return to general type-two Barnette graphs, i.e., cubic plane graphs with face sizes 3, 4, 5, or 6. As a first step, we analyze when a general cubic plane graph admits a distance-two four-coloring which has three colors on the vertices of every face of G .

Theorem 5.1. *A cubic plane graph G has a distance-two four-coloring with three colors per face if and only if*

- (1) *all faces in G have size which is a multiple of 3,*
- (2) *G is bi-connected, and*
- (3) *if two faces share more than one edge, the relative positions of the shared edges must be congruent modulo 3 in the two faces.*

The last condition means the following: if faces F_1, F_2 meet in edges e, e' and there are n_1 edges between e and e' in (some traversal of) F_1 , and n_2 edges between e and e' in (some traversal of) F_2 , then $n_1 \equiv n_2 \pmod{3}$.

Proof. Suppose G has a distance-two four-coloring with three colors in each face. The unique way to distance-two color a cycle with colors 1, 2, 3 is by repeating them in some order $(123)^*$ along one of the two traversals of the cycle. Therefore the length is a multiple of 3 so (1) holds. Moreover, there can be no bridge in G as that would imply a face that self-intersects and is traversed in opposite directions along any traversal of that face, disagreeing with the order $(123)^*$ in one of them; thus (2) also holds. Finally, (3) holds because the common edges must have the same colors in both faces.

Suppose the conditions hold, and consider the dual G^D of G . (Note that each face of G^D is a triangle.) We find a distance-two coloring of G as follows. Let F be a face in G ; according to conditions (1-2), its vertices can be distance-two colored with three colors. That takes care of the vertex F in G^D . Using condition (3), we can extend the coloring of G to any face F' adjacent to F in G^D . Note that we can use the fourth colour, 4, on the two vertices adjacent in F' to the two vertices of a common edge. In this way, we can propagate the distance-two coloring of G along the adjacencies in G^D . If this produces a distance-two coloring of all vertices of G , we are done. Thus it remains to show there is no inconsistency in the propagation. If there is an inconsistency, it will appear along a cycle C in G^D . If there is only one face inside of C , then C is a triangle corresponding to a vertex of G , and there is no inconsistency. Otherwise we can join some two vertices of C by a path P inside C , and the two sides of P inside C give two regions that are inside two cycles C', C'' . The consistency of C then follows from the consistency of each of C', C'' by induction on the number of faces inside the cycle. \square

It turns out that conditions (1 - 3) are automatically satisfied for cubic plane graphs with faces of sizes 3 or 6.

Corollary 5.2. *Type-two Barnette graphs with faces of sizes 3 or 6 are distance-two four-colorable.*

Proof. Such a graph must be bi-connected, i.e., cannot have a bridge, since no triangle or hexagon can self-intersect. Moreover, only two hexagons can have two common edges, and it is easy to check that they must indeed be in relative positions congruent modulo 3 on the two faces. (Since all vertices must have degree three.) Thus the result follows from Theorem 5.1. \square

Theorem 5.3. *Let G be type-two Barnette graph. Then G is distance-two four-colorable if and only if it is one of the graphs $C_k, k \geq 0$, or all faces of G have sizes 3 or 6.*

Proof. If there are faces of size both 3 and 4 (and possibly size 6), then there must be (by Euler's formula) two triangles and three squares, and as in the proof of Theorem 4.2, the squares must be joined by chains of hexagons, which is not possible with just three squares.

If there is a face of size 5, then there is no distance-two four-coloring since all five vertices of that face would need different colors. \square

6. DISTANCE-TWO COLORING OF QUARTIC GRAPHS

A *quartic graph* is a regular graph with all vertices of degree four. Thus any distance-two coloring of a quartic graph requires at least five colors. A *four-graph* is a plane quartic graph whose faces have sizes 3 or 4. The argument to view these as analogues of type-two Barnette graphs is as follows. For cubic plane Euler's formula limits the numbers of faces that are triangles, squares, and pentagons, but does not limit the number of hexagon faces. Similarly, for plane quartic graphs, Euler's formula implies that such a graph must have 8 triangle faces, but places no limits on the number of square faces.

We say that two faces are *adjacent* if they share an edge.

Lemma 6.1. *If a four-graph can be distance-two five-colored, then every square face must be adjacent to a triangle face. Thus G can have at most 24 square faces.*

Proof. We view the numbers 1, 2, 3, 4 modulo 4, and number 5 is separate. Let $u_1u_2u_3u_4$ be a square face that has no adjacent triangle face. (This is depicted in Figure 9 as the square in the middle.) Color u_i by i . Let the adjacent square faces be $u_iu_{i+1}w_{i+1}v_i$. One of v_i, w_i must be colored 5 and the other one $i + 2$. Then either all v_i or all w_i are colored 5, say all w_i are colored 5, and all v_i are colored $i + 1$. Then $v_iu_iw_i$ cannot be a triangle face, or w_i, w_{i+1} would be both colored 5 at distance two. Therefore $t_iv_iu_iw_i$ must be a square face. (In the figure, this is indicated by the corner vertices being marked by smaller circles; these must exist to avoid a triangle face.) This means that the original square is surrounded by eight square faces for $u_1u_2u_3u_4$, and t_i must have color $i + 3$, since u_i, v_{i+3}, v_i, w_i have colors $i, i + 1, i + 2, 5$.

But then there cannot be a triangle face $x_iv_iw_{i+1}$, since x_i is within distance two of $u_i, u_{i+1}, v_i, t_i, w_{i+1}$ of colors $i, i + 1, i + 2, i + 3, 5$, so each of the adjacent square faces $u_iu_{i+1}w_{i+1}v_i$ for $u_1u_2u_3u_4$ has adjacent square faces as well. This process of moving to adjacent square faces eventually reaches all faces as square faces, contrary to the fact that there are 8 triangle faces. \square

It follows that there are only finitely many distance-two five-colorable four-graphs.

Corollary 6.2. *The distance-two five-coloring problem for four-graphs is polynomial.*

In fact, we can fully describe all four-graphs that are distance-two five-colorable. Consider the four-graphs G_0, G_1 given in Figure 10. The graph G_0 has 8 triangle faces and 4 square faces, the graph G_1 has 8 triangle faces and 24 square faces. Note that G_0 is obtained from the cube by inserting two vertices of degree four in two opposite square faces. Similarly, G_1 is obtained from the cube by replacing each vertex with a triangle and inserting into each face of the cube a suitably connected degree four vertex. (In both figures, these inserted vertices are indicated by smaller size circles.)

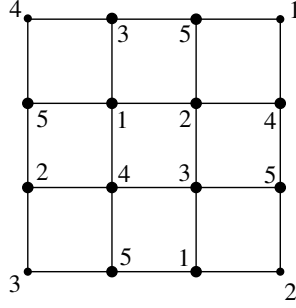


FIGURE 9. One square without adjacent triangles implies all faces must be squares

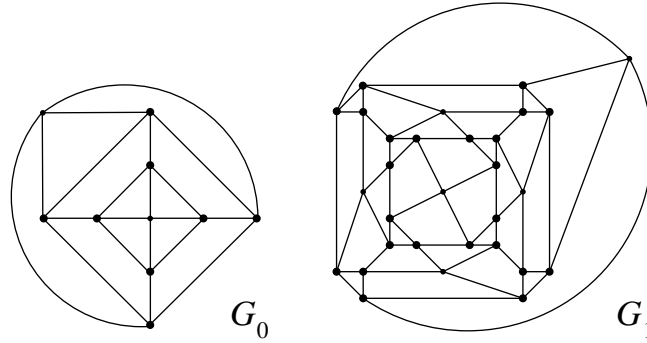


FIGURE 10. The only four-graphs that admit a distance-two five-coloring

Theorem 6.3. *The only four-graphs G that can be distance-two five-colored are G_0, G_1 . These two graphs can be so colored uniquely up to permutation of colors.*

Proof. We show that if G can be so colored, then either G is G_0 or every triangle in G must be surrounded by six square faces, in which case G is G_1 .

Suppose G has two adjacent triangles $u_5u_1u_2$ and $T = u_5u_2u_3$. The vertices adjacent to T must be given two colors other than those of u_5, u_2, u_3 . If T has two adjacent squares, then it has five adjacent vertices, which must be given the two colors in alternation, a contradiction. Similarly if T is adjacent to three triangles then the three vertices adjacent would need three new colors, a contradiction.

We may thus assume a triangle $u_5u_3u_4$. If there is a square $u_1u_5u_4t$, this square plus the two adjacent triangles would need six colors, a contradiction, so $u_5u_4u_1$ is a triangle, completing u_5 adjacent to the four-cycle $u_1u_2u_3u_4$. Color u_i with color i . Then the additional vertex v_i adjacent to u_i for $1 \leq i \leq 4$ must be given color $i + 2$ (modulo 4), so these v_i form a 4-cycle, and any additional vertex adjacent to a v_i must get color 5, so there is a single additional v_5 with color 5. This gives a uniquely colored G_0 , up to permutation of colors.

In the remaining case, each triangle $T' = u_1u_2u_3$ has adjacent squares $u_iu_{i+1}w_{i+1}v_i$, with addition modulo 3. The vertices v_i, w_{i+1} must be given the two colors different from those of T' , and in alternation around T' , so there cannot be a triangle $u_i v_i w_i$ else w_i, w_{i+1} with the

same color would be at distance two. So there are squares $u_i v_i t_i w_i$, and T' is surrounded by six squares.

By Lemma 6.1, we must have a triangle adjacent to the square $u_i v_i t_i w_i$, either $v_i t_i x_i$ or $w_i t_i y_i$, but not both since six colors would be needed. Let such a triangle be T_i , and we link T' to the three T_i . These triangles viewed as vertices linked form a cubic graph without triangles G' , since a triangle face would be three triangles joined in G , which would need to have only three squares inside by Lemma 6.1. The graph G' has 8 vertices for the 8 triangles, so this graph is the cube C . Replacing each vertex corresponding to a triangle by the corresponding triangle gives a graph D .

Suppose the triangles adjacent to T' are $v_i t_i x_i$ for $1 \leq i \leq 3$. Then going around a face of C we notice only one vertex inside this face by Lemma 6.1, giving the construction of G_1 . If we assign to the vertex inside this face the color 5, we notice that the surrounding triangles in D must use three colors at most 4, and each must omit a different color of 4. This implies that all vertices in centers of square faces must be 5, and only opposite triangles for C use the same 3 out of 4 colors. This proves existence and uniqueness up to permutation of colors of the distance-two 5-coloring of G_1 .

Suppose instead the adjacent triangles are $T_1 = w_1 t_1 y_1$, $T_2 = v_2 t_2 x_2$, and $T_3 = v_3 t_3 x_3$. If there is no triangle $v_1 w_2 x$, then the three squares Q_1, Q_2, Q_3 between T_1 and T_2 are respectively adjacent to squares Q'_1, Q'_2, Q'_3 , and Q'_2 must be adjacent to a triangle by Lemma 6.1. There must be triangles at both ends of the Q'_i and these are adjacent to T_1 and T_2 , a contradiction.

Finally, suppose again the adjacent triangles are $T_1 = w_1 t_1 y_1$, $T_2 = v_2 t_2 x_2$, and $T_3 = v_3 t_3 x_3$, but there is a triangle $v_1 w_2 x$. This triangle faces T' , and T_1 faces T_3 . Triangles facing each other give two diagonals in the square faces of C , which implies two opposite faces without such diagonals in C , while the four sets of two diagonals form a matching of the 8 vertices of C . If the center of a face without diagonals gets assigned 5, then the adjacent triangles will be assigned a subset of $1 \leq i \leq 4$. Then joining the sets of two diagonals assigns a 5 to a vertex of each remaining triangle, which is not possible to the center of the remaining face without diagonals. □

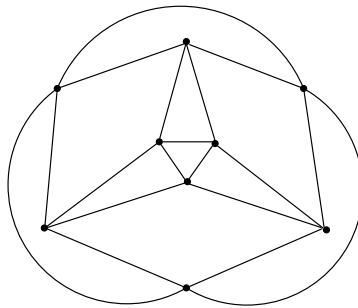


FIGURE 11. A four-graph requiring nine colors in any distance-two coloring

We close with a few remarks and open problems.

Wegner’s conjecture [25] that any planar graph with maximum degree $d = 3$ can be distance-two seven-colored has been proved in [14, 23]. That bound is actually achieved by a type-two

Barnette graph, namely the graph obtained from K_4 by subdividing three incident edges. Thus the bound of 7 cannot be lowered even for type-two Barnette graphs.

Wegner's conjecture for $d = 4$ claims that any planar graph with maximum degree four can be distance-two nine-colored. The four-graph in Figure 11 actually requires nine colors in any distance-two coloring. Thus if Wegner's conjecture for $d = 4$ is true, the bound of 9 cannot be lowered, even in the special case of four-graphs. It would be interesting to prove Wegner's conjecture for four-graphs, i.e., to prove that *any four-graph can be distance-two colored with nine colors*.

We've conjectured that any bipartite cubic planar graph can be distance-two colored with six colors (a special case of a conjecture of Hartke, Jahanbekaam and Thomas [14]). The hexagonal prism (a cyclic prism with $k = 3$, which is a Goodey graph), actually requires six colors. Hence if our conjecture is true, the bound of 6 cannot be lowered even for Goodey graphs. It would be interesting to prove our conjecture for Goodey graphs, i.e., to prove that *any Goodey graph can be distance-two colored with six colors*.

Finally, for the Barnette graphs we have focused on the complexity of the distance-two coloring problem with four colors, which is the minimum number of colors possible in any distance-two coloring. We might ask what is the complexity of the distance-two five-coloring problem for these classes.

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268 WAVERLEY STREET, PALO ALTO, CA 94301, USA
E-mail address: tomas@theory.stanford.edu

SCHOOL OF COMPUTING SCIENCE, SIMON FRASER UNIVERSITY, BURNABY, B.C., CANADA V5A 1S6
E-mail address: pavol@sfu.ca

LOS ALTOS HILLS
E-mail address: carlos.subi@hotmail.com