Is Edge 3-Colorability Reconstructible?

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1 Introduction

We follow the presentation in Locke [5]. Let G be a graph on at least three vertices and v be a vertex of G. Then $G - v$ is a *vertex deleted subgraph* of G. The collection of vertex deleted (unlabelled) subgraphs of G, together with their multiplicities, is the *deck* of G. Thus, the deck for a graph of n vertices consists of n graphs, each of which has $n - 1$ vertices. Each of the graphs in this deck could be called a *card*.

Given a deck of a graph G, certain parameters of G are easily determined. G has one more vertex than any graph in the deck. Every edge of G is missing from exactly two cards, thus each edge show up on $n-2$ cards. The number of edges of G can be determined by summing the numbers of edges on each card and dividing the total by $n - 2$.

Similarly, for any graph H on fewer vertices than G, the number of subgraphs of G which are isomorphic to H can be determined by summing the numbers of such subgraphs on each card and dividing the result by $|V(G)| - |V(H)|$. This is known as *Kelly's Lemma*.

A parameter of G which can be deduced from the deck is called *reconstructible*. Since the number of edges of G is reconstructible, the degree sequence of G is reconstructible. Thus, if G is a regular graph, G itself is reconstructible.

Disconnected graphs are reconstructible. Trees are reconstructible. Separable graphs without endvertices are reconstructible.

In 1967, Tutte proved that the dichromatic (rank) and Tutte (dichromate) polynomials are reconstructible. The chormatic polynomial, chromatic number, the flow polynomial, and the number of spanning trees are easily deduced from the dichromatic polynomial. Tutte also showed how to count Hamiltonian cycles, which implies by a result of Pouzet that the characteristic polynomial is reconstructible. The idiosyncratic polynomial is also reconstructible.

A useful tool is a lemma due to Kocay. Let H and F be disjoint, connected graphs with $|V(H)| + |V(F)| + |V(G)|$. The number of spanning subgraphs of G which are isomorphic to the union of H and F is reconstructible. Therefore the number of matchings of a given size in G is reconstructible.

We shall show that the size of a maximum matching is *polynomial time reconstructible*. We do not know of any other such resuts on the efficiency of the reconstruction process. We then make partial progress towards the reconstruction of edge 3-colorability. Recall that the vertex chromatic number is reconstructible.

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2 The Size of a Maximum Matching is Polynomial Time Reconstructible

Lemma 1 *Suppose that all cards in the deck of G have the same maximum matching size. Suppose also that the connected components of G have sizes $s_1, s_2, \ldots, s_r$. Then a maximum matching of G has size $\lfloor s_1/2 \rfloor + \lfloor s_2/2 \rfloor + \cdots + \lfloor s_r/2 \rfloor$, leaving at most one unmatched vertex in each component of G.*

Proof. Let M be a maximum matching of G, and suppose M leaves at least two vertices u and v unmatched in the same component R of G. Let w be a neighbor of u. If w is unmatched, then we coud match u and w, contrary to M being a maximum matching. If w is matched to x, then $G - w$ must have a maximum matching of the same size, and thus an augmenting path from some y matching x. We could then match w to u and increase the size of the maximum matching M, unless $y = u$. In this last case we have replaced the unmatched u by the unmatched neighbor w. We may repeat this process of going from the unmatched u to the arbitrarily chosen newly unmatched neighbor w, until we reach a w' that is a neighbor of v. But then we could match w' to v, contrary to M being a maximum matching. This contradiction shows that the maximum matching leaves at most one vertex of each component R unmatched, as required. □

Lemma 2 *The sizes $s_1, s_2, \ldots, s_r$ of the components of G are polynomial time reconstructible.*

Proof. If some component of G consists of an isolated vertex, then we could remove a vertex v of degree zero and from the resulting card infer the s_i by adding an $s_j = 1$. If G has no isolated vertices, then we could consider all cards with a minimal number of components, obtained by removing just one vertex v from any one component without separating the component. From all these choices of cards, choose one that has a component of least size, and add one to this size to infer the set of s_i. □

Theorem 1 *The size of a maximum matching of G is polynomial time reconstructible.*

Proof. If the cards of G have different matcing sizes k and $k+1$ then the size of a maximum matching of G is $k+1$. Otherwise the preceeding two Lemmas reconstruct the size of a maximum matching of G. □

Notice as remarked in the introduction that if we do not care about polynomial time then the number of matchings of a given size in G is reconstructible as well.

3 On Edge 3-Colorability Reconstruction

A graph G is *difficult* if it satisfies the following properties: (1) all vertices of G have degree 2 or 3; (2) at least 3 vertices of G have degree 2; and (3) all vertices of degree 2 in G are at distance at least 3 from one another.

Theorem 2 *If G is not difficult then edge 3-colorability of G is reconstructible.*

Proof. Suppose G has a vertex v of degree at least 4. Then G is not edge 3-colorable. If G has a vertex v of degree 0 or 1, then G is edge 3-colorable if and only if $G - v$ is edge 3-colorable. We are thus left with graphs satisfying property (1), namely all vertices have degree 2 or 3.

If G has all vertices of degree 3, then G itself is reconstructible. If G has a single vertex v of degree 2, then $G-v$ has two vertices x, y of degree 2 (adjacent to v in G), and we must edge 3-color $G-v$ in such a way that the two edges incident to x and y are not colored in the same way for x and for y for G to be edge 3-colorable, which is not possible. If G has two vertices u, v of degree 2, then G has an even number of vertices, and the colors of the two edges incident to u and v must be the same for both u and v. Thus G must decompose its vertices into a union of disjoint even length cycles (formed by the two colors of the edges incident to u or to v), which is reconstructible by Kocay's lemma from the Introduction if there are at least two such cycles, or by the result of Tutte if there is only one cycle giving Hamiltonicity. We are thus left with graphs satisfying property (2) as well, namely G has at least 3 vertices of degree 2.

If G has two adjacent vertices u, v of degree 2, then G is edge 3-colorable if and only if $G-u$ is edge 3-colorable. If G has two vertices u, v of degree 2 at distance 2 from each other, say w of degree 3 is adjacent to both u and v, then G is edge 3-colorable if and only if $G-w$ is edge 3-colorable. This follows immediately unless the colors at the single edge left for u and v is both say green, and the colors at the third neighbor x of w are blue and green. We can leave this case by finding an alternating path starting at u between green and red, and exchanging these two colors in the path. We are thus left with graphs satisfying property (3) as well, namely G has all vertices of degree 2 at distance at least 3 from each other. These are the difficult graphs. $\square$

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