Extension Problems with Degree Bounds

Tomas Feder*, Pavol Hell†, and Jing Huang‡

Abstract

We have proved in an earlier paper that the complexity of the list homomorphism problem, to a fixed graph H, does not change when the input graphs are restricted to have bounded degrees (except in the trivial case when the bound is two). By way of contrast, we show in this paper that the extension problem, again to a fixed graph H, can, in some cases, become easier for graphs with bounded degrees.

1 Background

We consider undirected graphs without multiple edges, but with loops allowed. A graph without loops is called *irreflexive*, and a graph in which each vertex has a loop is called *reflexive*. Note that a bipartite graph is, by definition, irreflexive.

A *homomorphism* $f : G \to H$ is a mapping $f : V(G) \to V(H)$ such that $f(g)f(g')$ is an edge of H for each edge gg' of G. Every graph H gives rise to a decision problem HOM_H in which one is to decide whether or not a given input irreflexive graph G admits a homomorphism to the fixed graph H. Such a homomorphism is also called an *H-colouring of G*, and the problem HOM_H is referred to as the *H-colouring problem* or the *homomorphism problem* to H. It is shown in [14] that each H-colouring problem HOM_H is polynomial-time solvable (if H is bipartite or contains a loop), or NP-complete (if H is irreflexive and nonbipartite).

The problem LHOM_H, known as the *list H-colouring problem* or the *list homomorphism problem* to H, has each instance consist of an irreflexive graph G together with lists, $L(g) \subseteq V(H), g \in V(G)$, and the question

*tomas@theory.stanford.edu

†School of Computing Science, Simon Fraser University, Burnaby, B.C., Canada, V5A 1S6; pavol@cs.sfu.ca

‡Department of Mathematics and Statistics, University of Victoria, P.O. Box 3045, Victoria, B.C., Canada, V8W 3P4; jing@math.uvic.ca

to decide is whether or not there exists a homomorphism f of G to H in which each $g \in V(G)$ has $f(g) \in L(g)$. Such a homomorphism will be called a *list homomorphism* (or a list H-colouring) with respect to the lists L. In a sequence of papers [6, 7, 8], we have classified the complexity of these problems. The problem LHOM_H is polynomial-time solvable when H is a 'bi-arc graph', and is NP-complete otherwise. Bi-arc graphs are defined in [8]; they are common generalization of reflexive interval graphs and (irreflexive) bipartite graphs whose complements are circular arc graphs.

Note that, unless stated otherwise, the input graphs G are always considered to be irreflexive.

A number of variants of these basic problems have been considered.

- EXT_H, called the *extension problem* for H, is the restriction of LHOM_H to inputs with lists $L(g), g \in V(G)$, which are either singletons ($|L(g)| = 1$), or the entire set ($L(g) = V(H)$). Thus the extension problem for H asks whether or not a partial mapping of $V(G)$ to $V(H)$ can be extended to a homomorphism of G to H.

- CLHOM_H, the *connected list homomorphism problem* to H, is the restriction of LHOM_H to inputs where each list $L(g), g \in V(G)$, induces a connected subgraph of H.

- BLHOM_H, the *balanced list homomorphism problem* to H, is the restriction of LHOM_H to inputs where each list $L(g), g \in V(G)$, satisfies $|L(g)| \geq deg_G(g)$.

When $H = K_n$, homomorphisms to H coincide with the usual notion of n-colourings. In this case, the extension problem EXT_{K_n} has been studied by many authors under the name 'pre-colouring extension' [1, 18, 19]. To conform to this interpretation, we shall call the vertices g with $|L(g)| = 1$ *pre-coloured*. A special case of the extension problem, when the input graph G contains H as a subgraph, and the singleton lists are exactly $L(h) = \{h\}, h \in V(H)$ has been studied under the name of *retraction problem* [6, 11]. In fact, when there are no degree constraints, it is easy to see that the retraction problem is equivalent to the retraction problem [6]. Of course, our polynomial algorithms on degree restricted extension problems apply also to retraction problems.

In [12, 15] the authors investigated the effect of restricting the degrees of the input graphs G. In particular, [15] sets up a common framework for all these variants, which we use here, slightly adapted for the purposes of this paper. Namely, we put a superscript Δ to indicate that the input graphs G are restricted to have all degrees at most Δ. For instance, LHOM_H^Δ is the restriction of LHOM_H to graphs G with all degrees at most Δ.

We shall generally assume that $\Delta \geq 3$, since graphs with degrees bounded by 2 are unions of paths and cycles, and all the problems are polynomial-time solvable by easy or standard techniques, cf. [4, 21]. When $\Delta \geq 3$ restricting the degrees can have a significant impact on the complexity of the problem. For instance, it is well known [13] that the problem HOM_H with $H = K_3$ (the classical problem of 3-colourability) is NP-complete, while the problem HOM_H^Δ with $\Delta = 3$ (the restriction to inputs with all degrees at most three) is polynomial-time solvable, since, by the theorem of Brooks, a connected graph with maximum degree three is either 3-colourable or isomorphic to K_4. In [12], there are more complex examples of hard HOM_H problems that become easy when a degree bound is imposed; it is also shown there that when H is an odd cycle of length at least five, the problem HOM_H^Δ remains NP-complete even for $\Delta = 3$.

In [15], the authors considered the problems LHOM_H^Δ. They observed that it is of course still the case that LHOM_H^Δ is polynomial-time solvable when H is a bi-arc graph, and posed as an open problem the question of classifying the complexity of LHOM_H^Δ for other graphs. This problem was solved in our earlier paper [9]; it turns out that the complexity of list homomorphisms does not change when degree constraints are imposed.

Theorem 1 *[9] Let $\Delta \geq 3$ be fixed. The problem LHOM_H^Δ is polynomial-time solvable when H is a bi-arc graph, and is NP-complete otherwise.*

In [10] we have given a polynomial time algorithm for the problems BLHOM_H in case the graph H is *nearly complete*, in the sense that for each vertex h of H there is at most one other vertex, possibly itself, to which h is not adjacent. It is not difficult to see, cf. [10], that for a nearly complete graph H, an instance of BLHOM_H with at least one vertex g having $deg_G(g) < |L(g)|$ must have a list homomorphism to H. Thus we may focus on instances in which all $deg_G(g) = |L(g)|$. In [10], we have shown that such instances either admit a list homomorphism to H, or have a very special structure. This yields a polynomial time algorithm for BLHOM_H.

In this paper we will focus on the problems EXT_H^Δ and CLHOM_H^Δ. The problem BLHOM_H will play an auxiliary role.

It is clear that each CLHOM_H^Δ is a restriction of LHOM_H^Δ, and that, if H is connected, EXT_H^Δ is a restriction of CLHOM_H^Δ. When $\Delta \leq |V(H)|$, there is a simple polynomial-time reduction from EXT_H^Δ to the problem BLHOM_H

- each pre-coloured vertex g of the input G is replaced by $deg_G(g)$ vertices of degree one, each attached to one neighbour of g, and all pre-coloured by the same colour as g. It is also clear that HOM_H is a restriction of EXT_H (*all* lists are $V(H)$). Figure 1 illustrates the *containment* of the problems, i.e., a lower placed problem is a restriction of an adjacent higher placed problem. Assuming $\Delta \leq |V(H)|$, there is an analogous figure for the Δ-restricted versions LHOM_H^{Δ}, CLHOM_H^{Δ}, EXT_H^{Δ}, and HOM_H^{Δ}. The inclusion (1) only applies if H is connected (in both versions), and the dashed inclusion (2) only applies to the Δ-restricted version EXT_H^{Δ}.

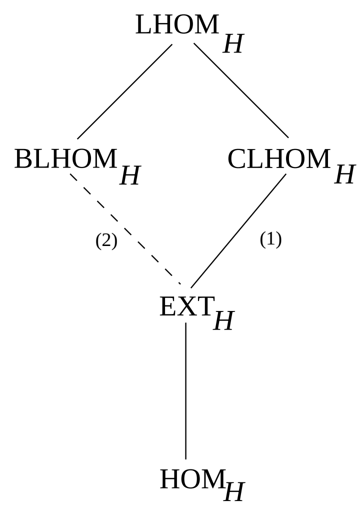

Figure 1: Various list homomorphism problems (see the above explanations)

The results of [10] open the way to classifying the complexity of the problems BLHOM_H, and therefore of CLHOM_H^{Δ}, and EXT_H^{Δ}, for all reflexive and irreflexive cycles H. Interestingly, in these situations restricting the degree Δ can have an important effect on the complexity of the problem.

In [6], the first two authors proved that, for a reflexive graph H, the problem CLHOM_H is polynomial-time solvable when H is a chordal graph (contains no induced cycle of length greater than three), and is NP-complete otherwise. As a byproduct of our results, we show that the complexity of the same problems does not change when degree constraints are imposed.

Theorem 2 *Let $\Delta \geq 3$ and a reflexive graph H be fixed. The problem $CLHOM_H^{\Delta}$ is polynomial-time solvable when H is a chordal graph, and is NP-complete otherwise.*

The following two tables summarize our results about the degree restricted homomorphism problems for reflexive and irreflexive cycles – showing the gradation of complexity of the various homomorphism problems, in terms of maximum degrees of the input graphs. (We omitted the superscripts Δ from the problem names in the table headings.) In each table, the last row ($\Delta \geq 4$ respectively $\Delta \geq 5$) also describes the situation for unrestricted degrees.

Table I: Irreflexive cycles

MAXIMUM DEGREE Δ	LENGTH k	HOM	EXT	CLHOM	LHOM
$\Delta = 3$	$k = 3$	P	P	NP-c	NP-c
	$k = 4$	P	P	P	P
	$k \geq 5$ odd	NP-c	NP-c	NP-c	NP-c
	$k = 6$	P	P	NP-c	NP-c
	$k \geq 8$ even	P	NP-c	NP-c	NP-c
$\Delta \geq 4$	$k = 3$	NP-c	NP-c	NP-c	NP-c
	$k = 4$	P	P	P	P
	$k \geq 5$ odd	NP-c	NP-c	NP-c	NP-c
	$k = 6$	P	P	NP-c	NP-c
	$k \geq 8$ even	P	NP-c	NP-c	NP-c

Table II: Reflexive cycles

MAXIMUM DEGREE Δ	LENGTHS k	HOM	EXT	CLHOM	LHOM
$\Delta = 3$	$k = 3$	P	P	P	P
	$k = 4, 5$	P	P	NP-c	NP-c
	$k \geq 6$	P	NP-c	NP-c	NP-c
$\Delta = 4$	$k = 3$	P	P	P	P
	$k = 4$	P	P	NP-c	NP-c
	$k \geq 5$	P	NP-c	NP-c	NP-c
$\Delta \geq 5$	$k = 3$	P	P	P	P
	$k \geq 4$	P	NP-c	NP-c	NP-c

2 Extension Problems

As noted above, the extension problem EXT_H^4 for the reflexive four-cycle H can be reduced to the problem BLHOM_H, which is solved in polynomial time, since the reflexive four-cycle is a nearly complete graph [9].

Corollary 1 *The problem EXT_H^4 for the reflexive four-cycle H can be solved in polynomial time.*

We also introduce a different tool.

Theorem 3 *Let H be a graph (with loops allowed) such that any two vertices a, b have a common neighbor c in H. Then the problem EXT_H^3 can be solved in polynomial time.*

Proof. Let G be a connected instance of EXT_H^3. As before, we may assume that all pre-coloured vertices g of G have degree one. (Otherwise we may replace a pre-coloured vertex g of degree k by k vertices of degree one, pre-coloured by the same colour, adjacent to the k different neighbours of g.) Now we have a balanced instance, and so we may assume there are no other (not pre-coloured) vertices of degree smaller than three. Indeed, if g of degree one or two is not pre-coloured, then a list homomorphism (extension) exists by the following argument. We can assign colours G greedily in the order of decreasing distance to g - whenever a vertex $x \neq g$ is being coloured, it has at most two previously coloured neighbors, and, by assumption, whatever two colours a, b occur on the neighbours of x, some colour c will be suitable for x. To complete the colouring we note that since $deg_G(g) \leq 2$, the vertex g also has at most two previously coloured neighbours and so the same argument applies.

Suppose first that G is a block. It could be an edge uv with pre-coloured vertices u, v, which may or may not be legally coloured. On the other hand, if G has three vertices, then none are pre-coloured, and all have degree three. If G has four vertices, then it is a K_4, which may or may not be H-colourable. (Recall that all lists are $V(H)$, thus this only depends on whether or not H itself contains a loop or a K_4.) Otherwise, G has at least five vertices. It is easy to argue that in this case G contains a vertex g such that $G - g$ is also a block. (According to a theorem of Kaugars, every critical block other than an edge contains a vertex of degree two, cf. [2], page 49.) Since G is cubic and has more than four vertices, we can find vertices u, v be such that u is a neighbor of g, and v is a neighbor of u, but v is not a neighbor of g. We now note that $G - g - v$ is connected, since $G - g$ is a block. We colour g and v by the same colour a (recalling that all lists are $V(H)$), and colour the remaining vertices greedily in order of decreasing distance to u in $G - g - v$. Every vertex other than u has at most two previosuly coloured neighbours, and u has only two previously assigned colours on its three neighbours (as the colours of v and g are the same), whence we can apply our assumption.

If G is not a block, then the blocks may not contain internal (noncutpoint) vertices of degree two, but a cutpoint could have degree two in some of its incident blocks. Suppose first that no block of G has a vertex of degree three in the block. Then every block B is either a single edge or a cycle. We may repeatedly consider a leaf block B with cutpoint w, and determine which colours of w can be extended to a colouring of all vertices u in B, by traversing the cycle B in one direction starting from w, or examining the single edge B. The problem then reduces to a problem for $G - (B - w)$. Repeatedly removing leaf blocks B' in this way leads to the situation where G is a block which is solved in the previous paragraph. Finally, assume that at least one block B contains at least one vertex v of degree three in B. Then we may assume v has a neighbour w that is a cutpoint. Let G' be the component containing w of the subgraph of G obtained by removing the neighbors of w in B. We may greedily colour the vertices of $G' - w$ in order of decreasing distance from w. Let a be the colour assigned to a neighbor u of w in G'. Colour v by a, and colour the vertices of $G - (G' - w) - v$ greedily in order of decreasing distance from w. □

The algorithm can declare that an extension exists unless the instance has the very special structure described in the proof above. Specifically, we don't have to check anything unless G is an edge with both vertices pre-coloured, or a four-clique, or has only pre-coloured vertices of degree one and non-pre-coloured vertices of degree three and consists of blocks which are either edges or cycles. In these instances, it is a simple matter to check whether or not an extension exists.

Let H be a reflexive graph of diameter at most two. Then the condition in Theorem 3 is satisfied and hence EXT_H^3 can be solved in polynomial time.

Corollary 2 *The problem EXT_H^3 for the reflexive five-cycle H can be solved in polynomial time.*

If H is an irreflexive graph of diameter at most two and each edge belongs to a triangle, the condition in Theorem 3 is also satisfied, and hence EXT_H^3 can be solved in polynomial time.

Corollary 3 *The problem EXT_H^3 for the irreflexive three-cycle H can be solved in polynomial time.*

Recall that the extension problems EXT, being special list homomorphism problems LHOM, were formulated for instances with irreflexive graphs G. For the list homomorphism problems, this is a natural restriction, since we can accommodate loops in G by modifying the lists. (If $g \in V(G)$ has a loop, delete the loop and remove from $L(g)$ any vertices $h \in V(H)$ without loops.) For the extension problems, we have the following result. We note that a loop in G contributes only one to the degree of its vertex.

Corollary 4 *Let H be a graph with loops allowed such that any two vertices a, b have a common neighbor c in H, and any vertex u in H has a neighbor v that has a loop. Then EXT_H^3 can be solved in polynomial time even for graphs G with loops allowed.*

PROOF. If a connected G has a vertex v of degree two we may greedily assign colours to the vertices of G, in the order of decreasing distance to v. If G is a block then either it is an edge or it has no pre-coloured vertices; in the latter case it can be mapped to a loop.

If G is not a block but has a block B that contains a vertex v without loop of degree three in B, then we may choose such v so that some neighbor u of v has a loop or has an edge $e = uw$ incident to u that belongs to a different block B'. In the latter case we may greedily colour the subgraph G' attached at u via the edge e in the order of decreasing distance to u. Once we have coloured $G' - u$, we may assign to v the same colour as to w, and finally colour $G - G' + u - v$ greedily by decreasing distance to u. If u has a loop, then v has a neighbor z in B different from u such that z has degree three, where z may have a loop (with two other incident edges) or not, such that $G - u - z$ is connected. We then assign the same colour s to both u and z, where s is a loop in H, and colour $G - u - z$ in the order of decreasing distance to v.

We are thus left with the case where each block B of G is either an edge or a cycle, possibly with some loops. This case is solved as before. □

3 Irreflexive Cycles

The graphs in this section are restricted to be irreflexive (but not necessarily bipartite). We shall focus on the the problems LHOM_H^Δ and EXT_H^Δ, in the case when H is the irreflexive cycle of length k. Both the unrestricted versions LHOM_H and EXT_H have the same time complexity - polynomial when $k = 4$, and NP-complete otherwise (when $k = 3$ or $k \geq 5$). Indeed, when $k = 4$, LHOM_H (and hence also EXT_H) is obviously polynomial-time solvable. If the input is not bipartite, no (list) homomorphism can exist; otherwise, we may assume that black vertices of the input graph have black lists and white vertices have white lists. In the four-cycle any black vertex is adjacent to any white vertex - thus we can simply assign to each vertex of G any vertex of its list. On the other hand, when k is odd, HOM_H is NP-complete [14], and hence so is EXT_H and LHOM_H. (HOM_H is the list homomorphism problem in which all lists are $V(H)$.) It was proved independently by Gary MacGillivray (personal communication) and Tomas Feder, see for instance [7], that EXT_H (and thus also LHOM_H) is NP-complete for $k = 2q > 4$.

This classification remains unchanged for LHOM_H^Δ:

Proposition 1 *Let H be the irreflexive cycle of length k, and let $\Delta \geq 3$.*
If $k = 4$, then the problem $LHOM_H^\Delta$ is polynomial-time solvable.
Otherwise ($k = 3$ or $k \geq 5$), the problem $LHOM_H^\Delta$ is NP-complete.

Proof. The case of $k = 4$, follows from the general remarks above. When k is odd, $k \neq 3$, this follows from the fact that HOM_H^Δ is NP-complete [12].

The cases of k even, $k \geq 6$, are covered by Propositions 1 and 2 of [9].

The NP-completeness of LHOM_H^Δ when $k = 3$ follows from Proposition 2 of [9], which proves the NP-completeness of LHOM_H^Δ when H is an irreflexive six-cycle. Indeed, there is a natural transformation which changes an arbitrary graph H into a bipartite graph H^* in such a way that H is a bi-arc graph if and only if the complement of H^* is a circular arc graph, [8]. The graph H^* is called the *associated bipartite graph* of the graph H, and is defined to have the vertex set $\{n_h, s_h : h \in V(H)\}$ and the edge set $\{n_h s_{h'}, s_h n_{h'} : hh' \in E(H)\}$. It is immediate from the definitions that a bi-arc representation of H is a circular arc representation of the complement of H^*. It is shown in [8] that if LHOM_{H^*} is NP-complete then LHOM_H is also NP-complete, and it is easy to see that the proof given there also implies that if $\mathrm{LHOM}_{H^*}^\Delta$ is NP-complete then so is LHOM_H^Δ, for any Δ. It now only remains to note that when H is the irreflexive three-cycle, then H^* is the irreflexive six-cycle. □

However, restricting the degrees has an effect on the complexity of extension:

Proposition 2 *Let H be the irreflexive cycle of length k, and let $\Delta = 3$.*
If $k = 3, 4, 6$, then EXT_H^Δ is polynomial-time solvable.
Otherwise ($k = 5$ or $k \geq 7$), EXT_H^Δ is NP-complete.

Proof. For $k = 4$ we note that EXT_H^Δ is a restriction of LHOM_H^Δ which is polynomial-time solvable by the preceding proposition. For $k = 3$, we apply Corollary 3. A similar algorithm takes care of the problem EXT_H^Δ when $k =$

6: assume that H is the six-cycle with consecutive vertices $1, 2', 3, 1', 2, 3'$. We may assume that the input graph G is bipartite (with, say, black and white vertices), and that all pre-coloured vertices are either black with lists $\{i\}(i = 1, 2, 3)$, or white, with lists $\{j'\}(j = 1, 2, 3)$. Replacing each pre-coloured vertex with three vertices of degree one, each adjacent to one of the neighbours, and all of the same colour, again reduces the problem to an instance of BLHOM_H. Although H is not nearly complete in the standard definition, we can view it as nearly complete in the bipartite sense - each white vertex is nonadjacent to just one black vertex and conversely. (By adding all loops and edges except for $11', 22', 33'$ we obtain a nearly complete graph.) It can be easily checked that the polynomial algorithm for BLHOM still applies in this context.

The NP-completeness of EXT_H^Δ for cycles of odd length $k \geq 5$ follows from [12]. Thus it remains to show NP-completeness of EXT_H^Δ for cycles H of even length $k \geq 8$.

In [7] (Theorem 3.1) we have shown a simple reduction of the problem of r-colourability to EXT_H, where H is the $2r$-cycle. We now explain a similar, but more elaborate, reduction which also ensures that all degrees are at most three, i.e., reduce r-colourability to EXT_H^Δ. Let H be the cycle $01 \ldots (2r)0$, with $r \geq 4$. For any graph F, we shall construct (in polynomial time) a graph G with lists $L(v) \subseteq V(H)$, $v \in V(G)$, each of which is either a singleton or the whole set $V(H)$, in such a way that F is r-colourable if and only if G admits a list homomorphism to H with respect to the list L.

Assume first that r is even, i.e., $2r$ divisible by four. The first step in the reduction is to replace each edge xy of F with a separate copy $Z(x, y)$ of the following gadget Z. In Z there is a $2r$-cycle $Z_1 = H$ with vertices $0, 1, \ldots, 2r - 1$. There are also additional $2r$-cycles $Z_2, Z_3, \ldots, Z_r$ (each isomorphic to Z_1), with edges joining corresponding odd vertices between Z_{2i} and Z_{2i+1} and joining corresponding even vertices between Z_{2i+1} and Z_{2i+2}. Two opposite vertices x', y' of degree two are chosen in Z_r, an edge xx' and a path from y to y' of length $r - 3$ are attached. We now define the lists for the resulting graph G' - each vertex v of Z_1 has the list $L(v) = \{v\}$ and all other vertices have the list $V(H)$.

The connections between the consecutive cycles Z_i assure that in any list homomorphism the vertex j of Z_i can only be identified (have the same image) with the vertex $(j - 1)$ or $(j + 1)$ of Z_{i-1}. This is clear if j is adjacent to Z_{i-1}, and follows easily if j is not adjacent to Z_{i-1} because its two neighbours are adjacent to Z_{i-1} and so if j didn't map to $j - 1$ or $j + 1$ it would not be adjacent to the image of one of these neighbours. (Here we use the fact that r is at least 4.)

Thus each Z_i must rotate one step clockwise or one step counterclockwise to Z_{i-1}. Therefore x' and y' can only map to even vertices of H, and any two opposite even vertices of H are possible images of x', y'. It follows that any two distinct odd vertices are possible images of x and y. Hence F is r-colourable if and only if G' admits a list homomorphism.

When r is odd we do not have two opposite vertices x', y' of degree two in Z_r. It is not difficult to find other ways to interconnect the consecutive cycles Z_i, to obtain two opposite vertices of degree two in Z_r, which assure that each Z_i must rotate one step clockwise or counterclockwise to Z_{i-1}. For instance we may connect vertices $1, r-1, r+1$ and $r+3$ between Z_i and Z_{i+1} when i is even, and vertices $2, r, r+2$ and $r+4$ when i is odd.

We note that G contains all the vertices of F, and that all other vertices of G' have degree at most three. The second step in the reduction makes sure all the degrees are at most three. Thus we again replace in G' each vertex v of F by its own gadget $Y(v)$ described below. Suppose the degree of v in G' is d. Then $Y(v)$ is contains a cycle Y_1 with $2rd$ vertices and consecutive lists $\{0\}, \{1\}, \ldots, \{2r-1\}, \{0\}, \{1\}, \ldots, \{2r-1\}, \ldots \subseteq V(H)$. Isomorphic copies $Y_2, Y_3, \ldots, Y_r$ of the cycle Y_1 are joined to Y_1 in the same way as cycles were joined in the first part of the proof. The vertices of all these cycles have lists $V(H)$. It is easy to see that the vertices of degree two that are $2r$ apart in Y_r all must map to the same even (respectively odd) vertex of H, and any even (respectively odd) vertex of H is a possible image. Thus we may construct G by attaching each of the edges of G' at v to a different vertex of degree two in Y_r of $Y(v)$. Then F admits an r-colouring if and only if G admits a list homomorphism, and all degrees in G are at most three. □

However, when the degree restriction is even slightly weaker, $\Delta \geq 4$, the classification reverts back to the one enjoyed by unrestricted degrees:

Corollary 5 *Let H be the irreflexive cycle of length k, and let $\Delta \geq 4$.*
If $k = 4$, then EXT_H^{Δ} is polynomial-time solvable.
Otherwise ($k = 3$ or $k \geq 5$), EXT_H^{Δ} is NP-complete.

Proof. Only the case when H is the irreflexive three-cycle and the irreflexive six-cycle require an explanation: When H is the three-cycle, then already the problem HOM_H^{Δ} is NP-complete [12]. When H is the six-cycle the result follows from the proof in the previous section, because we can assume all vertices in a list have the same parity, and a vertex with list of size two given by $i-1, i+1$ can be represented by including an edge to i. □

4 Reflexive Cycles

We now consider the case of reflexive cycles:

Proposition 3 *Let H be the reflexive cycle of length k, and let $\Delta \geq 3$.*
If $k = 3$, then $CLHOM_H^\Delta$ is polynomial-time solvable.
Otherwise ($k \geq 4$), $CLHOM_H^\Delta$ is NP-complete.

The lists used in the NP-completeness proof are either a single vertex, two adjacent vertices, or all the vertices of H; lists of three vertices are also used in the case $k = 4$.

Proof. For a reflexive triangle every instance has a solution. We show NP-completeness for $k \geq 4$.

We shall give a polynomial reduction from the NP-complete problem 3-SAT. Thus assume we have an instance of 3-SAT with d clauses. We take a cycle C of length dk for each Boolean variable x_i. We shall consider each vertex with list $i, i+1$ as describing a Boolean variable with corresponding values $0, 1$, with i corresponding to 0 and $i+1$ corresponding to 1. A cycle of length dk whose ith vertex has list $i, i+1$ (modulo k) allows us to represent a vertex with list $i, i+1$ of degree d and establishes the correspondence between such lists for different i.

We shall encode a clause $x \vee \overline{y} \vee \overline{z}$ on these boolean variables. Analogously, we can obtain a clause $\overline{x} \vee y \vee z$ on these boolean variables. Combining these two types of clauses with the single literal clauses x and $\overline{x}$ encodes the 3SAT problem, which is NP-complete.

For $k = 2r$ or $k = 2r - 1$, we use x with list $0, 1$, y with list $1, 2$, and z with list $r, r+1$. We add a vertex t adjacent to both x and y, with t joined by a path of length $r - 1$ to z. These added vertices have full lists, with the following exceptions: For $k = 4$ the list of t is $0, 1, 2$ (3 is excluded); for $k = 2r - 1$, the list of the vertex u adjacent to t on the path from t to z excludes the value 0 (we can represent this with a path of length $r - 2$ from u to a vertex v with list $r - 1, r$). The effect of this gadget is to forbid the assignment $x = 0$, $y = 2$, $z = r + 1$, and only this assignment. In terms of boolean variables, the forbidden assignment is $xyz = 011$; thus the clause $x \vee \overline{y} \vee \overline{z}$ is obtained. □

The problems CLHOM_H for reflexive graphs H were considered by the first two authors [6], where it is shown that CLHOM_H is polynomial-time solvable when H is a chordal graph, and is NP-complete otherwise. Combining it with the above proposition, we see that Theorem 2 holds.

We now settle in detail the complexity of EXT_H^Δ, when H is the reflexive cycle of length k. It was shown independently by MacGillivray and Feder that without degree restrictions EXT_H is polynomial-time solvable when $k = 3$, and is NP-complete otherwise ($k \geq 4$), cf. [6, 7]. Here the effect of restricting the degree is quite pronounced:

Proposition 4 *Let H be the reflexive cycle of length k.*

If $\Delta = 3$, then EXT_H^Δ is polynomial-time solvable when $k = 3, 4, 5$ and NP-complete otherwise ($k \geq 6$).

If $\Delta = 4$, then EXT_H^Δ is polynomial-time solvable when $k = 3, 4$ and NP-complete otherwise ($k \geq 5$).

If $\Delta \geq 5$, then EXT_H^Δ is polynomial-time solvable when $k = 3$ and NP-complete otherwise ($k \geq 4$).

The proposition will follow from the following smaller pieces:

Proposition 5 *Let H be the reflexive cycle of length k.*

Then EXT_H^Δ is NP-complete if $k = 4$ and $\Delta \geq 5$, if $k = 5$ and $\Delta \geq 4$, and if $k \geq 6$ and $\Delta \geq 3$.

Proof. If $k = 4$ and the maximum degree is five, then for a vertex of degree at most three in Proposition 3, a list $i, i+1$ can be replaced with two edges to i and to $i+1$, and a list $i-1, i, i+1$ can be replaced with an edge to i; the degree increases by two.

If $k \geq 5$, consider the cycle of length kd with lists $i, i+1$. Only one of these kd lists needs to be represented with two edges to i and to $i+1$, and thus have degree 4 if we count the two edges of the cycle as well. The next vertex with list $i+1, i+2$ need only have an edge to $i+2$, because the fact that it is adjacent to a vertex with list $i, i+1$ and to $i+2$ implies that it is adjacent to $i+1$ as well. The degree two for these elements from the cycle increases thus only to three (increases by one) for the $kd-1$ vertices that will be used elsewhere in the instance.

If $k \geq 7$, we only need to be able to simulate a vertex of degree $d = 4$ with degree 3. We use a construction similar to the one in Proposition 2, and construct a gadget with d vertices of degree two that must all map to the same vertex (any vertex) H. The gadget starts with a cycle D_1 of length $2dk$ whose vertices are viewed as integers $0, 1, 2, \ldots, dk-1$ modulo dk. A vertex of D_1 in position i is given a list of size 1 assigning to it the vertex of H in position i modulo k. Add a cycle D_2 of length dk so that a vertex in position $2i$ of D_1 is connected to the vertex in position $2i+1$ of D_2. Thus

D_2 can be mapped to H in three possible ways. Connect similarly D_2 to a cycle D_3, so that D_3 can now be mapped to H in five possible ways, and so on all the way up to D_r for $r = \lceil \frac{k+1}{2} \rceil$, which can be mapped in all k possible ways of the correct orientation and parity. The d vertices in positions $2ik$ of D_r they all map to the same vertex of H, which can be any vertex, and they all have degree two.

The remaining case is $k = 6$ with maximum degree 3. The reflexive cycle of length 6 is $-2, -1, 0, 1, 2, 3$. We show that we can represent a vertex with list $-1, 1$ and arbitrary degree. The gadget R consists of four paths $0a_1b_1c_13$, $0a_2b_2c_23$, $0d_1c_1$, $0d_2c_2$, and three additional edges a_1e, a_2e, $e3$. Inspection of this gadget shows that a_1, a_2, d_1, d_2 can only have values $-1, 1$, that c_1, c_2, e can only have values $-2, 2$, and that b_1, b_2 can only have values $-1, -2, 1, 2$. Furthermore, the values for these nine vertices are either all from $-1, -2$ or all from $1, 2$.

We make copies R^j for $1 \le j \le d$ of the gadget R, and add edges $b_1^j b_2^{j+1}$. The result is $2d$ vertices of degree 2, namely d_1^j, d_2^j, that must all take value -1 or all take value 1. This simulates a vertex x_0 with list $-1, 1$ and of degree $2d$.

We encode not-all-equal SAT. We can use the above construction to simulate vertices x_i with list $i-1, i+1$ of arbitrary degree. A cycle $x_{-2}x_{-1}x_0x_1x_2x_3$ forces the x_i to all have value $i-1$ or all have value $i+1$. We can then view the x_i as boolean variables with $i-1, i+1$ corresponding to boolean values $0, 1$. Now consider the vertices x_{-2} with list $3, -1$, x_0 with list $-1, 1$, and x_2 with list $1, 3$. If these three vertices are made adjacent to a vertex v, then v can be assigned a value unless x_{-2}, x_0, x_2 have values $3, -1, 1$ respectively or $-1, 1, 3$ respectively. This means in terms of the corresponding boolean variables that the two assignments 000 and 111 are forbidden. Thus not-all-equal SAT is represented. □

In the companion paper [9] we have conjectured that all the homomorphism type problems HOM_H, LHOM_H, CLHOM_H, EXT_H, etc., have the same classifications even when restricted to graphs with degrees at most Δ, as long as Δ is chosen large enough. (In fact, this is true for LHOM_H because of the results in [9].) All results obtained in this paper support this conjecture.

We have recently learned that Mark Siggers [20] has proved the conjecture for the case of the problems HOM_H.

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